SIAM J. CONTROL AND OPTIMIZATION
Vol. 20, No. 4, July 1982

0363-0129/82/2004-0009 $01.00/0

CONTROLLABILITY FOR DISTRIBUTED BILINEAR SYSTEMS*

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Abstract. This paper studies controllability of systems of the form $dw/dt = \mathcal{A}w + p(t)\mathcal{B}w$ where $\mathcal{A}$ is the infinitesimal generator of a C^0 semigroup of bounded linear operators $e^{\mathcal{A}t}$ on a Banach space X, $\mathcal{B}: X \to X$ is a C^1 map, and $p \in L^1([0, T]; \mathbb{R})$ is a control. The paper (i) gives conditions for elements of X to be accessible from a given initial state w_0 and (ii) shows that controllability to a full neighborhood in X of w_0 is impossible for $\dim X = \infty$. Examples of hyperbolic partial differential equations are provided.

1. Introduction.

The purpose of this paper is to discuss controllability for abstract evolution equations of the form

$$\dot{w}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}(w(t)), \tag{1.1}$$

$$w(0) = w_0, \tag{1.2}$$

where $\mathcal{A}$ generates a C^0 semigroup of bounded linear operators on a (possibly complex) Banach space X, $\mathcal{B}: X \to X$ is a C^1 map, and $p \in L^1([0, T]; \mathbb{R})$ is a control defined on a specified interval $[0, T]$. Usually we assume that $\mathcal{B}$ is linear, so that (1.1) is *bilinear* in the pair (p, w); note that even in this case the solution w of (1.1), (1.2) is a *nonlinear function of p*. A motivating example is the rod equation

$$u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, \qquad 0 < x < 1, \tag{1.3}$$

with hinged end conditions

$$u = u_{xx} = 0 \quad \text{at } x = 0, 1, \tag{1.4}$$

which can be put in the form (1.1) by setting $w = \binom{u}{u_t}$ with $X = (H^2(0,1) \cap H_0^1(0,1)) \times L^2(0,1)$. Here the control $p(t)$ is the axial load.

The main tool used in our analysis is the generalized inverse function, or "local onto" theorem. In finite dimensions, the well-known controllability results for bilinear systems have been obtained in this way (see, for example, Brockett [1972] and Hermes [1974]). In infinite dimensions, however, new phenomena arise. Perhaps the most interesting of these is our result (Theorem 3.6) which shows that for $\mathcal{B}$ linear and $\dim X = \infty$, the set of states accessible from w_0 for $p \in L^r_{loc}([0, \infty); \mathbb{R})$, $1 < r \leqq \infty$, has *dense complement* in X. Hence we can *never* expect to control to an open neighborhood of w_0 for controls in L^r_{loc}. (Using L^1 controls doesn't help, at least for examples such as (1.3), (1.4); see Theorem 5.5.) This stands in direct contrast to the available positive results on controllability when $\dim X < \infty$.

* Received by the editors March 24, 1981, and in revised form September 25, 1981.

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Given the impossibility of controlling the system (1.1) to a full neighborhood of w_0 with p's in L^r, we investigate two alternative procedures. One approach generalizes an idea of Hermes [1979]; we show that it is often possible to *control with respect to finite-dimensional observations* in a neighborhood of w_0. Our second idea is based upon the concept of *approximate controllability*, i.e., we identify a dense subset of X, depending on w_0 and t, to which $w(t)$ belongs, and show that with respect to a strengthened topology one can control to a neighborhood of $e^{\mathcal{A}t}w_0$ (the "free solution" of (1.1), (1.2) corresponding to $p \equiv 0$) in this set, provided t is suitably chosen. For (1.3), (1.4) we prove that $t>0$ can be taken arbitrarily small, whereas for the wave equation

$$u_{tt} - u_{xx} + p(t)u = 0, \qquad 0 < x < 1, \tag{1.5}$$

with either the boundary conditions

$$u = 0 \quad \text{at } x = 0, 1,$$

or the boundary conditions

$$u = 0 \quad \text{at } x = 0, \qquad u + \alpha u_x = 0 \quad \text{at } x = 1, \qquad \alpha > 0,$$

t has to exceed some number $T>0$. This study of local approximate controllability involves technicalities concerning nonharmonic Fourier series in the spirit of Russell [1967] and Ball and Slemrod [1979]. The delicacy of these questions has the unfortunate consequence that we have only been able to obtain positive results in cases, such as these described above, in which (1.1) is an abstract hyperbolic equation that is "diagonal"; i.e., is reducible to an infinite set of uncoupled ordinary differential equations (each, of course, containing the control $p(t)$). Since we have to control infinitely many ordinary differential equations simultaneously, however, the problem is still not trivial. Nevertheless, our assumptions exclude some important nondiagonal examples such as (1.3) with clamped end conditions

$$u = u_x = 0 \quad \text{at } x = 0, 1.$$

In special cases, such as (1.3), (1.4), our local approximate controllability theory leads to a global approximate controllability result; thus, for example, for suitable initial data, we prove that the attainable set for (1.3), (1.4) is dense in X.

The paper is divided into six sections. Section 2 assembles the machinery for studying (1.1), (1.2) in the form of various abstract existence and smoothness theorems. Section 3 provides an abstract controllability theorem and the result on noncontrollability mentioned above. In § 4 we discuss the general theory of control with respect to finite-dimensional observers. In § 5 we consider abstract hyperbolic equations, apply the theory of § 4 to this case, and develop our theory of approximate controllability. We conclude in § 6 with specific applications to partial differential equations, such as (1.3), (1.4).

2. Abstract existence and smoothness theorems. In this section we give some basic results on nonlinear evolution equations which will be useful in our later analysis. Let X be a Banach space with norm $\|\cdot\|$, let $\mathcal{A}$ generate a C^0 semigroup of bounded linear operators on X, and let $\mathcal{B}: X \to X$ be a C^k mapping, $k \geqq 1$. Let $Z(T)$ be a Banach space continuously and densely included in $L^1([0, T]; \mathbb{R})$, where $T>0$ is given.

For a given $w_0 \in X$ and $p \in Z(T)$, consider the initial value problem associated with (1.1) written in integrated form, i.e.,

$$w(t) = e^{\mathcal{A}t}w_0 + \int_0^t e^{\mathcal{A}(t-s)}p(s)\mathcal{B}(w(s))\,ds. \tag{2.1}$$

Solutions of (2.1) are often called "mild solutions" of (1.1), (1.2). The question as to when solutions of (2.1) are actually solutions of (1.1) is discussed in Remark 2.7 at the end of this section.

PROPOSITION 2.1. *For each* $w_0 \in X$, *and* $p \in Z(T)$ *there exists* t_0, $0 < t_0 \leqq T$, *such that* (2.1) *has a unique solution* $w \in C([0, t_0]; X)$.

Proof. Let $\mathcal{F} = \{w \in C([0, t_0]; X) \| w(t) - w_0\| \leqq R\}$, and define $T_p : \mathcal{F} \to C([0, t_0]; X)$ by

$$(T_p w)(t) = e^{\mathcal{A}t} w_0 + \int_0^t e^{\mathcal{A}(t-s)} p(s) \mathcal{B}(w(s))\, ds.$$

Since $\|e^{\mathcal{A}t}\| \leqq M e^{\beta t}$ for positive constants β, M, an easy estimate shows that T maps $\mathcal{F}$ to $\mathcal{F}$ provided

$$\|e^{\mathcal{A}t} w_0 - w_0\| + M e^{\beta t_0} C \int_0^{t_0} |p(s)|\, ds \leqq R, \qquad 0 \leqq t \leqq t_0,$$

where C is such that $\|Bw\| \leqq C$ for $\|w - w_0\| \leqq R$. This condition is achieved for R, t_0 sufficiently small via the continuity of $\mathcal{B}$, $e^{\mathcal{A}t} w_0$ and the fact that $p \in L^1([0, T]; \mathbb{R})$. Similarly, T_p is a contraction map of $\mathcal{F}$ to $\mathcal{F}$ provided that

$$KM e^{\beta t_0} \int_0^{t_0} |p(s)|\, ds < 1,$$

where K is a Lipschitz constant for $\mathcal{B}$ on the ball $\|w - w_0\| \leqq R$. Again this holds for R and t_0 sufficiently small. The result now follows from the contraction mapping principle. □

Of course the above proposition is a special case of many more general results on existence and uniqueness of solutions to semilinear evolution equations (see, for example, Segal [1963], Pazy [1974], Balakrishnan [1976] and Tanabe [1979b]). The point for us here is that use of the contraction mapping principle leads to other important features of the solution map w, as we now see.

PROPOSITION 2.2. *Fix* $p_0 \in Z(T)$. *Then there exist an open neighborhood* U *of* p_0 *in* $Z(T)$ *and* $t_0 > 0$ *such that for any* $p \in U$, (2.1) *has a unique solution* $w(t; p, w_0)$, $0 \leqq t \leqq t_0$. *Moreover* $w(t; p, w_0)$ *is a* C^k *map from* U *to* $C([0, t_0]; X)$.

Proof. The proof of Proposition 2.1 shows that if R and t_0 are sufficiently small and p is close enough to p_0 in L^1-norm then T_p is a uniform contraction. Also, T_p is a C^k function of w and p on the interior of $\mathcal{F}$, so that the C^1 result follows from Hale [1969, Thm. 3.2, p. 7]. The C^k result is then obtained by induction. □

COROLLARY 2.3. *The map* $w(t_0; \cdot, w_0): U \to X$ *is* C^k.

Proof. This follows from the chain rule, Proposition 2.2 and the fact that the map $w(\cdot) \mapsto w(t_0)$ is smooth (since it is continuous and linear from $C([0, t_0]; X)$ to X). □

In the same way we see that the solution $w(t; \cdot, \cdot)$ is a C^k function of w_0 and p. However, in this paper we are primarily concerned with differentiability in p. The proof of the theorem in Hale [1969] cited above shows that the derivative can be obtained by formally linearizing. Thus we get the following result.

COROLLARY 2.4. *The (Fréchet) derivative* $D_p w(t; p_0, w_0) \cdot p$ *of* $w(t; p, w_0)$ *with respect to* p *at* p_0 *in the direction* p *is the unique solution of the equation*

$$\begin{aligned} D_p w(t; p_0, w_0) \cdot p &= \int_0^t e^{\mathcal{A}(t-s)} p(s) \mathcal{B}(w(s; p_0, w_0))\, ds \\ &\quad + \int_0^t e^{\mathcal{A}(t-s)} p_0(s) D\mathcal{B}(w(s; p_0, w_0)) D_p w(s; p_0, w_0) \cdot p\, ds. \end{aligned} \tag{2.2}$$

Here $D\mathscr{B}(w(s;p_0,w_0))$ denotes the Fréchet derivative of $\mathscr{B}$ at $w(s;p_0,w_0)$. In particular, at $p_0=0$, $D_pw(t;0,w_0)\cdot p$ is given explicitly by

$$D_pw(t;0,w_0)\cdot p=\int_0^t e^{\mathscr{A}(t-s)}p(s)\mathscr{B}(e^{\mathscr{A}s}w_0)\,ds. \tag{2.3}$$

Next we show that solutions are globally defined under a sublinear growth condition.

