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THE HAMILTONIAN STRUCTURE OF THE MAXWELL–VLASOV EQUATIONS

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Morrison [25] has observed that the Maxwell–Vlasov and Poisson–Vlasov equations for a collisionless plasma can be written in Hamiltonian form relative to a certain Poisson bracket. We derive another Poisson structure for these equations by using general methods of symplectic geometry. The main ingredients in our construction are the symplectic structure on the co-adjoint orbits for the group of canonical transformations, and the symplectic structure for the phase space of the electromagnetic field regarded as a gauge theory. Our Poisson bracket satisfies the Jacobi identity, whereas Morrison's does not [37]. Our construction also shows where canonical variables can be found and can be applied to the Yang–Mills–Vlasov equations and to electromagnetic fluid dynamics.

1. Introduction

In this paper we show how to construct a Poisson structure for the Maxwell–Vlasov and Poisson–Vlasov equations for collisionless plasmas by using general methods of symplectic geometry. We shall compare our structure to that obtained by Morrison [25].

We consider a plasma consisting of particles with charge e and mass m moving in Euclidean space $\mathbb{R}^3$ with positions $\boldsymbol{x}$ and velocities $\boldsymbol{v}$. For simplicity we consider only one species of particle; the general case is similar. Let $f(\boldsymbol{x}, \boldsymbol{v}, t)$ be the plasma density at time t, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ the electric and magnetic fields. The *Maxwell–Vlasov equations* are:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \frac{e}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{1.1}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{curl} \boldsymbol{E}, \tag{1.2a}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{curl} \boldsymbol{B} - \frac{e}{c}\int \boldsymbol{v} f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}, \tag{1.2b}$$

$$\operatorname{div} \boldsymbol{E} = \rho_f, \quad \text{where} \quad \rho_f = e \int f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}, \tag{1.3a}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{1.3b}$$

Letting $c \to \infty$ leads to the *Poisson–Vlasov equation*:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{e}{m}\frac{\partial \phi_f}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{1.4}$$

where

$$\Delta \phi_f = -\rho_f, \quad \text{and} \quad \Delta = \nabla^2 \text{ is the Laplacian.} \tag{1.5}$$

In what follows we shall set $e = m = c = 1$.

The Hamiltonian for the Maxwell–Vlasov system is

$$H(f, \boldsymbol{E}, \boldsymbol{B}) = \int \frac{1}{2}|\boldsymbol{v}|^2 f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v} + \int \frac{1}{2}[|\boldsymbol{E}(\boldsymbol{x}, t)|^2 + |\boldsymbol{B}(\boldsymbol{x}, t)|^2]\, \mathrm{d}\boldsymbol{x}, \tag{1.6}$$

while that for the Poisson–Vlasov equation is

$$H(f) = \int \frac{1}{2}|\boldsymbol{v}|^2 f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v} + \frac{1}{2}\int \phi_f(\boldsymbol{x})\rho_f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{1.7}$$

The Poisson bracket used by Morrison is defined on functions $F(f, \boldsymbol{E}, \boldsymbol{B})$ of the fields $f, \boldsymbol{E}, \boldsymbol{B}$ by

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$$\{\{F, G\}\} = \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$$
$$+ \int \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \operatorname{curl} \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \operatorname{curl} \frac{\delta F}{\delta \boldsymbol{B}}\right) \mathrm{d}\boldsymbol{x}$$
$$+ \int \frac{\delta F}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f} \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$$
$$+ \int \frac{\delta F}{\delta \boldsymbol{B}} \cdot \left(\frac{\partial f}{\partial \boldsymbol{v}} \times \boldsymbol{v}\right) \frac{\delta G}{\delta f} - \frac{\delta G}{\delta \boldsymbol{B}} \cdot \left(\frac{\partial f}{\partial \boldsymbol{v}} \times \boldsymbol{v}\right) \frac{\delta F}{\delta f} \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}, \tag{1.8}$$

where in the first term $\{\ ,\ \}$ denotes the standard Poisson bracket for functions of $(\boldsymbol{x}, \boldsymbol{v})$, given by

$$\{h, k\} = \sum_{i=1}^{3} \left(\frac{\partial h}{\partial x_i} \frac{\partial k}{\partial v_i} - \frac{\partial h}{\partial v_i} \frac{\partial k}{\partial x_i}\right).$$

The functional derivatives, such as $\delta F/\delta f$, are defined in terms of the usual (Frèchet) derivative by

$$(D_f F) \cdot f' = \int \frac{\delta F}{\delta f} f' \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}, \quad \text{etc..}$$

For the Poisson–Vlasov equation, one keeps only the first term of (1.8). The eqns. (1.1) and (1.2) (or (1.4)) are then equivalent to

$$\dot{F} = \{\{F, H\}\}, \tag{1.9}$$

with H given by (1.6) (or (1.7) for the Poisson–Vlasov equation).