THEOREM 2.5. *If there are constants C and K such that $\|\mathscr{B}(x)\|\leqq C+K\|x\|$ for all $x\in X$, then (2.1) has solutions defined for $0\leqq t\leqq T$. These solutions are unique within the class $C([0,T];X)$. Moreover, the solution $w(t;p,w_0)$ is a C^k function of $p\in Z(T)$ and $w_0\in X$ with (Fréchet) derivative in p given by (2.2) (or (2.3) if $p_0=0$).*

The proof is based on the following version of Gronwall's inequality (see, for example, Carroll [1969, p. 124]).

LEMMA 2.6. *Let $p\in L^1([a,b];\mathbb{R})$ and let $v\in L^\infty([a,b];\mathbb{R})$ with $v\geqq 0$. If there exists a constant $C\geqq 0$ such that for all $t\in[a,b]$*

$$v(t)\leqq C+\int_0^t|p(s)|v(s)\,ds,$$

then

$$v(t)\leqq C\exp\left(\int_a^t|p(s)|\,ds\right).$$

Proof of Theorem 2.5. Suppose $w(t)$ solves (2.1) and is defined for $0\leqq t<a\leqq T$. Then

$$\|w(t)\|\leqq Me^{\beta a}\left(\|w_0\|+\int_0^t|p(s)|(C+K\|w(s)\|)\,ds\right),$$

and so, assuming $K>0$ without loss of generality, we get

$$\|w(t)\|\leqq(Me^{\beta a}\|w_0\|+CK^{-1})\exp\left(Me^{\beta a}K\int_0^t|p(s)|\,ds\right)-CK^{-1}\leqq C_1.$$

Therefore, for $s,t\in[0,a)$ we have

$$\|w(t)-w(s)\|\leqq\|e^{\mathscr{A}t}w_0-e^{\mathscr{A}s}w_0\|+\left\|\int_s^t e^{\mathscr{A}(t-\tau)}p(\tau)\mathscr{B}(w(\tau))\,d\tau\right\|$$

$$\leqq\|e^{\mathscr{A}t}w_0-e^{\mathscr{A}s}w_0\|+Me^{\beta a}(C+KC_1)\int_s^t|p(\tau)|\,d\tau.$$

Thus $\lim_{t\to a-}w(t)$ exists, so that by Proposition 2.1 $w(t)$ can be continued beyond $t=a$. Hence solutions are defined for $0\leqq t\leqq T$.

For global uniqueness, we use the standard argument: suppose $w(t)$ and $\bar{w}(t)$ solve (2.1) for $0\leqq t\leqq T$. Let $S=\{a\in[0,T]\,|\,w(t)=\bar{w}(t)\text{ for }t\in[0,a]\}$. The local uniqueness assertion in Proposition 2.1 shows that S is relatively open in $[0,T]$. If $a_n\in S$ and $a_n\to a\leqq T$ then $a\in S$ since $\lim_{n\to\infty}w(a_n)=\lim_{n\to\infty}\bar{w}(a_n)$. Thus S is closed, so that $S=[0,T]$.

Thus there is a globally defined semiflow $F_t^p(w_0)$, $F_t^p(\cdot):\mathbb{R}^+\times X\to X$, which depends parametrically on p. Proposition 2.2 shows that $F_t^p(w_0)$ is C^k in p and w_0 for t sufficiently small. Let $\tilde{S}=\{a\in[0,T]\,|\,F_t^p(w_0)\text{ is }C^k\text{ in }(w_0,p)\text{ for }t\in[0,a]\}$. We claim that $\tilde{S}$ is open. Indeed, if $a\in\tilde{S}$ and k is small,

$$F_{a+h}^p(w_0)=F_h^p(F_a^p(w_0))$$

is C^k in p and w_0, because by Proposition 2.2 $F_h^p(w)$ is C^k in p and w for w near $F_a^{p_0}(w_0)$. The local uniformity of the time interval on which Proposition 2.2 holds shows that $\tilde{S}$ is closed, and hence $\tilde{S} = [0, T]$.

Thus we have shown that $w(t; p, w_0)$ is C^k in p and w_0. By differentiating (2.1) we obtain (2.2). □

Remark 2.7. Suppose $w_0 \in D(A)$ and $p \in C^1([0, T]; \mathbb{R})$. Then $w(t) \in D(A)$ and $w(t)$ is differentiable and satisfies (1.1). This assertion follows from Segal [1963, Lemma 3.1] or from Tanabe [9, p. 102]. If merely $w_0 \in X$ and $p \in L^1([0, T]; \mathbb{R})$ then w is a "weak solution" of (1.1) (see Balakrishnan [1976] and Ball [1977]).

3. An abstract controllability theorem and a negative result. Define the linear operator $L_T : Z(T) \to X$ by

$$L_T p = \int_0^T e^{\mathcal{A}(T-s)} p(s) \mathcal{B}(e^{\mathcal{A}s} w_0)\, ds.$$

Then by (2.3) we have

$$D_p w(T; 0, w_0) \cdot p = L_T p. \tag{3.1}$$

A natural consequence of Theorem 2.5 is the following.

THEOREM 3.1. *Let $\mathcal{A}$ be the infinitesimal generator of a C^0 semigroup of bounded linear operators on the Banach space X, and let $\mathcal{B} : X \to X$ be a C^k map, $k \geqq 1$, which satisfies $\|Bx\| \leqq C + K\|x\|$ for all $x \in X$, where C and K are constants. Suppose that* Range$(L_T) = X$. *Then there is an $\varepsilon > 0$ such that $w(T; p, w_0) = h$ for some $p \in Z(T)$, provided $\|h - e^{\mathcal{A}T} w_0\| < \varepsilon$.*

This result follows easily from the (generalized) inverse function theorem; a convenient reference is Luenberger [1969, p. 240]. The p that controls w_0 to hit h will be in a neighborhood of zero in $Z(T)$.

We note that if $\mathcal{A}$ generates a group, surjectivity of L_T is equivalent to surjectivity of $\hat{L}_T : Z(T) \to X$, where

$$\hat{L}_T p = \int_0^T e^{-\mathcal{A}s} p(s) \mathcal{B}(e^{\mathcal{A}s} w_0)\, ds. \tag{3.2}$$

A major difficulty with Theorem 3.1 is that is is not usually an easy matter to check the surjectivity of L_T (or $\hat{L}_T$). In fact, as we shall prove in Theorem 3.6, if dim $X = \infty$, L_T *will not* in general be surjective, though it may have dense range. This prevents us from applying Theorem 3.1 to partial differential equations.

We now present a basic criterion for L_T to have dense range.

PROPOSITION 3.2. *Suppose that*

$$\langle l, e^{\mathcal{A}(t-s)} \mathcal{B}(e^{\mathcal{A}s} w_0) \rangle = 0$$

for all s, $0 \leqq s \leqq T$, where $l \in X^*$ (the dual space of X), implies $l = 0$. Then Range (L_T) *is dense in X.*

Proof. Range (L_T) is dense if the only $l \in X^*$ annihilating the range is $l = 0$. But

$$\langle l, L_T p \rangle = \int_0^T \langle l, e^{\mathcal{A}(T-s)} \mathcal{B}(e^{\mathcal{A}s} w_0) \rangle p(s)\, ds.$$

If this vanishes for all $p \in Z(T)$, then the continuous function $\langle l, e^{\mathcal{A}(T-s)} \mathcal{B}(e^{\mathcal{A}s} w_0) \rangle$ must vanish. This follows because $Z(T)$ is dense in $L^1([0, T]; \mathbb{R})$. Our hypothesis then gives $l = 0$.

Remark 3.3. If $\mathcal{B}$ is linear and $\mathcal{A}$ is a *bounded* linear operator, then

$$e^{-\mathcal{A}s}\mathcal{B}\,e^{\mathcal{A}s}w_0 = \mathcal{B}w_0 + s[\mathcal{A},\mathcal{B}]w_0 + \frac{s^2}{2}[\mathcal{A},[\mathcal{A},\mathcal{B}]]w_0 + \cdots$$

(i.e., the Campbell–Baker–Hausdorff formula), where $[\mathcal{A},\mathcal{B}] = -\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}$. From Proposition 3.2, we see that Range (L_T) is dense in X for all $T>0$ if the closure of the span of $\mathcal{B}w_0, [\mathcal{A},\mathcal{B}]w_0, [\mathcal{A},[\mathcal{A},\mathcal{B}]]w_0, \cdots$ is dense in X.

The next two well-known controllability results now follow for $X = \mathbb{R}^n$ and $\mathcal{B}$ linear.

COROLLARY 3.4 (Hermes [1974], Lobry [1970]). *Assume $X=\mathbb{R}^n$ and that* dim span $\{\mathcal{B}w_0, [\mathcal{A},\mathcal{B}]w_0, [\mathcal{A},[\mathcal{A},\mathcal{B}]]w_0, \cdots\} = n$. *Then for every $T>0$ there is an $\varepsilon_T>0$ with the property that if $\|e^{\mathcal{A}t}w_0 - h\| < \varepsilon_T$, we can find a $p \in Z(T)$ such that $w(T; p, w_0) = h$.*

Here one can choose $Z(T) = L^q([0,T];\mathbb{R})$ for any q, $1 \leqq q \leqq \infty$, or $Z(T) = C^k([0,T];\mathbb{R})$, for example.

COROLLARY 3.5 (Lobry [1970], Jurdjevic and Quinn [1978]). *Let the hypotheses of Corollary* 3.4 *hold. Assume $e^{\mathcal{A}t}w_0$ is almost periodic. Then for any $k \geqq 0$, there exist $T>0$ and $\varepsilon>0$ such that $\|h - w_0\| < \varepsilon$ implies $w(T; p, w_0) = h$ for some $p \in C^k([0,T];\mathbb{R})$.*

Proof. Let $T_1 > 0$ be fixed and let $\varepsilon_{T_1} > 0$ be as in Corollary 3.4. We show that if $\|h - w_0\| < \varepsilon_{T_1}/2$, then there exists $\tau > 0$ such that $w(T_1+\tau; p, w_0) = h$ for some $p \in C^k([0, T_1+\tau];\mathbb{R})$. First, by the almost periodicity of $e^{\mathcal{A}t}w_0$, there exists $\tau>0$ such that

$$\|e^{\mathcal{A}\tau}w_0 - e^{-\mathcal{A}T_1}w_0\| < \frac{\varepsilon_{T_1}}{2}\|e^{\mathcal{A}T_1}\|^{-1}.$$

We run (2.1) from time $t=0$ until $t=\tau$ with $p\equiv 0$, so that $w(\tau) = e^{\mathcal{A}\tau}w_0$. By Corollary 3.4, we can hit h in additional time T_1, using a C^k control which vanishes together with its first k derivatives at τ, provided $\|e^{\mathcal{A}T_1}w(\tau) - h\| < \varepsilon_{T_1}$. But this is true, since

$$\begin{aligned}\|e^{\mathcal{A}T_1}w(\tau) - h\| &= \|e^{\mathcal{A}T_1}e^{\mathcal{A}\tau}w_0 - h\| = \|e^{\mathcal{A}T_1}(e^{\mathcal{A}\tau}w_0 - e^{-\mathcal{A}T_1}w_0 + e^{-\mathcal{A}T_1}w_0) - h\| \\ &\leqq \|e^{\mathcal{A}T_1}\|\|e^{\mathcal{A}\tau}w_0 - e^{-\mathcal{A}T_1}w_0\| + \|w_0 - h\| < \varepsilon_{T_1}.\end{aligned}$$ □

In the case dim $X = \infty$ things are quite different. Specifically, we shall now show that for a large class of spaces $Z(T)$, the map $w(T;\cdot, w_0): Z(T) \to X$ *will never cover* an open neighborhood of $e^{\mathcal{A}T}w_0$ (and consequently L_T cannot be onto). Thus, for these $Z(T)$'s, Theorem 3.1 will be vacuous unless dim $X<\infty$.