Our purpose is to show how another Poisson structure can be constructed by a general procedure involving reduction (Marsden and Weinstein [22]) and coupling of Hamiltonian systems to gauge fields (Weinstein [36]). Our Poisson bracket is written out in full in equation (7.1) below. It differs from (1.8) in that the last integral is replaced by

$$\int f\boldsymbol{B} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$$
$$= \int \frac{\delta F}{\delta f} \frac{\partial f}{\partial \boldsymbol{v}} \cdot \left[\boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\delta G}{\delta f}\right)\right] \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$$
$$- \int \frac{\delta G}{\delta f} \frac{\partial f}{\partial \boldsymbol{v}} \cdot \left[\boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\delta F}{\delta f}\right)\right] \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}. \tag{1.10}$$

Both structures yield the correct equations of motion for the Hamiltonians we have specified; however, Morrison's does not satisfy the Jacobi identity (Weinstein and Morrison [37]), while this identity follows for our structure from the general theory of reduction. In addition, the equations $\operatorname{div} \boldsymbol{E} = \rho_f$ and $\operatorname{div} \boldsymbol{B} = 0$ arise naturally from the gauge symmetry of the problem and need not be postulated separately.

Our Poisson structure fits into a pattern, special cases of which have been found by others. For example, Arnold [4] showed that the Euler equations for a perfect incompressible fluid are a Hamiltonian system in the canonical Poisson structure associated with the group of volume preserving diffeomorphisms of a region in $\mathbb{R}^3$. Using Arnold's methods, one can also see that the compressible equations are associated to the semidirect product of the group of diffeomorphisms and the additive group of densities on $\mathbb{R}^3$. (This fits into the schemes of Guillemin and Sternberg [15] and Ratiu [31].) It is easy to check that this approach yields the same Poisson structure found for perfect fluids by Morrison and Greene [27] and ought to be extendible to the MHD equations by the methods of this paper. The KdV equation is associated with the Lie algebra of the group of canonical transformations in the work of, for example, Adler [2]. (See Davidson [11] for a link between the Maxwell–Vlasov equations and the KdV equation.) In Ebin and Marsden [12], the functional analytic machinery required to fully justify Arnold's approach was given. It was proved, for example, that the volume preserving diffeomorphisms form a C^∞ infinite-dimensional manifold which is, in an appropriate sense, a Lie group. It was also shown that the group of canonical transformations is a Lie group as well, but no physical interpretation was given. The Vlasov equation provides one.

In this paper we shall not deal with the delicate functional analytic issues needed to make precise all the infinite-dimensional geometry behind the Vlasov equations, nor shall we deal with equations of existence and uniqueness (cf. Braun and Hepp [10], Batt [6], Ukai–Okabe [33], Horst [18, 19], Wollman

[38, 39] and references therein). We expect that this gap can be filled by using techniques of Ebin and Marsden [12] and Ratiu and Schmid [32].

All of the clues above suggest that it is fruitful to find a more geometric and group-theoretic framework for the basic equations of plasma physics. Such a framework is provided here.

The subject matter of this paper requires considerable background material. A more leisurely exposition is planned for the near future in Marsden, Weinstein, Schmid and Spencer [23].

A Poisson bracket for special relativistic plasmas which agrees with ours in the non-relativistic limit has been recently obtained by Bialynicki-Birula and Hubbard [8]. They also point out that the brackets for electrodynamics go back to Pauli [28] and Born and Infeld [9], and that Pauli also gives brackets for interacting discrete particles and electromagnetic fields. A canonical formulation of relativistic hydrodynamics was given by Bialynicki-Birula and Iwiński [7]. In none of these references are the Poisson brackets derived from canonical brackets as we do.

2. Poisson structures for Lie algebras

We begin by reviewing the Poisson structure on the dual of a Lie algebra. This material is largely available in Guillemin and Sternberg [15] so will be only quickly treated, but in notation suitable for this paper.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. The pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ is denoted $\langle\ ,\ \rangle$. We wish to define a bracket $\{\{F, G\}\}$ on functions from $\mathfrak{g}^*$ to $\mathbb{R}$. There are three ways to do this:

Method 1 (Direct). We just write down a formula for $\{\{F, G\}\}$ due to Berezin [40]; closely related formulas were used by Kirillov and Arnold – see [4], [5]. The formula depends on the notation of "functional" derivative defined as follows: For $F: \mathfrak{g}^* \to \mathbb{R}$, define $\delta F/\delta\mu \in \mathfrak{g}$ (μ denotes the variable in $\mathfrak{g}^*$) by

$$DF(\mu)\cdot\nu = \left\langle \nu, \frac{\delta F}{\delta\mu}\right\rangle, \quad \text{for all} \quad \nu \in \mathfrak{g}^*, \tag{2.1}$$

i.e. we identify $\mathfrak{g}^{**}$ with $\mathfrak{g}$ so that $DF(\mu) \in \mathfrak{g}^{**}$ becomes an element of $\mathfrak{g}$. Then the bracket is defined by

$$\{\{f, g\}\}(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta F}{\delta\mu}\right]\right\rangle, \tag{2.2}$$

where $[\ ,\ ]$ is the Lie algebra operation on $\mathfrak{g}$.

The bracket (2.2) defines a *Poisson structure*; i.e. $\{\{F, G\}\}$ is bilinear, antisymmetric, satisfies Jacobi's identity, and is a derivation in each argument. This can be proved directly or by noting the equivalence of (2.2) to the other formulas (2.3) and (2.4) below, from which it is obvious that one obtains a Poisson structure.

Method 2. (Restriction). The Kirillov–Kostant–Souriau theorem asserts that the orbits of the co-adjoint representation in $\mathfrak{g}^*$ are symplectic manifolds. (See, for instance Arnold [5] or Abraham and Marsden [1] for the proof). Thus, $\mathfrak{g}^*$ is a disjoint union of symplectic manifolds. For $F, G: \mathfrak{g}^* \to \mathbb{R}$ a Poisson bracket is thus defined by

$$\{\{F, G\}\}(\mu) = \{F|\mathcal{O}_\mu, G|\mathcal{O}_\mu\}_\mu(\mu), \tag{2.3}$$

where $\mu \in \mathfrak{g}^*$, $\mathcal{O}_\mu$ is the orbit through μ, $F \mid \mathcal{O}_\mu$ is the *restriction* of F to $\mathcal{O}_\mu$, and $\{\ ,\ \}_\mu$ is the bracket on $\mathcal{O}_\mu$.

Method 2 shows that the bracket $\{\{\ ,\ \}\}$ is degenerate; however it determines a symplectic foliation, on each leaf of which it is nondegenerate. The leaves are just the co-adjoint orbits.

Method 3 (Extension). Given $F, G: \mathfrak{g}^* \to \mathbb{R}$, extend them to maps $\hat{F}, \hat{G}: T^*G \to \mathbb{R}$ by left invariance. Then, using the canonical bracket structure on T^*G, form $\{\hat{F}, \hat{G}\}$. Finally, regarding $\mathfrak{g}^*$ as $T^*_eG \subset T^*G$, restrict to $\mathfrak{g}^*$:

$$\{\{F, G\}\} = \{\hat{F}, \hat{G}\} \,|\, \mathfrak{g}^*. \tag{2.4}$$

Methods 2 and 3 are related by reduction; i.e. the reduced symplectic manifolds for the action of G on T^*G by left translation are the co-adjoint orbits (Marsden and Weinstein [22]).

2.1. *Proposition.* The formulas (2.2), (2.3) and (2.4) all define the same Poisson structure on $\mathfrak{g}^*$.

This result is implicit in the literature cited, so we omit the proof; however additional details in the case that concerns us will be given in the next section.

3. The Poisson structure for the density variables

We are now ready to explain the geometric meaning of the term $\int f\{(\delta F/\delta f), (\delta F/\delta f)\}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$ in (1.8). In the following sections, we shall explain the term for Maxwell's equations (the second integral in (1.8)), and then finally the coupling terms (the remaining two integrals).