THEOREM 3.6. *Let X be a Banach space with* dim $X=\infty$. *Let $\mathcal{A}$ generate a C^0 semigroup of bounded linear operators on X and let $\mathcal{B}: X \to X$ be a bounded linear operator. Let $w_0 \in X$ be fixed and let $w(t; p, w_0)$ denote the unique solution of* (2.1) *for $p \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R})$. If $T>0$ and $p_n \to p$ weakly in $L^1([0,T];\mathbb{R})$, then $w(\cdot\,; p_n, w_0) \to w(\cdot\,; p, w_0)$ strongly in $C([0,T];X)$. Moreover, the set of states accessible from w_0 defined by*

$$S(w_0) = \bigcup_{\substack{t\geqq 0\\ p\in L^r_{\mathrm{loc}}([0,\infty);\mathbb{R})\\ r>1}} w(t; p, w_0)$$

is contained in a countable union of compact subsets of X, and in particular has dense complement.

Proof. Let $p_n \to p$ weakly in $L^1([0, T]; \mathbb{R})$. Write $w_n(t) = w(t; p_n, w_0)$, $w(t) = w(t; p, w_0)$, and $z_n(t) = w_n(t) - w(t)$. Then

$$w_n(t) = e^{\mathcal{A}t} w_0 + \int_0^t p_n(s)\, e^{\mathcal{A}(t-s)} \mathcal{B} w_n(s)\, ds$$

and

$$w(t) = e^{\mathcal{A}t} w_0 + \int_0^t p(s)\, e^{\mathcal{A}(t-s)} \mathcal{B} w(s)\, ds,$$

so that

$$z_n(t) = \int_0^t [p_n(s) - p(s)]\, e^{\mathcal{A}(t-s)} \mathcal{B} w(s)\, ds + \int_0^t p_n(s)\, e^{\mathcal{A}(t-s)} \mathcal{B} z_n(s)\, ds. \tag{3.3}$$

We now need the following:

LEMMA 3.7. *Let*

$$\varepsilon_n = \sup_{t \in [0, T]} \left\| \int_0^t [p_n(s) - p(s)]\, e^{\mathcal{A}(t-s)} \mathcal{B} w(s)\, ds \right\|.$$

Then $\lim_{n \to \infty} \varepsilon_n = 0$.

Proof of Lemma 3.7. Suppose the lemma is false. Then there exist $\varepsilon > 0$, a subsequence $\{p_\mu\}$ of $\{p_n\}$ and a sequence $\{t_\mu\} \subset [0, T]$, $t_\mu \to t \in [0, T]$, such that for all μ

$$\left\| \int_0^t [p_\mu(s) - p(s)]\, e^{\mathcal{A}(t_\mu - s)} \mathcal{B} w(s)\, ds \right\| > \varepsilon. \tag{3.4}$$

We can suppose without loss of generality that either $t_\mu \leqq t$ for all μ, or $t_\mu \geqq t$ for all μ. In the case $t_\mu \leqq t$ let

$$c_\mu = \sup_{s \in [0, t_\mu]} \| (e^{\mathcal{A}(t_\mu - s)} - e^{\mathcal{A}(t-s)}) \mathcal{B} w(s) \|.$$

The joint continuity of the map $(x, \tau) \mapsto e^{\mathcal{A}\tau} x$ and the continuity of $w(\cdot)$ together imply that $c_\mu \to 0$ as $\mu \to \infty$. Hence

$$\begin{aligned} &\lim_{\mu \to \infty} \left\| \int_0^{t_\mu} [p_\mu(s) - p(s)] (e^{\mathcal{A}(t_\mu - s)} - e^{\mathcal{A}(t-s)}) \mathcal{B} w(s)\, ds \right\| \\ &\qquad \leqq \lim_{\mu \to \infty} c_\mu \int_0^{t_\mu} |p_\mu(s) - p(s)|\, ds = 0. \end{aligned} \tag{3.5}$$

Furthermore, since $p_\mu \to p$ weakly in $L^1([0, T]; \mathbb{R})$, $|p_\mu - p|$ is uniformly equi-integrable over $[0, T]$ (see Dunford and Schwartz [1964, pp. 293–294]), and hence

$$\lim_{\mu \to \infty} \left\| \int_{t_\mu}^t [p_\mu(s) - p(s)]\, e^{\mathcal{A}(t-s)} \mathcal{B} w(s)\, ds \right\| \leqq \text{const} \cdot \lim_{\mu \to \infty} \int_{t_\mu}^t |p_\mu(s) - p(s)|\, ds = 0. \tag{3.6}$$

Combining (3.5) and (3.6), we deduce that

$$\lim_{\mu \to \infty} \left\| \int_0^{t_\mu} [p_\mu(s) - p(s)]\, e^{\mathcal{A}(t_\mu - s)} \mathcal{B} w(s)\, ds - \int_0^t [p_\mu(s) - p(s)] v(s)\, ds \right\| = 0, \tag{3.7}$$

where $v(s)$ is defined by $v(s) = e^{\mathcal{A}(t-s)} \mathcal{B} w(s)$. A similar argument shows that (3.7) holds if $t_\mu \geqq t$ for all μ.

Let $\rho = \sup_\mu \int_0^t |p_\mu(s) - p(s)|\, ds$. Since $v \in C([0, T]; X)$ there exists a step function g such that $\|g - v\|_{L^\infty([0,T];X)} < \varepsilon/4\rho$. Suppose $g(s) = \sum_{i=1}^M \chi_{I_i}(s)\, e_i$, where the I_i are disjoint intervals and $e_i \in X$. Then

$$\int_0^t [p_\mu(s) - p(s)] g(s)\, ds = \sum_{i=1}^M \int_{I_i \cap [0,t]} [p_\mu(s) - p(s)]\, ds\, e_i,$$

which tends to zero as $\mu \to \infty$ from the weak convergence of p_μ. Therefore

$$\left\| \int_0^t [p_\mu(s) - p(s)] v(s)\, ds \right\| \leqq \frac{\varepsilon}{4\rho} \int_0^t |p_\mu(s) - p(s)|\, ds + \left\| \int_0^t [p_\mu(s) - p(s)] g(s)\, ds \right\| \leqq \frac{\varepsilon}{2} \tag{3.8}$$

for large enough μ. We now combine (3.8) with (3.4) and (3.7) to reach a contradiction, which proves the lemma. □

Continuation of proof of Theorem 3.6. From (3.3) we have

$$\|z_n(t)\| \leqq \varepsilon_n + \int_0^t |p_n(s)| \|e^{\mathscr{A}(t-s)}\| \|\mathscr{B}\| \|z_n(s)\|\, ds \leqq \varepsilon_n + C \int_0^t |p_n(s)| \|z_n(s)\|\, ds,$$

where C is a positive constant independent of $t \in [0, T]$. By Gronwall's inequality

$$\|z_n(t)\| \leqq \varepsilon_n \exp\left(C \int_0^t |p_n(s)|\, ds \right),$$

which by the lemma tends to zero uniformly in $[0, T]$ as $n \to \infty$. This proves the first part of the theorem.

To prove the second part, given positive integers m, n and r, define

$$S_{mnr}(w_0) = \bigcup_{\substack{t \in [0,m] \\ \|p\|_{L^{1+1/r}([0,m];\mathbb{R})} \leqq n}} w(t; p, w_0).$$

Let $w(t_j; p_j, w_0) \in S_{mnr}(w_0)$. Since $L^{1+1/r}([0, m]; \mathbb{R})$ is reflexive there exist subsequences $\{t_\mu\} \subset [0, n]$ and $\{p_\mu\} \subset L^{1+1/r}([0, m]; \mathbb{R})$, such that $t_\mu \to t$ and $p_\mu \to p$ weakly in $L^{1+1/r}([0, m]; \mathbb{R})$. By the first part of the theorem, $w(t_\mu; p_\mu, w_0) \to w(t; p, w_0)$ in X. Hence $S_{mnr}(w_0)$ is precompact in X. But $S(w_0) \subset \bigcup_{m,n,r=1}^\infty S_{mnr}(w_0)$ so that $S(w_0)$ is contained in a countable union of compact sets.

Since $\dim X = \infty$, $S_{mnr}(w_0)$ is nowhere dense. By the Baire category theorem, $S(w_0)$ has dense complement. □

Remark 3.8. The theorem leaves open the question of whether

$$\{w(t; p, w_0);\, t \geqq 0,\, p \in L^1_{\text{loc}}([0, \infty); \mathbb{R})\}$$

has dense complement. We show in Theorem 5.5 that this holds in an important special case.

4. Finite-dimensional observability. In this section we consider the restricted problem of trying to control only a finite-dimensional projection of the state variable $w(t; p, w_0)$; i.e., we try to control only a "finite number of modes." This problem was discussed originally by Hermes [1979], and our first result is analogous to his.

THEOREM 4.1. *Let $\mathscr{A}$, $\mathscr{B}$ be as in Theorem 3.1. Suppose $G: X \to \mathbb{R}^n$ is a bounded linear map. Suppose that for given $T > 0$ and $\lambda \in (\mathbb{R}^n)^*$,*

$$\langle \lambda, G e^{\mathscr{A}(T-s)} \mathscr{B}(e^{\mathscr{A}s} w_0) \rangle = 0$$

for all s, $0 \leqq s \leqq T$ implies $\lambda = 0$. Then there is an $\varepsilon_T > 0$ such that $\|q - G e^{\mathscr{A}T} w_0\|_{\mathbb{R}^n} < \varepsilon_T$ implies $Gw(T; p, w_0) = q$ for some $p \in Z(T)$.

Proof. The derivative of the map $p \mapsto Gw(t; p, w_0)$ from $Z(T)$ to the range of G, evaluated at $p = 0$ is the operator GL_T. To show this is surjective, let $\lambda \in (\mathbb{R}^n)^*$ and assume λ annihilates the range of GL_T. An argument similar to the proof of Proposition 3.2 shows that $\lambda = 0$.

COROLLARY 4.2. *Let* $\mathcal{A}$, $\mathcal{B}$ *and* G *be as in Theorem* 4.1, *where* G *is now assumed to be surjective. Suppose the hypothesis of Proposition* 3.2 *holds. Then there is an* $\varepsilon_T > 0$ *such that*

$$\|q - Ge^{\mathcal{A}T}w_0\|_{\mathbb{R}^n} < \varepsilon_T \text{ implies } Gw(T; p, w_0) = q \text{ for some } p \in Z(T).$$

Proof. Set $l = G^*\lambda$, where G^* is the adjoint of G, and use Theorem 4.1. □

The usefulness of Corollary 4.2 is that it applies to *all* surjective bounded maps $G: X \to \mathbb{R}^n$, n arbitrary.