In the absence of a magnetic field, by normalizing mass, we can identify velocity with momentum; hence we let $\mathbb{R}^6$ denote the usual position-momentum phase space with coordinates $(x_1, x_2, x_3, p_1, p_2, p_3)$ and symplectic structure $\Sigma\, \mathrm{d}x_i \wedge \mathrm{d}p_i$. (See Abraham and Marsden [1] or Arnold [5].) Let $\mathfrak{S}$ denote the group of canonical transformations of $\mathbb{R}^6$ which have polynomial growth at infinity in the momentum directions. The Lie algebra $\mathfrak{s}$ of $\mathfrak{S}$ consists of the Hamiltonian vector fields on $\mathbb{R}^6$ with polynomial growth in the momentum directions. We can identify elements of $\mathfrak{s}$ with their generating functions,* so that $\mathfrak{s}$ consists of the C^∞ functions on $\mathbb{R}^6$ and the (right) Lie algebra structure is given by $[f, g] = -\{f, g\}$, the *negative* of the usual Poisson bracket on phase space. (This follows from Exercise 4.1G and Corollary 3.3.18 of Abraham and Marsden [1]).

The dual space $\mathfrak{s}^*$ can be identified with the distribution densities on $\mathbb{R}^6$ which are rapidly decreasing in the momentum directions; the pairing between $h \in \mathfrak{s}$ and $f \in \mathfrak{s}^*$ is given by integration:

$$\langle h, f\rangle = \int hf\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{p}.$$

(The "density" is really $f\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{p}$, but we denote it simply by f.) Now as for any Lie algebra, the dual space $\mathfrak{s}^*$ carries a natural Poisson structure which is non-degenerate on the co-adjoint orbits (see Section 2 above). In our case the orbit through $f \in \mathfrak{s}^*$ is

$$\mathcal{O}_f = \{\eta^* f \mid \eta \in \mathfrak{S}\}. \tag{3.1}$$

It follows that a tangent vector to $\mathcal{O}_f$ at f has the form $\{f, h\}$ for $h \in \mathfrak{s}$†. The Kirillov–Kostant symplectic form ω on $\mathcal{O}_f$ is given at f by the bilinear pairing ω_f defined by

$$\omega_f\,\{f, h\}, \{f, k\}) = \langle f, \{h, k\}\rangle. \tag{3.2}$$

(See p. 303 of Abraham and Marsden [1]—two minus signs have cancelled here.) The Hamiltonian vector field X_f on $\mathcal{O}_f$ determined by a smooth function $F: \mathfrak{s}^* \to \mathbb{R}$ satisfies the defining relation

$$\omega_f(\{f, h\}, \{f, k\}) = \mathrm{d}F(f) \cdot \{f, k\}, \tag{3.3}$$

for all $k \in \mathfrak{s}$.

3.1. *Lemma.* We have

$$X_F(f) = -\left\{f, \frac{\delta F}{\delta f}\right\}. \tag{3.4}$$

Proof. By (3.3) and (3.2) we need only check that

*The generating function of a Hamiltonian vector field is determined only up to an additive constant. The "correct" group $\mathfrak{S}$ is really the group of transformations of $\mathbb{R}^6 \times \mathbb{R}$ preserving the 1-form $\Sigma\, p_i\, \mathrm{d}q_i + \mathrm{d}\tau$ (Van Hove [34]), but we can ignore this technical point here without encountering any essential difficulties.

†By $\{f, h\}$, we mean the Lie derivative of the density f by the Hamiltonian vector field of h. This is the "infinitesimal co-adjoint representation". We are implicitly identifying $\mathfrak{s}^*$ with $\mathfrak{s}$ via the integration pairing.

$$-\left\langle f, \left\{\frac{\delta F}{\delta f}, k\right\}\right\rangle = \mathrm{d}F(f)\cdot\{f,k\} = \left\langle \frac{\delta F}{\delta f}, \{f,k\}\right\rangle. \tag{3.5}$$

But (3.5) follows by integration by parts. It is a special case of the following useful identity:

$$\langle f, \{k, h\}\rangle = \langle \{f, k\}, h\rangle,$$

i.e.

$$\int f\{k, h\}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} = \int \{f, k\} h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p}. \tag{3.6}$$