COROLLARY 4.3. *Assume that either the hypotheses of Theorem* 4.1 *or those of Corollary* 4.2 *hold for some* $T_1 > 0$ *and that* $e^{\mathcal{A}t}$ *is a group with* $e^{\mathcal{A}t}w_0$ *an almost periodic function of* t. *Then for any* $k \geqq 0$ *there exist* $T > 0$ *and* $\varepsilon_T > 0$ *such that* $\|q - Gw_0\|_{\mathbb{R}^n} < \varepsilon_T$ *implies* $Gw(T; p, w_0) = q$ *for some* $p \in C^k([0, T]; \mathbb{R})$.

Proof. This is very similar to the proof of Corollary 3.5. □

We note that the above results could be extended to nonlinear $G \in C^1(X; \mathbb{R}^n)$ in the obvious way.

One approach to trying to obtain full state controllability might be to solve an infinite sequence of finite-dimensional controllability problems by letting $n \to \infty$. This possibility will be precluded by Theorem 3.6. More specifically, we note:

COROLLARY 4.4. *Let* $\{X_n\}$ *be an increasing sequence of subspaces of* X, *with* $\dim X_n = n$ *for each* n *such that* Closure $(\bigcup_{n=1}^{\infty} X_n) = X$, *and with corresponding continuous projections* G_n *of* X *onto* X_n *having uniformly bounded norms. If*

$$H = \{h \in X;\ \text{there exist } T > 0,\ r > 1 \text{ and } \{p_n\} \subset L^r([0, T]; \mathbb{R}) \text{ such that } G_n w(T; p_n, w_0) = G_n h \text{ and } \|p_n\|_{L^r([0,T];\mathbb{R})} \leqq \text{const (independent of } n),\ n = 1, 2, \cdots\},$$

then H *has dense complement in* X.

Proof. Let $h \in H$. Then there exists a corresponding sequence $\{p_n\} \subset L^r([0, T]; \mathbb{R})$, $r > 1$. Since $\{p_n\}$ is bounded, there exists a subsequence, also denoted by $\{p_n\}$, such that $p_n \to p$ weakly in $L^1([0, T]; \mathbb{R})$. Now

$$\begin{aligned} \|w(T; p, w_0) - h\| \leqq {} & \|w(t; p, w_0) - G_n w(T; p, w_0)\| \\ & + \|G_n w(T; p, w_0) - G_n w(T; p_n, w_0)\| \\ & + \|G_n w(T; p_n, w_0) - G_n h\| + \|G_n h - h\|. \end{aligned} \tag{4.1}$$

Since the G_n are projections having uniformly bounded norms the first and last terms on the right-hand side of (4.1) tend to zero as $n \to \infty$. By hypothesis the third term is identically zero. As to the second term, $w(T; p_n, w_0) \to w(T; p, w_0)$ by Theorem 3.6 and $\|G_n\| \leqq$ const, so that this tends to zero also. Hence $h = w(T; p, w_0)$ and so H is a subset of the attainable set $S(w_0)$, which by Theorem 3.6 has dense complement. □

In practical terms Corollary 4.4 says that, in general, approximation of the problem $w(T; p, w_0) = h$ by a sequence of finite-dimensional problems will inevitably lead to the need for ever larger controls p_n as $n \to \infty$. In this sense, finite-dimensional approximations can be misleading for control of the full problem.

5. Abstract hyperbolic equations. We now investigate systems of the form

$$\ddot{u} + Au + p(t)Bu = 0, \tag{5.1}$$

$$u(0) = u_0 \in D(A^{1/2}), \qquad \dot{u}(0) = u_1 \in H, \tag{5.2}$$

where A is a positive definite self-adjoint operator with dense domain $D(A)$ in the real Hilbert space H, B is a bounded linear operator from $D(A^{1/2})$ to H, and p is a real-valued control. The inner product in H is denoted $(\cdot,\cdot)$. We suppose that A^{-1} is compact, and that A has simple eigenvalues λ_n^2, $n=1,2,\cdots$, where $0<\lambda_1<\lambda_2<\cdots$. Then there exists a corresponding complete orthonormal basis $\{\phi_n\}$ of eigenfunctions: $A\phi_n=\lambda_n^2\phi_n$, $(\phi_n,\phi_m)=\delta_{mn}$.

To investigate controllability of (5.1) we could rewrite (5.1) in first order form

$$w=\begin{pmatrix} u \\ \dot u\end{pmatrix},\quad \mathscr{A}=\begin{pmatrix} 0 & I \\ -A & 0\end{pmatrix},\quad \mathscr{B}=\begin{pmatrix} 0 & 0 \\ -B & 0\end{pmatrix}$$

and set $X=D(A^{1/2})\times H$ with inner product

$$\langle (u_1,u_2),(v_1,v_2)\rangle_X=(\mathscr{A}^{1/2}u_1,A^{1/2}v_1)+(u_2,v_2).$$

With this set-up, we see that $\mathscr{A}$ generates a C^0 group of isometries on X and the hypotheses of Theorem 2.5 are satisfied. Controllability then hinges on the operator $\hat{L}_T$. To facilitate computations, however, it is advantageous to introduce a different first order form. We therefore set up a complex structure in a way that is standard for Hamiltonian systems (see Chernoff and Marsden [1974, § 2.7]).

Let $\mathscr{H}$ denote the complexified Hilbert space $H\oplus iH$ with inner product defined by

$$\langle x_1+iy_1,x_2+iy_2\rangle_{\mathscr{H}}=(x_1,x_2)+(y_1,y_2)+i[(y_1,x_2)-(x_1,y_2)]$$

for $x_1,x_2,y_1,y_2\in H$. The map $\psi:X\to\mathscr{H}$ defined by

$$\psi(u_1,u_2)=A^{1/2}u_1+iu_2$$

is an isometry. Let $z=A^{1/2}u+i\dot u$, so that (5.1), (5.2) become

$$i\dot z=A^{1/2}z+p(t)BA^{-1/2}\,\mathrm{Re}\,z, \tag{5.3}$$

$$z(0)=z_0, \tag{5.4}$$

where

$$z_0=A^{1/2}u_0+iu_1\in\mathscr{H}. \tag{5.5}$$

Of course, in (5.3) $p(t)$ is still *real*. Writing $\hat{\mathscr{A}}=-i\mathscr{A}^{1/2}$ (regarded as a complex operator) and $\hat{\mathscr{B}}=-iBA^{-1/2}\,\mathrm{Re}$ (a real-linear bounded operator from $\mathscr{H}$ into $\mathscr{H}$), we see that the hypotheses of Theorem 2.5 are satisfied.

The basis $\{\phi_n\}$ of H may also be regarded as a basis of $\mathscr{H}$. For any $z\in\mathscr{H}$, let $\{z_n\}$ be the (complex) components of z relative to this basis, i.e.,

$$z=\sum_{n=1}^{\infty}z_n\phi_n, \tag{5.6}$$

so that $\{z_n\}\in l_2$. Thus we have

$$e^{\hat{\mathscr{A}}s}z=\sum_{n=1}^{\infty}z_n e^{-i\lambda_n s}\phi_n. \tag{5.7}$$

Let $B_{mn}=(B\phi_m,\phi_n)$, so that the B_{mn} are real and

$$B\phi_m=\sum_{n=1}^{\infty}B_{mn}\phi_n.$$

Thus (5.7) gives

$$\hat{\mathscr{B}}e^{\hat{\mathscr{A}}s}z=-i\sum_{m,n=1}^{\infty}\frac{B_{mn}}{\lambda_m}\mathrm{Re}\,(e^{-i\lambda_m s}z_m)\phi_n,$$

and so

$$e^{-\hat{\mathcal{A}}s}\hat{\mathcal{B}}\, e^{\hat{\mathcal{A}}s} z = -\frac{i}{2}\sum_{m,n=1}^{\infty} \frac{B_{mn}}{\lambda_m}\left(e^{i(\lambda_n-\lambda_m)s} z_m + e^{i(\lambda_n+\lambda_m)s}\bar{z}_m\right)\phi_n. \tag{5.8}$$

5.1. Riesz bases.

DEFINITION. A sequence of elements $\{\omega_i\}_{i=1}^{\infty}$ of a (real or complex) Hilbert space Z is called a *Riesz basis* of Z if every $\theta \in Z$ has a unique expansion

$$\theta = \sum_{i=1}^{\infty} a_i \omega_i$$

that is convergent in Z, and

$$C_1 \sum_{m=1}^{\infty} |a_i|^2 \leqq \|\theta\|^2 \leqq C_2 \sum_{i=1}^{\infty} |a_i|^2$$

for absolute positive constants C_1, C_2.

We collect together some useful facts concerning Riesz bases.

LEMMA 5.1. *Let $\{\omega_i\}$ be a Riesz basis of Z, and let $\{e_i\}$ be any complete orthonormal basis of Z. Then:*

(i) *the formula $T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i \omega_i$ defines an isomorphism*

$$T : Z \to Z;$$

(ii) *for any $\theta \in Z$,*

$$\sum_{i=1}^{\infty} |(\theta, \omega_i)|^2 \leqq \|T^*\|^2 \|\theta\|^2;$$

(iii) *given any sequence $\{a_n\} \in l_2$ there exists a unique solution $\theta \in Z$ of the equations*

$$(\theta, \omega_j) = a_j, \qquad j = 1, 2, \cdots. \tag{5.9}$$

Proof. For a proof of (i) see Gohberg and Krein [1969, p. 310]. To prove (ii) note that $(\theta, \omega_i) = (\theta, Te_i) = (T^*\theta, e_i)$, so that

$$\sum_{i=1}^{\infty} |(\theta, \omega_i)|^2 = \|T^*\theta\|^2 \leqq \|T^*\|^2 \|\theta\|^2.$$

Finally, the equations (5.9) are equivalent to

$$(T^*\theta, e_j) = a_j,$$

and thus have the unique solution

$$\theta = (T^*)^{-1} \sum_{j=1}^{\infty} a_j e_j. \qquad \square$$

A useful criterion for the construction of a Riesz basis is as follows.

THEOREM 5.2. *Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, $\mu_{-k} = -\mu_k$, and suppose that*

$$\lim_{k\to\infty} (\mu_{k+1} - \mu_k) \geqq \gamma > 0.$$

Then for any $T > 2\pi/\gamma$ the functions $\{e^{i\mu_k t}\}_{k=-\infty}^{\infty}$ may be extended to a Riesz basis of $L^2([0, T]; \mathbb{C})$.