Thus, the Poisson bracket on $\mathfrak{g}^*$ is given by

$$\begin{aligned}\{\{F, G\}\}(f) &= \omega_f(X_F(f), X_G(f)) \qquad \text{(by definition)}\\ &= \omega_f\left(\left\{f, \frac{\delta F}{\delta f}\right\}, \left\{f, \frac{\delta G}{\delta f}\right\}\right) \qquad \text{(by (3.4))}\\ &= \left\langle f, \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}\right\rangle \qquad \text{(by 3.2).}\end{aligned}$$

We have thus proved the equivalence of (2.2) and (2.3) in the present case:

3.2. *Proposition.* The natural Poisson structure on the dual of the Lie algebra of the group of canonical transformations is given by

$$\{\{F, G\}\}(f) = \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p}. \tag{3.7}$$

Remarks. 1. Notice that (3.7) coincides with the first term for the Poisson structure (1.8) if $\boldsymbol{p}$ is replaced by $\boldsymbol{v}$.

2. The bracket (3.7) automatically satisfies the Jacobi identity since it coincides with the Poisson bracket on each of the symplectic manifolds $\mathcal{O}_f$.

3. (a) If f is a delta density, $\mathcal{O}_f$ "coincides" with $\mathbb{R}^6$. (In a similar way, every symplectic manifold is a co-adjoint orbit.) For f a sum of n delta functions, $\mathcal{O}_f$ is the phase space for n particles. For continuous plasmas, f is taken to be a continuous density, in which case $\mathcal{O}_f$ can be shown to be a smooth infinite dimensional manifold.

(b) If f is a density concentrated along a curve, then $\mathcal{O}_f$ is identifiable with all curves having a fixed action integral. This is a reduced form of the loop space, a symplectic manifold used in the variational principle of Weinstein [35]. If f is concentrated on a lagrangian torus, then $\mathcal{O}_f$ consists of lagrangian tori with fixed action integrals. This is related to a variational principle of Percival [29].

4. By using an appropriate Darboux theorem, (see Marsden [21], lecture 1), one can show that $\mathcal{O}_f$ admits canonically conjugate coordinates.

5. The Vlasov–Poisson equation is a Hamiltonian system on $\mathfrak{g}^*$ with energy function given by (1.7). If f evolves according to (1.4) then (1.9) is true. This is a simple direct calculation, already noted by Morrison [25] and Gibbons [41]. More can be learned from our derivation of the Poisson structure: the vector field in eq. (1.4) is tangent to each orbit $\mathcal{O}_f$, so it defines a Hamiltonian system on each orbit. The preservation of $\mathcal{O}_f$ and the validity of (1.9) can be seen by noting that (1.4) can be written in terms of ordinary Poisson brackets as

$$\frac{\partial f}{\partial t} = -\{f, \mathcal{H}(f)\}, \tag{1.4$'$}$$

where

$$\mathcal{H}(f) = \frac{1}{2}|\boldsymbol{v}|^2 + \phi_f(\boldsymbol{x}).$$

In view of (3.4), (1.4)$'$ is equivalent to saying that f evolves by $\dot{f} = X_{\mathcal{H}}(f)$, where $\mathcal{H}(f) = \delta H/\delta f$, the "self-consistent Hamiltonian". Thus the evolution of f can be described by

$$f_t = \eta_t^* f_0,$$

where f_0 is the initial value of f, f_t is its value at time t, and $\eta_t \in \mathfrak{S}$. In particular, if $\mathcal{F}$ is a function of a single real variable, we get the well-known conservation laws

$$\int \mathcal{F}(f_t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} = \text{constant in time},$$

by the change of variables formula and the fact that each $\eta \in \mathfrak{S}$ is volume preserving. (These conservation laws are useful in proving existence and uniqueness theorems since, as in the case of two-dimensional ideal incompressible flow, they lead to a priori L^p-estimates.)

6. In Ebin and Marsden [12] the convective term $v \cdot \nabla v$ in fluid mechanics led to a crucial difference between working spatially (in the Lie algebra—the "Euler" picture) or materially (on the group... the "Lagrange" picture). Here there is no such term, since it would be given by $\{f, f\}$, which vanishes.

7. Analogies with fluid mechanics raise several interesting analytical problems: (A) if δ times a dissipation term associated with collisions is added, do the solutions converge to those of the Vlasov equation as $\delta \to 0$? (Analogous to the limit of zero viscosity). Standard techniques (Ebin and Marsden [12], Kato [20]) can probably be used to answer this affirmatively for short time.

(B) Can the Hamiltonian structure be used to study chaotic or turbulent dynamics, as was done in, for example, Holmes and Marsden [17]?

(C) Is the time-t map for the Poisson–Vlasov or Maxwell–Vlasov equations smooth? See Ratiu [30] for a discussion of why this question is of interest for the KdV equation.

4. Maxwell's equations and reduction

Before coupling the Vlasov equation to the electromagnetic field equations, we shall review the Hamiltonian description of Maxwell's equations. The appropriate Poisson bracket for the E and B fields (the second term in (1.8)) will be constructed by reduction (Marsden and Weinstein [22]).

As the configuration space for Maxwell's equations, we take the space $\mathfrak{A}$ of vector fields $\boldsymbol{A}$ on $\mathbb{R}^3$. (These are the "vector potentials", In more general situations, one should replace $\mathfrak{A}$ by the set of connections on a principal bundle over configuration space.) The corresponding phase space is then the cotangent bundle $T^*\mathfrak{A}$, with the canonical symplectic structure. Elements of $T^*\mathfrak{A}$ may be identified with pairs (A, Y), where Y is a vector field density on $\mathbb{R}^3$. (As usual, we do not distinguish Y and $Y\,\mathrm{d}x$.) The pairing between A's and Y's is given by integration, so that the canonical symplectic structure ω on $T^*\mathfrak{A}$ is given by

$$\omega((\boldsymbol{A}_1, \boldsymbol{Y}_1), (\boldsymbol{A}_2, \boldsymbol{Y}_2)) = \int (\boldsymbol{Y}_2 \cdot \boldsymbol{A}_1 - \boldsymbol{Y}_1 \cdot \boldsymbol{A}_2)\,\mathrm{d}\boldsymbol{x}, \tag{4.1}$$

with associated Poisson bracket

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \boldsymbol{A}}\frac{\delta G}{\delta \boldsymbol{Y}} - \frac{\delta F}{\delta \boldsymbol{Y}}\frac{\delta G}{\delta \boldsymbol{A}}\right)\mathrm{d}\boldsymbol{x}. \tag{4.2}$$

With the Hamiltonian

$$H(\boldsymbol{A}, \boldsymbol{Y}) = \frac{1}{2}\int |\boldsymbol{Y}|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int |\mathrm{curl}\ \boldsymbol{A}|^2\,\mathrm{d}\boldsymbol{x}, \tag{4.3}$$

Hamilton's equations are easily computed to be

$$\frac{\partial \boldsymbol{Y}}{\partial t} = -\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{A} \quad \text{and} \quad \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{Y}. \tag{4.4}$$

If we write $\boldsymbol{B}$ for curl $\boldsymbol{A}$ and $\boldsymbol{E}$ for $-\boldsymbol{Y}$, the Hamiltonian becomes the usual field energy

$$\frac{1}{2}\int |\boldsymbol{E}|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int |\boldsymbol{B}|^2\,\mathrm{d}\boldsymbol{x} \tag{4.5}$$

and the equations (4.4) imply Maxwell's equations

$$\frac{\partial \boldsymbol{E}}{\partial t} = \mathrm{curl}\,\boldsymbol{B} \quad \text{and} \quad \frac{\partial \boldsymbol{B}}{\partial t} = -\mathrm{curl}\,\boldsymbol{E}. \tag{4.6}$$

The remaining two Maxwell equations will appear as a consequence of gauge invariance. The gauge group $\mathfrak{G}$ consists of real valued functions on $\mathbb{R}^3$; the group operation is addition. An element $\psi \in \mathfrak{G}$ acts on $\mathfrak{A}$ by the rule†

†Notice that we work directly with three-dimensional fields. Four dimensionally, one has an extra "degree" of gauge freedom associated with the time derivative $\partial_t\psi$. We have already eliminated this freedom and the corresponding non-dynamical field A_4 (whose conjugate momentum vanishes). This is the standard Dirac procedure for a relativistic field theory such as Maxwell's equations. In the context of principal bundles, $\mathfrak{G}$ is defined to be the group of bundle automorphisms (covering the identity).

$$\mathbf{A} \mapsto \mathbf{A} + \nabla\psi. \tag{4.7}$$

This "translation" of A extends in the usual way to a canonical transformation ("extended point transformation") of $T^*\mathfrak{A}$ given by

$$(\mathbf{A}, \mathbf{Y}) \to (\mathbf{A} + \nabla\psi, \mathbf{Y}). \tag{4.8}$$