Proof. Let S denote the closed linear span of the set of functions $\{e^{i\mu_k t}\}$ in $L^2([0, T]; \mathbb{C})$. It follows from Ball and Slemrod [1979, Thm. 2.1] (the essential idea is due to Ingham) that for any finite sum

$$f(t) = \sum_{|k| \leq N} a_k\, e^{i\mu_k t},$$

we have

$$C_1 \sum_{|k|\leq N} |a_k|^2 \leq \frac{1}{T} \int_0^T |f(t)|^2\, dt \leq C_2 \sum_{|k|\leq N} |a_k|^2.$$

It follows that any $f \in S$ has a unique expansion

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{i\mu_k t}$$

convergent in $L^2([0, T]; \mathbb{C})$, and that

$$C_1 \sum_{k=-\infty}^{\infty} |a_k|^2 \leq \frac{1}{T} \int_0^T |f(t)|^2\, dt \leq C_2 \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Let $\{e_i\}$ be an orthonormal basis of $S^\perp$. It follows readily that $\{e_i\} \cup \{e^{i\mu_k t}\}$ is a Riesz basis of $L^2([0, T]; \mathbb{C})$. □

The above discussion is a slightly different presentation of results summarized in Russell [1967].

5.2. Finite-dimensional observers. We now employ Theorem 4.1 to discuss when (5.1) is controllable relative to a finite-dimensional observer.

THEOREM 5.3. *Assume the initial data u_0, u_1 in* (5.2) *satisfy*

$$\text{(i)} \qquad B_{nn}[(u_0, \phi_n)^2 + (u_1, \phi_n)^2] \neq 0, \qquad n = 1, 2, \cdots$$

and that $T > 0$ is such that

$$\text{(ii)} \qquad \{e^{2i\lambda_n s}\}_{n=1}^{\infty} \cup \{e^{i(\lambda_p - \lambda_q)s}, e^{i(\lambda_p + \lambda_q)s} \mid p \neq q \text{ and } B_{pq} \neq 0\}$$

can be extended to a Riesz basis of $L^2([0, T]; C)$.

Then (5.3) *satisfies the hypotheses of Proposition* 3.2. *In particular, for any $T_1 \geq T$ and bounded surjective maps $G_1 : D(A^{1/2}) \to \mathbb{R}^m$, $G_2 : H \to \mathbb{R}^n$, there exists ε_{T_1} such that if*

$$\|q_1 - G_1 u(T_1; 0, u_0, u_1)\|_{\mathbb{R}^m} < \varepsilon_{T_1}, \qquad \|q_2 - G_2 \dot{u}(T_1; 0, u_0, u_1)\|_{\mathbb{R}^n} < \varepsilon_{T_1},$$

then

$$G_1 u(T_1; p, u_0, u_1) = q_1, \qquad G_2 \dot{u}(T_1; p, u_0, u_1) = q_2$$

for some $p \in Z(T_1)$. Here $u(t; p, u_0, u_1)$ is the solution of (5.1), (5.2).

Proof. Let $l = \sum_{n=1}^{\infty} l_n \phi_n$ be an arbitrary element of $\mathcal{H}$. Then $\langle l, e^{-\mathcal{A}s} \hat{\mathcal{B}} e^{\mathcal{A}s} z_0 \rangle_{\mathcal{H}}$ may be computed for (5.3) by using (5.8). Specifically we have

$$(5.10) \qquad \begin{aligned} 2i\langle e^{-\mathcal{A}s} \hat{\mathcal{B}} e^{\mathcal{A}s} z_0, l\rangle_{\mathcal{H}} = &\sum_{n=1}^{\infty} \bar{l}_n (z_{0n} + \bar{z}_{0n} e^{2i\lambda_n s}) \frac{B_{nn}}{\lambda_n} \\ &+ \sum_{\substack{m \neq n \\ m,n=1}}^{\infty} \bar{l}_n (z_{0m} e^{i(\lambda_n - \lambda_m)s} + \bar{z}_{0m} e^{i(\lambda_n + \lambda_m)s}) \frac{B_{mn}}{\lambda_m}, \end{aligned}$$

where z_0 is given by (5.5) and where

$$z_0 = \sum_{n=1}^{\infty} z_{0n} \phi_n,$$

so that $z_{0n} = \lambda_n (u_0, \phi_n) + i(u_1, \phi_n)$. Thus, if $\langle e^{-\mathcal{A}s} \mathcal{B} e^{\mathcal{A}s} z_0, l\rangle_{\mathcal{H}} = 0$ for all s such that $0 \leq s \leq T_1$, the right-hand side of (5.10) will equal zero on $[0, T]$. By assumption (ii) the coefficients of $\{e^{2i\lambda_n s}\}$ vanish; that is,

$$\frac{\bar{l}_n \bar{z}_{0n} B_{nn}}{\lambda_n} = 0 \quad \text{for } n = 1, 2, \cdots.$$

By (i) this implies $l_n = 0$ for $n = 1, 2, \cdots$, and hence $l = 0$. Therefore, the hypothesis of Proposition 3.2 is satisfied, and by Corollary 4.2 the result follows. □

COROLLARY 5.4. *Assume the hypotheses of Theorem 5.3 are satisfied, and let* G_1, G_2 *be bounded surjective linear maps,* $G_1 : D(A^{1/2}) \to \mathbb{R}^m$, $G_2 : H \to \mathbb{R}^n$. *Then for any* $k \geqq 0$ *there exist* $T_1 > 0$ *and* $\varepsilon_{T_1} > 0$ *such that*

$$\|q_1 - G_1 u_0\|_{\mathbb{R}^m} < \varepsilon_{T_1}, \qquad \|q_2 - G_1 u_1\|_{\mathbb{R}^n} < \varepsilon_{T_1},$$

imply

$$G_1 u(T_1; p, u_0, u_1) = q_1, \qquad G_2 \dot{u}(T_1; p, u_0, u_1) = q_2$$

for some $p \in C^k([0, T_1]; \mathbb{R})$.

Proof. The result follows immediately from Theorem 5.3 and Corollary 4.3. □

Hypothesis (ii) of Theorem 5.3 is difficult to verify unless $B_{pq} = 0$ for $p \neq q$. Sufficient conditions for it to hold may be deduced from Theorem 5.2, but they are not revealing except in the case just mentioned.

5.3. Approximate controllability. In this subsection we study approximate controllability, in a sense to be made precise, of (5.1), (5.2). As above we work with the equivalent first order system

$$\dot{z} = \hat{\mathscr{A}} z + p(t) \hat{\mathscr{B}} z \tag{5.11}$$

where $\hat{\mathscr{A}} = -iA^{1/2}$, $\hat{\mathscr{B}} = -iBA^{-1/2}\,\mathrm{Re}$. In addition, to simplify matters we make the assumption

$$B_{mn} = b_m \delta_{mn} \tag{D1}$$

for nonzero constants b_m, where δ_{mn} is the Kronecker delta. Since (D1) implies that $B_{mn} = 0$ for $m \neq n$, we shall refer to (D1) as the *diagonal case.*

Writing

$$z(t) = \sum_{n=1}^{\infty} z_n(t) \phi_n,$$

we see that in the diagonal case, (5.11) reduces to the infinite system of uncoupled ordinary differential equations

$$\dot{z}_n = -i\lambda_n z_n - ip(t) \frac{b_n}{\lambda_n} \mathrm{Re}\, z_n, \qquad n = 1, 2, \cdots. \tag{5.12}$$

The corresponding initial conditions are

$$z_n(0) = z_{0n}. \tag{5.13}$$

We note that the fact that $BA^{-1/2}$ is a bounded linear operator from $H \to H$ is equivalent to the condition

$$\left\{ \frac{b_n}{\lambda_n} \right\} \in l_\infty. \tag{5.14}$$

We first strengthen Theorem 3.6 in the diagonal case by showing that even when L^1 controls are allowed, exact controllability is in general impossible.

THEOREM 5.5. *Given* $\{z_{0n}\} \in l_2$, *the set*

$$\bigcup_{\substack{t \geqq 0 \\ p \in L^1_{\mathrm{loc}}([0,\infty); \mathbb{R})}} \{z_n(t; p, z_0)\}$$

is contained in a countable union of compact sets of l_2, *and thus has dense complement.*

Here, $\{z_n(t; p, z_0)\}$ denotes the unique mild solution of (5.12), (5.13) with $z_0 = \{z_{0n}\}$. Consequently the attainability set $\{u(t; p, u_0, u_1), u_t(t; p, u_0, u_1) \mid t \geqq 0, p \in L^1_{\text{loc}}([0, \infty))\}$ is contained in the countable union of compact sets in $D(A^{1/2}) \times H$ and so has a dense complement.

Proof. Since

$$z_n(t) = e^{-i\lambda_n t} z_{0n} = i\frac{b_n}{\lambda_n}\int_0^t e^{-i\lambda_n(t-s)} p(s) \operatorname{Re} z_n(s)\, ds,$$

it follows that

$$|z_n(t)| \leqq |z_{0n}| + \left|\frac{b_n}{\lambda_n}\right| \int_0^t |p(s)||z_n(s)|\, ds,$$

and hence, by Gronwall's inequality and (5.14)

$$|z_n(t)| \leqq |z_{0n}| \exp\left(\kappa \int_0^t |p(s)|\, ds\right),$$

where $\kappa = \|\{b_n/\lambda_n\}\|_{l_\infty}$. Thus $\{z_n(t)\} \in \bigcup_{N=1}^{\infty} S_N(z_0)$ for any $t \geqq 0$ and $p \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$, where S_N is defined by

$$S_N(z_0) = \{\{a_n\} \in l_2 : |a_n| \leqq N|z_{0n}|\}.$$

The result now follows from the next lemma.

LEMMA 5.6. *$S_N(z_0)$ is a compact subset of l_2.*

Proof.[1] Let $a^{(r)} \in S_N(z_0)$, $r = 1, 2, \cdots$. Then

$$\sum_{n=1}^{\infty} |a_n^{(r)}|^2 \leqq N^2 \sum_{n=1}^{\infty} |z_{0n}|^2 = N^2 \|z_0\|_{l_2}^2.$$

So some subsequence $a^{(\mu)} \to a$ weakly in l_2, which implies in particular that $a_n^{(\mu)} \to a_n$ for each n. Also, given $\varepsilon > 0$

$$\sum_{n=M}^{\infty} |a_n^{(\mu)}|^2 \leqq N^2 \sum_{n=M}^{\infty} |z_{0n}|^2 < \varepsilon$$

for M sufficiently large. Therefore $\sum_{n=1}^{\infty} |a_n^{(\mu)}|^2 \to \sum_{n=1}^{\infty} |a_n|^2$, and so $a^{(\mu)} \to a$ strongly in l_2. Hence $S_N(z_0)$ is precompact. Since $S_N(z_0)$ is closed, the lemma is proved. □

We now make the following additional assumption

$$\text{(D2)} \qquad \frac{b_n}{\lambda_n} = c + \gamma_n \quad \text{for some } c \in \mathbb{R} \text{ and } \{\gamma_n\} \in l_2.$$

We write $P(t) = \int_0^t p(s)\, ds$ and make the following change of variables (motivated by averaging):

$$\zeta_n = \frac{\lambda_n}{b_n}\left[\frac{z_n}{z_{0n}} \exp i\left(\lambda_n t + \frac{b_n}{2\lambda_n} p(t)\right) - 1\right]. \tag{5.15}$$

Substitution of (5.15) into (5.12) yields

$$\dot{\zeta}_n(t) = -i\frac{p(t)}{2}\frac{\bar{z}_{0n}}{z_{0n}}\left(\frac{b_n}{\lambda_n}\bar{\zeta}_n(t) + 1\right) \exp\left[2i\left(\lambda_n t + \frac{b_n}{2\lambda_n} p(t)\right)\right], \tag{5.16}$$

$$\zeta_n(0) = 0. \tag{5.17}$$

---

[1] This lemma follows from Dunford and Schwartz [1964, p. 338]. We have included the proof for completeness.