Notice that the Hamiltonian (4.3) is invariant under the transformations (4.8). This means that we can use the gauge symmetries to reduce the degrees of freedom of our system. The action of $\mathfrak{G}$ on $T^*\mathfrak{A}$ has a momentum map $J: T^*\mathfrak{A} \to \mathfrak{g}^*$, where $\mathfrak{g}$, the Lie algebra of $\mathfrak{G}$, is identified with the real valued functions on $\mathbb{R}^3$. This map may be determined by a standard formula (Abraham and Marsden [1], Corollary 4.2.11): for $\phi \in \mathfrak{g}$,

$$\langle J(\mathbf{A}, \mathbf{Y}), \phi\rangle = \int (\mathbf{Y}\cdot\nabla\phi)\,\mathrm{d}\mathbf{x} = -\int (\operatorname{div}\mathbf{Y})\,\phi\,\mathrm{d}\mathbf{x}. \tag{4.9}$$

Thus we may write

$$J(\mathbf{A}, \mathbf{Y}) = -\operatorname{div}\mathbf{Y}. \tag{4.10}$$

If ρ is an element of $\mathfrak{g}^*$ (the densities on $\mathbb{R}^3$), $J^{-1}(\rho) = \{(\mathbf{A}, \mathbf{Y}) \in T^*\mathfrak{A} \mid \operatorname{div}\mathbf{Y} = -\rho\}$. In terms of $\boldsymbol{E}$, the condition $\operatorname{div}\mathbf{Y} = -\rho$ becomes the Maxwell equation $\operatorname{div}\boldsymbol{E} = \rho$, so we may interpret the elements of $\mathfrak{g}^*$ as charge densities.

By a general theorem on reduction (Marsden and Weinstein [22]), the manifold $J^{-1}(\rho)/\mathfrak{G}$ has a naturally induced symplectic structure.

4.1. *Proposition*. The reduced manifold $J^{-1}(\rho)/\mathfrak{G}$ can be identified with $\mathcal{M}ax = \{(\boldsymbol{E}, \boldsymbol{B}) \mid \operatorname{div}\boldsymbol{E} = \rho, \operatorname{div}\boldsymbol{B} = 0\}$, so that the Poisson bracket on $\mathcal{M}ax$ is given in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ by†

$$\{\{F, G\}\} = \int \left(\frac{\delta F}{\delta \boldsymbol{E}}\operatorname{curl}\frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}}\operatorname{curl}\frac{\delta F}{\delta \boldsymbol{B}}\right)\mathrm{d}\mathbf{x}. \tag{4.11}$$

equations with an ambient charge density ρ are Hamilton's equations for

$$H(\boldsymbol{E}, \boldsymbol{B}) = \frac{1}{2}\int \left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right)\mathrm{d}\mathbf{x} \tag{4.12}$$

on the space $\mathcal{M}ax$.

Proof. To each $(\mathbf{A}, \mathbf{Y})$ in $J^{-1}(\rho)$ we associate the pair $(\boldsymbol{B}, \boldsymbol{E}) = (\operatorname{curl}\mathbf{A}, -\mathbf{Y})$ in $\mathcal{M}ax$. Since two vector fields $\mathbf{A}_1$ and $\mathbf{A}_2$ on $\mathbb{R}^3$ have the same curl if and only if they differ by a gradient, and every divergence-free $\boldsymbol{B}$ is a curl this association gives a 1–1 correspondence between $J^{-1}(\rho)/\mathfrak{G}$ and $\mathcal{M}ax$‡.