The following existence and differentiability theorem gives conditions under which the solution $\{\zeta_n(t)\}$ of (5.16), (5.17) belongs to l_2, and thus gives more precise information on the attainable set (but under stronger hypotheses) than Theorem 5.5.

THEOREM 5.7. *Suppose $\{z_{0n}\}\in l_2$, $z_{0n}\neq 0$ for all $n=1,2,\cdots$, and that $\{e^{2i\lambda_n t}\}$ can be extended to a Riesz basis of $L^2([0,l];\mathbb{R})$ for some $l>0$. Let $p\in L^2_{\mathrm{loc}}([0,\infty);\mathbb{R})$. Then* (5.16), (5.17) *have a unique absolutely continuous solution $\zeta_n=\zeta_n(t;p)$ defined for all $t\geqq 0$, and $\{\zeta_n(\cdot;p)\}\in C([0,T];l_2)$ for $0<T\leqq l$. Furthermore, the mapping $p\mapsto\{\zeta_n(T;p)\}$ is C^1 from $L^2([0,T];\mathbb{R})$ to l_2 for each $0<T\leqq l$, and*

$$D_p\{\zeta_n(T;0)\}\cdot p=-\frac{i}{2}\frac{\bar z_{0n}}{z_{0n}}\int_0^T p(t)\exp(2i\lambda_n t)\,dt. \tag{5.18}$$

Proof. We write (5.16), (5.17) in integrated form:

$$\zeta_n(t)=-\frac{i}{2}\int_0^t p(s)\frac{\bar z_{0n}}{z_{0n}}\left(\frac{b_n}{\lambda_n}\bar\zeta_n(s)+1\right)\exp\left[2i\left(\lambda_n s+\frac{b_n}{2\lambda_n}p(s)\right)\right]ds. \tag{5.19}$$

We can solve these equations in a manner similar to Theorem 2.5, but for variety we shall adopt a standard device to get existence on an arbitrary time interval in a single step. Let $0<T\leqq l$. For any $\delta\geqq 0$ the norm

$$\|\zeta\|_\delta=\sup_{t\in[0,T]}e^{-\delta t}\|\zeta(t)\|_{l_2}$$

on $X_T=C([0,T];l_2)$ is equivalent to the usual one, namely $\|\cdot\|_0$. For $\zeta\in X_T$ define

$$((J_p\zeta)(t))_n=-\frac{i}{2}\int_0^t p(s)\frac{\bar z_{0n}}{z_{0n}}\left(\frac{b_n}{\lambda_n}\bar\zeta_n(s)+1\right)\exp\left[2i\left(\lambda_n s+\frac{b_n}{2\lambda_n}p(s)\right)\right]ds.$$

Then for $0\leqq\tau\leqq t\leqq T$

$$\begin{aligned}\sum_{n=1}^{\infty}&|(J_p\zeta(t)-J_p(\zeta(\tau))_n|^2\\ &=\frac{1}{4}\sum_{n=1}^{\infty}\left|\int_\tau^t p(s)\frac{\bar z_{0n}}{z_{0n}}\left(\frac{b_n}{\lambda_n}\bar\zeta_n(s)+1\right)\exp\left[2i\left(\lambda_n s+\frac{b_n}{2\lambda_n}p(s)\right)\right]ds\right|^2\\ &\leqq\frac{\kappa^2}{2}\sum_{n=1}^{\infty}\left(\int_\tau^t|p(s)||\bar\zeta_n(s)|\,ds\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}\left|\int_\tau^t p(s)\exp\left[2i\left(\lambda_n s+\frac{b_n}{2\lambda_n}p(s)\right)\right]ds\right|^2,\end{aligned} \tag{5.20}$$

where, as before, $\kappa=\|\{b_n/\lambda_n\}\|_{L^\infty}$. But

$$\frac{\kappa^2}{2}\sum_{n=1}^{\infty}\left(\int_\tau^t|p(s)||\bar\zeta_n(s)|\,ds\right)^2\leqq\frac{\kappa^2T}{2}\left(\int_\tau^t|p(s)|^2\,ds\right)\|\zeta\|_0^2$$

while

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{\infty}&\left|\int_\tau^t p(s)\exp\left[2i\left(\lambda_n s+\frac{b_n}{2\lambda_n}p(s)\right)\right]ds\right|^2\\ &=\frac{1}{2}\sum_{n=1}^{\infty}\left|\int_\tau^t p(s)\exp(icP(s))\exp(2i\lambda_n s)[1+i\gamma_n p(s)+o(|\gamma_n|)]\,ds\right|^2\\ &\leqq C\int_\tau^t|p(s)|^2\,ds\left[1+\sum_{n=1}^{\infty}|\gamma_n|^2\right],\end{aligned}$$

where C is a constant (depending on p), and where we have applied Lemma 5.1 (ii) to the function $\theta(s)=\chi_{[\tau,t]}(s)p(s)\exp(icP(s))$ with $Z=L^2([0,l];\mathbb{C})$. From (5.20) we thus deduce that J_p maps χ_T into itself.

Let $\zeta,\eta\in X_T$. Then

$$e^{-\delta t}\left(\sum_{n=1}^{\infty}|(J_p\zeta(t)-J_p\eta(t))_n|^2\right)^{1/2}\leqq\frac{\kappa^2}{2}e^{-\delta t}\left(\sum_{n=1}^{\infty}\left(\int_0^t|p(s)||\zeta_n(s)-\eta_n(s)|\,ds\right)^2\right)^{1/2}$$
$$\leqq\frac{\kappa}{2}\left(\int_0^t|p(s)|^2\,ds\right)^{1/2}\int_0^t e^{\delta(s-t)}\,ds\|\zeta-\eta\|_\delta.$$

Hence J_p is a uniform contraction with respect to the norm $\|\cdot\|_\delta$ provided δ is sufficiently large. Calculations similar to those above show that J_p is C^1 in p. The result then follows as in Propositions 2.1, 2.2. □

It is now easy to prove a local approximate controllability result.

THEOREM 5.8. *Suppose* $\{z_{0n}\}\in l_2$, $z_{0n}\neq 0$, $b_n\neq 0$, *for all* $n=1,2,\cdots$, *and that* $\{1,e^{\pm 2i\lambda_n t}\}$ *can be extended to a Riesz basis of* $L^2([0,l];\mathbb{C})$ *for some* $l>0$. *Then there exists* $\varepsilon_l>0$ *such that if* $\|h\|_{l_2}+|\theta|<\varepsilon_l$ *where* $h\in l_2$ *and* $\theta\in\mathbb{R}$, *then*

$$\text{(5.21)}\qquad \frac{\lambda_n}{b_n}\left(\frac{z_n(l)}{z_{0n}}\exp\left[i\left(\lambda_n l+\frac{b_n}{2\lambda_n}\theta\right)\right]-1\right)=h_n,\qquad n=1,2,\cdots,$$

for some $p\in L^2([0,l];\mathbb{R})$ *with* $\int_0^l p(t)\,dt=\theta$.

Proof. Consider the map $Q:L^2([0,l];\mathbb{R})\to l_2\times\mathbb{R}$ defined by

$$Q(p)=\left(\{\zeta_n(l;p)\},\int_0^l p(t)\,dt\right).$$

By (5.18),

$$D_pQ(0)\cdot p=\left(\left\{-\frac{i}{2}\frac{\bar z_{0n}}{z_{0n}}\int_0^l p(t)\exp(2i\lambda_n t)\,dt\right\},\int_0^l p(t)\,dt\right).$$

Since Q is C^1 by Theorem 5.7 it suffices to show that $D_pQ(0)$ is surjective. Let $\{a_n\}\in l_2$, $\alpha\in\mathbb{R}$. Write $b_n=2i(z_{0n}/\bar z_{0n})a_n$. By Lemma 5.1 (iii) we can solve the equations

$$\int_0^l q(t)\exp(2i\lambda_n t)\,dt=b_n,\qquad\int_0^l q(t)\exp(-2i\lambda_n t)\,dt=\bar b_n,\qquad n=1,2,\cdots,$$
$$\int_0^l q(t)\,dt=\alpha$$

for $q\in L^2([0,l];\mathbb{C})$. Setting $p(t)=\operatorname{Re}q(t)$ we see that $D_pQ(0)$ is surjective. □

Remark 5.9. Suppose that $\{1,e^{\pm 2i\lambda_n t},\phi_1(t),\cdots,\phi_N(t)\}$ can be extended to a Riesz basis of $L^2([0,l];\mathbb{C})$, where $\phi_i\in L^2([0,l];\mathbb{R})$, $1\leqq i\leqq N$. Then the proof shows that we can find a $p\in L^2([0,l];\mathbb{R})$ such that (5.21) holds, $\int_0^l p(t)\,dt=\theta$, and $\int_0^l p(t)\phi_i(t)\,dt=\theta_i$, $1\leqq i\leqq N$, provided that

$$\|h\|_{l_2}+|\theta|+\sum_{i=1}^N|\theta_i|$$

is sufficiently small. Thus, the more deficient the set $\{1,e^{\pm 2i\lambda_n t}\}$ is, the more controls there are such that (5.22) holds. If $\{1,e^{\pm 2i\lambda_n t}\}$ is already a Riesz basis, then p is unique.

COROLLARY 5.10. *Suppose* $\{z_{0n}\}\in l_2$ *with* $b_n\neq 0$, $z_{0n}\neq 0$ *for all* $n=1,2,\cdots$, *and*

$$\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)\geqq\nu>0.$$

Then given any $T>(\pi/\nu)$ there exists $\varepsilon_T>0$ such that for any $h\in l_2$, $\theta\in\mathbb{R}$, with $\|h\|_{l_2}+|\theta|<\varepsilon_T$, there is a $p\in L^2([0, T]; \mathbb{R})$ such that

$$\frac{\lambda_n}{b_n}\left(\frac{z_n(T)}{z_{0n}}\exp\left[i\left(\lambda_n T+\frac{b_n}{2\lambda_n}\theta\right)\right]-1\right)=h_n, \qquad n=1,2,\cdots$$

and $\int_0^T p(t)\,dt=\theta$.

Furthermore, if λ_n/σ is an integer for all n and some $\sigma>0$, then there exists an $\varepsilon>0$ such that if $\|h\|_{l_2}+|\theta|<\varepsilon$ then there is a $p\in L^2([0, 2\pi/\sigma]; \mathbb{R})$ such that

$$\frac{z_n(2\pi/\sigma)}{z_{0n}}=\exp\left(\frac{-ib_n\theta}{2\lambda_n}\right)\left(1+\frac{b_nh_n}{\lambda_n}\right), \qquad n=1,2,\cdots$$

and $\int_0^{2\pi/\sigma} p(t)\,dt=\theta$.

Proof. The first part follows immediately from Theorems 5.2, 5.8. The second part is then obvious. □

Remarks 5.11. 1. In Corollary 5.10 there exist infinitely many families of possible controls p. This follows from the fact that by Theorem 5.2 $\{1, e^{\pm 2i\lambda_n t}\}$ can be extended to a Riesz basis of $L^2([0, A]; \mathbb{C})$ for any $\pi/\nu<A<T$, so that there are infinitely many linearly independent real functions in the orthogonal complement of the subspace of $L^2([0, T]; \mathbb{C})$ spanned by $\{1, e^{\pm 2i\lambda_n t}\}$, and Remark 5.9.

2. The set of $z=\sum_{n=1}^{\infty} z_n\phi_n\in\mathcal{H}$ such that for some $\theta\in\mathbb{R}$

$$\zeta_n=\frac{\lambda_n}{b_n}\left(\frac{z_n}{z_{0n}}\exp\left[i\left(\lambda_n T+\frac{b_n}{2\lambda_n}\theta\right)\right]-1\right)$$

belongs to the ball $\|\zeta\|_{l_2}<\varepsilon$ is compact (use $|z_n|=(|b_n\zeta_n/\lambda_n|+1)|z_{0n}|\leqq(C\varepsilon^{1/2}+1)|z_{0n}|$ and Lemma 5.6). Hence the results of Theorem 5.8 and Corollary 5.10 do *not* say that we can control in finite time to points of a dense subset of some neighborhood of $e^{\mathcal{A}T}z(0)$ in $\mathcal{H}$. To prove such an approximate controllability result we would need to extend Theorem 5.8 by allowing ε_l to be arbitrarily large.

We now show how Corollary 5.10 can be applied to prove a *global* approximate controllability theorem. We restrict attention to the case when $e^{\mathcal{A}t}$ is periodic.

THEOREM 5.12. *Suppose that $z_0=\{z_{0n}\}\in l_2$ with $z_{0n}\neq 0$ for all $n=1,2,\cdots$, and let λ_n/σ be an integer for all n and some $\sigma>0$. Then for any $h\in l_2$ with $1+(b_n/\lambda_n)h_n\neq 0$ for all n, and any $\theta\in\mathbb{R}$, there exist a positive integer m and a control $p\in L^2([0, 2m\pi/\sigma]; \mathbb{R})$ such that*

$$\frac{z_n\left(\dfrac{2m\pi}{\sigma}\right)}{z_{0n}}=\exp\left(-\frac{ib_n\theta}{2\lambda_n}\right)\left(1+\frac{b_n}{\lambda_n}h_n\right), \qquad n=1,2,\cdots.$$

Proof. Let

$$A=\left\{(h,\theta)\in l_2\times\mathbb{R}\,\Big|\, z_n\left(\frac{2m\pi}{\sigma}\right)=\exp\left(-\frac{b_n\theta}{2\lambda_n}\right)\left(1+\frac{b_n}{\lambda_n}h_n\right)z_{0n}\text{ for all } n\right\},$$

some positive integer m, and some $p\in L([0, 2m\pi/\sigma]; \mathbb{R})$ and

$$B=\left\{(h,\theta)\in l_2\times\mathbb{R}\,\Big|\, 1+\frac{b_n}{\lambda_n}h_n\neq 0\text{ for all } n\right\}.$$

We show that $A=B$. By the backwards uniqueness of solutions to (5.13) and the assumption $z_{0n}\neq 0$ for all n we see that $A\subset B$. It therefore suffices to show that (i) A is open, (ii) $\partial A\cap B$ is empty, and (iii) B is arcwise connected.

To prove (i), let $(h, \theta) \in A$, so that

$$z_n\left(\frac{2m\pi}{\sigma}\right) = \exp\left(-\frac{ib_n\theta}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} h_n\right) z_{0n}, \qquad n = 1, 2, \cdots$$

for some m and $p \in L^2([0, 2m\pi/\sigma]; \mathbb{R})$. We apply Corollary 5.10, with initial data

$$\tilde{z}_{0n} = \exp\left(-\frac{ib_n\theta}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} h_n\right) z_{0n},$$

to deduce the following assertion: if

$$\|g\|_{l_2} + |\alpha| < \varepsilon \tag{5.22}$$

then there exists $p \in L^2([0, 2(m+1)\pi/\sigma]; \mathbb{R})$ such that

$$z_n\left(\frac{2(m+1)\pi}{\sigma}\right) = \exp\left(-\frac{ib_n\alpha}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} g_n\right) \tilde{z}_{0n}, \qquad n = 1, 2, \cdots.$$

But if $\|h - \tilde{h}\|_{l_2}$ and $|\theta - \tilde{\theta}|$ are sufficiently small then

$$g_n \equiv \frac{\tilde{h}_n - h_n}{1 + (b_n/\lambda_n)h_n} \quad \text{and} \quad \alpha \equiv \tilde{\theta} - \theta$$

satisfy (5.22) (note that $\{h_n\} \in l_2$ implies that $|1 + (b_n/\lambda_n)h_n| \geq k > 0$), and so for the corresponding p we have

$$z_n\left(\frac{2(m+1)\pi}{\sigma}\right) = \exp\left(-\frac{ib_n\tilde{\theta}}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} \tilde{h}_n\right) z_{0n}, \qquad n = 1, 2, \cdots.$$

Thus A is open.

Suppose that $(h, \theta) \in \partial A \cap B$. We show that the time reversibility properties of (5.1) lead to a contradiction. Let

$$w_{0n} = \exp\left(-\frac{ib_n\theta}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} h_n\right) z_{0n}.$$

By Corollary 5.10, if (5.22) holds, there exists $q \in L^2([0, 2\pi/\sigma]; \mathbb{R})$ with $\int_0^{2\pi/\sigma} q(t)\, dt = \alpha$, such that the solution of

$$\dot{v}_n(t) = -i\lambda_n v_n(t) - iq(t)\frac{b_n}{\lambda_n} \operatorname{Re} v_n(t), \qquad v_n(0) = \bar{w}_{0n},$$

satisfies

$$v_n\left(\frac{2\pi}{\sigma}\right) = \exp\left(-\frac{ib_n\alpha}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} \bar{g}_n\right) \bar{w}_{0n}, \qquad n = 1, 2, \cdots.$$

Hence

$$\tilde{z}_n(t) \equiv \bar{v}_n\left(\frac{2\pi}{\sigma} - t\right)$$

satisfies

$$\dot{\tilde{z}}_n(t) = -i\lambda_n \tilde{z}_n(t) - iq\left(\frac{2\pi}{\sigma} - t\right)\frac{b_n}{\lambda_n} \operatorname{Re} \tilde{z}_n(t),$$

$$\tilde{z}_n(0) = \exp\left(\frac{ib_n\alpha}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} g_n\right) w_{0n}, \tag{5.23}$$

$$\tilde{z}_n\left(\frac{2\pi}{\sigma}\right) = w_{0n}.$$

Since $(h, \theta) \in \partial A$, there exists a sequence $(h^{(r)}, \theta^{(r)}) \in A$ with $(h^{(r)}, \theta^{(r)}) \to (h, \theta)$ in $l_2 \times \mathbb{R}$. Define

$$\alpha = \theta - \theta^{(r)} \quad \text{and} \quad g_n = \frac{h_n^{(r)} - h_n}{1 + (b_n/\lambda_n)h_n},$$

for some fixed r large enough for (5.22) to hold. For this r there exist m and $p \in L^2([0, 2m\pi/\sigma]; \mathbb{R})$ such that

$$z_n\left(\frac{2m\pi}{\sigma}\right) = \exp\left(-\frac{ib_n\theta^{(r)}}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} h_n^{(r)}\right) z_{0n} = \exp\left(\frac{ib_n\alpha}{2\lambda_n}\right)\left(1 + \frac{b_n}{\lambda_n} g_n\right) w_{0n}.$$

Extending p to be $q(2(m+1)\pi/\sigma - t)$ on $[2m\pi/\sigma, 2(m+1)\pi/\sigma]$ we see that by (5.23)

$$z_n\left(\frac{2(m+1)\pi}{\sigma}\right) = w_{0n}.$$

Hence $(h, \theta) \in A$, a contradiction. This proves (ii).

To prove (iii), note that if $(h, \theta) \in B$ then $|(b_n/\lambda_n)h_n| < 1$ for $n > N$, say. Let $h^N = (h_1, \cdots, h_N, 0, \cdots)$. The arc $t \mapsto (h^N + t(h - h^N), t\theta)$, $t \in [0, 1]$ connects (h, θ) to $(h^N, 0)$ and lies in B. But $(h^N, 0)$ can be connected to $(0, 0)$ by an arc in B of the form $(s, 0)$ where $s \in R^N$ and runs from h^N to 0 and avoids $(-\lambda_1/b_1, -\lambda_2/b_2, \cdots, -\lambda_N/b_N)$. Thus B is arcwise connected. □

COROLLARY 5.13. *Let the hypotheses of Theorem* 5.12 *hold. Then the attainable set*

$$s(z_0) = \bigcup_{\substack{t \ge 0 \\ p \in L^2_{\text{loc}}([0, \infty); \mathbb{R})}} z(t; p, z_0)$$

is dense in $\mathcal{H}$.

Proof. The set $\{h \in l_2 \mid 1 + (b_n/\lambda_n)h_n \neq 0 \text{ for all } n\}$ is dense in l_2. □

Remark 5.14. Clearly the information provided by Theorem 5.12 implies global controllability with respect to suitable finite-dimensional observers. We leave the precise formulation of these results to the reader.

6. Applications to partial differential equations.

Example 1. *Wave equation with Dirichlet boundary conditions.* Consider the wave equation

$$u_{tt} - u_{xx} + p(t)u = 0, \qquad 0 < x < 1,$$

with boundary conditions

$$u = 0 \quad \text{at } x = 0, 1$$

and initial conditions

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad 0 < x < 1.$$

In the notation of (5.1), (5.2) we have

$$A = -\frac{d^2}{dx^2}, \quad B = I, \quad H = L^2(0, 1) = L^2([0, 1]; \mathbb{R}),$$

$$D(A) = H^2(0, 1) \cap H_0^1(0, 1), \qquad D(A^{1/2}) = H_0^1(0, 1),$$

$$\lambda_n = n\pi, \qquad \phi_n = \sqrt{2} \sin n\pi x, \qquad n = 1, 2, \cdots,$$

$$(B\phi_n, \phi_m) = \delta_{mn}.$$

We thus see that (D1) holds, and since $b_n = 1$ we have $b_n/\lambda_n = 1/n\pi$ so that (D2) also holds.

As before, we set

$$z(t) = A^{1/2}u(t) + i\dot{u}(t) \quad \text{and} \quad z_0 = A^{1/2}u_0 + iu_1,$$

so that

$$z_{0n} = \lambda_n(u_0, \phi_n) + i(u_1, \phi_n).$$

In this case $\mathscr{H} = L^2(0, 1) \oplus iL^2(0, 1)$. We suppose that $z_0 \in \mathscr{H}$. We note that $\{1, e^{\pm 2i\lambda_n t}\}$ forms a Riesz basis of $L^2([0, 1]; \mathbb{C})$ and can be extended to a Riesz basis of $L^2([0, l]; \mathbb{C})$ for any $l \geqq 1$. Then Theorem 5.3, Corollary 5.4, Theorem 5.7 and Theorem 5.8 are all applicable. For example, Theorem 5.3 says that if $z_{0n} \neq 0$ for all n we can control any finite-dimensional projection of the solution to take any value sufficiently close to the projection of the free solution ($p \equiv 0$) at time $T_1 \geqq 1$, while Theorem 5.8 holds for any $l \geqq 1$.

In particular, Theorem 5.5 shows that the set of $\{u, u_t\}$ in $H_0^1(0, 1) \times L^2(0, 1)$ accessible from $\{u_0, u_1\}$ with controls in $L^r_{\text{loc}}[0, \infty)$, $r \geqq 1$, given by

$$S(\{u_0, u_1\}) = \bigcup_{\substack{t \geqq 0 \\ p \in L^r_{\text{loc}}([0,\infty);\mathbb{R})}} \{u(t; p, u_0, u_1), u_t(t; p, u_0, u_1)\}$$

has dense complement in $H_0^1(0, 1) \times L^2(0, 1)$. On the other hand, by Theorem 5.12 and Corollary 5.13 we have global approximate controllability: thus the set S of states that can be reached using L^2 controls on a time interval of length at least one is dense in $H_0^1 \times L^2$, provided $z_{0n} \neq 0$, i.e., all modes of the initial data are active.

Example 2. *Wave equation with mixed boundary conditions.* Consider the wave equation

$$u_{tt} - u_{xx} + p(t)u = 0, \qquad 0 < x < 1,$$

with boundary conditions

$$u = 0 \text{ at } x = 0, \quad u - \alpha u_x = 0 \text{ at } x = 1, \quad \alpha > 0 \text{ constant},$$

and initial conditions

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad 0 < x < 1.$$

In the notation of (5.1) and (5.2) we have

$$A = -\frac{d^2}{dx^2}, \quad B = I, \quad H = L^2(0, 1),$$

$$D(A) = \{u \in H^2(0, 1) \mid u = 0 \text{ at } x = 0,\ u + \alpha u_x = 0 \text{ at } x = 1\},$$

$$D(A^{1/2}) = \{u \in H^1(0, 1) \mid u = 0 \text{ at } x = 0\},$$

$$\tan \lambda_n + \alpha\lambda_n = 0, \qquad \phi_n(x) = (\sin \lambda_n x) \Big/ \left(\int_0^1 \sin^2 \lambda_n x\right)^{1/2}, \qquad n = 1, 2, \cdots,$$

and $(B\phi_m, \phi_n) = \delta_{mn}$.

In this case,

$$\lambda_n = \frac{n\pi}{2} + \varepsilon_n(\alpha), \qquad n = 1, 2, \cdots,$$

where $|\varepsilon_n(\alpha)| \to 0$ as $n + \alpha \to \infty$. Thus, since $b_n = 1$, $\{b_n/\lambda_n\} \in l_2$. Hence (D1) and (D2) hold.

As usual, we set

$$z(t)=A^{1/2}u(t)+i\dot{u}(t), \qquad z_0=A^{1/2}u_0+iu_1,$$

so that

$$z_{0n}=\lambda_n(u_0,\phi_n)+i(u_1,\phi_n).$$

As in Example 1, $\mathscr{H}=L^2(0,1)\oplus iL^2(0,1)$, and we let $z_0\in\mathscr{H}$. Theorem 5.3, Corollary 5.4, Theorem 5.7, Theorem 5.8 and the first part of Corollary 5.10 are all applicable. By Theorem 5.2 $\{e^{2i\lambda_n t}\}$ can be extended to a Riesz basis of $L^2([0,T];\mathbb{C})$ for any $T>2$, so that in the above results the assertions of finite-dimensional or approximate controllability apply to time intervals of length greater than 2. Actually, for α sufficiently large we can take $T_1\geqq 2$ in Theorem 5.3 and $T\geqq 2$ in Corollary 5.10. (This is because $\sup_n|\varepsilon_n(\alpha)|=|\varepsilon_1(\alpha)|<\frac{1}{2}\log 2$ for α sufficiently large, so that

$$\sup_n|2\lambda_n-n\pi|<\log 2,$$

which implies by Riesz and Nagy [1955, p. 209] that $\{1,e^{\pm 2i\lambda_n t}\}$ forms a Riesz basis of $L^2(0,2)$.)

Example 3. *Rod equation with hinged ends.* Consider the system

$$u_{tt}+u_{xxxx}+p(t)u_{xx}=0, \qquad 0<x<1, \tag{6.1}$$

with boundary conditions

$$u=u_{xx}=0 \quad \text{at } x=0,1 \tag{6.2}$$

and initial conditions

$$u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x). \tag{6.3}$$

In the notation of (5.1), (5.2) we set

$$A=\frac{d^4}{dx^4},\quad B=\frac{d^2}{dx^2},\quad H=L^2(0,1),$$

$$D(A)=\{u\in H^4(0,1)\,|\,u,u_{xx}\in H_0^1(0,1)\},$$

$$D(A^{1/2})=H^2(0,1)\cap H_0^1(0,1), \qquad \lambda_n=n^2\pi^2,$$

$$\phi_n=\sqrt{2}\sin n\pi x, \qquad n=1,2,\cdots,$$

$$(B\phi_m,\phi_n)=0,\quad n\neq m, \qquad (B\phi_n,\phi_n)=-n^2\pi^2.$$

In this case $b_n/\lambda_n=-1$, so that (D1), (D2) are again satisfied. As usual we write $z(t)=A^{1/2}u(t)+i\dot{u}(t)=\sum_{n=1}^{\infty}z_n(t)\phi_n$, $z_n(0)=z_{0n}$. Note that

$$\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=\infty. \tag{6.4}$$

Theorem 5.2 therefore implies that $\{1,e^{\pm 2i\lambda_n t}\}$ can be extended to a Riesz basis of $L^2([0,T];\mathbb{C})$ for *any* $T>0$. Theorem 5.3 is therefore applicable with any $T_1>0$, Corollary 5.4 holds, Theorems 5.7 and 5.8 hold for any $l>0$, both conclusions of Corollary 5.10 are valid, and Theorem 5.12 and Corollary 5.13 hold. We summarize the approximate controllability results in the following theorem.

THEOREM 6.1. *Let* $u_0\in H^2(0,1)\cap H_0^1(0,1)$, $u_1\in L^2(0,1)$ *and suppose that*

$$z_{0n}\equiv n^2\pi^2(u_0,\phi_n)+i(u_1,\phi_n)\neq 0 \quad \textit{for all } n=1,2,\cdots.$$

For any $p \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ a unique mild solution

$$\{u, \dot{u}\} \in C([0, \infty); X)$$

of (6.1)–(6.3) exists, where $X = (H^2(0, 1) \cap H^1_0(0, 1)) \times L^2(0, 1)$, and if $p \in L^2_{\text{loc}}([0, \infty); \mathbb{R})$ then

$$\left\{\frac{z_n(t)}{z_{0n}} \exp\left[i\left(\lambda_n t - \frac{1}{2}\int_0^t p(s)\, ds\right)\right] - 1\right\} \in C([0, \infty); l_2).$$

Conversely, for any $T > 0$ there exists $\varepsilon_T > 0$ such that if $\|h\|_{l_2} + |\alpha| < \varepsilon_T$ then

$$\frac{z_n(T)}{z_{0n}} e^{i(\lambda_n T - \alpha)} - 1 = h_n, \qquad n = 1, 2, \cdots$$

for infinitely many $p \in L^2_{\text{loc}}([0, T]; \mathbb{R})$ with $\int_0^T p(t)\, dt = 2\alpha$. In particular, setting $T = 2/\pi$, there exists $\varepsilon > 0$ such that if $\|h\|_{l_2} + |\alpha| < \varepsilon$ then

$$z_n\left(\frac{2}{\pi}\right) = e^{i\alpha}(1 + h_n) z_{0n}, \qquad n = 1, 2, \cdots$$

for infinitely many $p \in L^2([0, 2/\pi]; \mathbb{R})$ with $\int_0^{2/\pi} p(t)\, dt = 2\alpha$. Furthermore, if $(h, \alpha) \in l_2 \times \mathbb{R}$ with $h_n \neq -1$ for all n, there exist a positive integer m and a control $p \in L^2([0, 2m/\pi]; \mathbb{R})$ such that

$$z_n\left(\frac{2m}{\pi}\right) = e^{i\alpha}(1 + h_n) z_{0n}, \qquad n = 1, 2, \cdots, \tag{6.5}$$

so that the set of states accessible from $\{u_0, u_1\}$ is dense in X.

Remark 6.2. Our method of proof shows that given $\varepsilon > 0$ we can find m and p such that (6.5) holds and $\|p\|_{L^2(I;\mathbb{R})} < \varepsilon$ for any interval $I \subset [0, 2m/\pi]$ of length 1. Of course m will need to be large if ε is small.

Example 4. *Rod equation with clamped ends.* Consider (6.1) with boundary conditions

$$u = u_x = 0 \quad \text{at } x = 0, 1$$

and initial conditions (6.3). As is well known, this case is much more delicate than (6.1) with hinged boundary conditions (6.2). We now have

$$A = \frac{d^4}{dx^4}, \quad B = \frac{d^2}{dx^2}, \quad H = L^2(0, 1),$$

$$D(A) = H^4(0, 1) \cap H^2_0(0, 1), \qquad D(A^{1/2}) = H^2_0(0, 1),$$

$$\cosh \lambda_n^{1/2} \cos \lambda_n^{1/2} = 1, \qquad n = 1, 2, \cdots.$$

The usual graphical analysis shows that

$$\lambda_n = (n - \tfrac{1}{2})^2 \pi^2 + \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. (Very precise estimates for ε_n are given in Ball and Slemrod [1979].) The corresponding orthonormal eigenfunctions ϕ_n do *not* satisfy $(B\phi_m, \phi_n) = 0$, $m \neq n$, and so none of the results in § 5.3 are applicable. Furthermore, hypothesis (ii) of Theorem 5.3 does not hold, since $2\lambda_n - (\lambda_p + \lambda_q)$ can be arbitrarily small for arbitrarily large n, p and q (cf. Ball and Slemrod [1979], especially pp. 560, 574). So it is not obvious that (6.1), (6.6) is controllable locally with respect to finite-dimensional observers. It is possible that estimates on the lines of those in the preceding reference for the λ_n might establish local controllability relative to G of the form

$$G\left(\sum_{n=1}^{\infty} a_n z_n\right) = L\left(\sum_{n=1}^{N} a_n z_n\right).$$

The only results in this paper applicable to (6.1), (6.6) are the basic existence theorem, Theorem 2.5, which just gives the standard result that for $\{u_0, u_1\} \in D(A^{1/2}) \times H = X$ there exists for each $p \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ a unique mild solution with initial data $\{u_0, u_1\}$, and Theorem 3.6, which demonstrates the general impossibility of exact controllability using controls $p \in L^r_{\text{loc}}([0, \infty); \mathbb{R})$, $r > 1$.

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