Now let F and G be functionals on $\mathcal{M}ax$. To compute their Poisson bracket $\{\{F, G\}\}$, we must pull them back to $J^{-1}(\rho)$, extend them to $T^*\mathfrak{A}$, take the canonical Poisson bracket in $T^*\mathfrak{A}$, restrict to $J^{-1}(\rho)$, and "push down" the resulting $\mathfrak{G}$-invariant function to $\mathcal{M}ax$. The result does not depend upon the choice of extension made, and in fact we can do the computation without mentioning the extension again. Given $F(\boldsymbol{B}, \boldsymbol{E})$, we define the pull back $\hat{F}(\mathbf{A}, \mathbf{Y})$ by

$$\hat{F}(\mathbf{A}, \mathbf{Y}) = F(\operatorname{curl}\mathbf{A}, -\mathbf{Y}). \tag{4.13}$$

Using the canonical bracket (4.2) on $T^*\mathfrak{A}$, we have

$$\begin{aligned}\{\{F, G\}\} = \{\hat{F}, \hat{G}\} &= \int \left(\frac{\delta \hat{F}}{\delta \mathbf{A}}\frac{\delta \hat{G}}{\delta \mathbf{Y}} - \frac{\delta \hat{G}}{\delta \mathbf{A}}\frac{\delta \hat{F}}{\delta \mathbf{Y}}\right)\mathrm{d}\mathbf{x}\\ &= -\int \left(\frac{\delta \hat{F}}{\delta \mathbf{A}}\frac{\delta \hat{G}}{\delta \boldsymbol{E}} - \frac{\delta \hat{G}}{\delta \mathbf{A}}\frac{\delta \hat{F}}{\delta \boldsymbol{E}}\right)\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.14}$$

The chain rule, the definition of functional derivatives, and integration by parts give the identity

$$\begin{aligned}\int \frac{\delta \hat{F}}{\delta \mathbf{A}}\cdot\mathbf{A}'\,\mathrm{d}\mathbf{x} &= \int \frac{\delta F}{\delta \boldsymbol{B}}\cdot\operatorname{curl}\mathbf{A}'\,\mathrm{d}\mathbf{x}\\ &= \int \mathbf{A}'\cdot\operatorname{curl}\frac{\delta F}{\delta \boldsymbol{B}}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.15}$$

Substitution of (4.15) in (4.14) gives (4.11). The rest of the proposition follows by direct calculation or from the general theory of reduction. ■

†Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are constrained in $\mathcal{M}ax$, the functional derivatives in (4.11) must be defined by extension of F and G to all $\boldsymbol{E}$ and $\boldsymbol{B}$, followed by restriction.

‡If $\mathbb{R}^3$ is replaced by a Riemannian manifold, $\boldsymbol{B}$ is replaced by the divergence-free part of $\mathbf{A}$ (also called the transverse component) in the Hodge decomposition.

This formalism generalizes readily to Yang–Mills fields and to these fields coupled to gravity; see Arms [3].

5. The Maxwell–Vlasov equations before reduction

The Hamiltonian structure for the Maxwell–Vlasov system is very simple if we choose as our variables densities on $(\boldsymbol{x}, \boldsymbol{p})$ space (rather than $(\boldsymbol{x}, \boldsymbol{v})$ space) and elements $(\mathbf{A}, \mathbf{Y})$ of $T^*\mathfrak{A}$. To avoid confusion with densities f on $(\boldsymbol{x}, \boldsymbol{v})$ space, we shall use the notation f_{mom} for densities on $(\boldsymbol{x}, \boldsymbol{p})$ space.

The Poisson structure on $\mathfrak{g}^* \times T^*\mathfrak{A}$ is just the sum of those on $\mathfrak{g}^*$ and $T^*\mathfrak{A}$: for functions $\bar{F}$ and $\bar{G}$ of f_{mom}, A, and Y, set

$$\{\{\bar{F}, \bar{G}\}\}(f_{\text{mom}}, \mathbf{A}, \mathbf{Y}) = \int f_{\text{mom}} \left\{\frac{\partial \bar{F}}{\delta f_{\text{mom}}}, \frac{\delta \bar{G}}{\delta f_{\text{mom}}}\right\} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{p} + \int \left(\frac{\delta \bar{F}}{\delta \mathbf{A}} \frac{\delta \bar{G}}{\delta \mathbf{Y}} - \frac{\delta \bar{G}}{\delta \mathbf{A}} \frac{\delta \bar{F}}{\delta \mathbf{Y}}\right) \mathrm{d}\boldsymbol{x} \tag{5.1}$$

and the Hamiltonian is just (1.6) written in terms of these variables. Using the classical relation $\boldsymbol{p} = \boldsymbol{v} + \mathbf{A}$ between momentum and velocity:

$$\bar{H}(f_{\text{mom}}, \mathbf{A}, \mathbf{Y}) = \frac{1}{2}\int |\boldsymbol{p} - \mathbf{A}(\boldsymbol{x})|^2 f_{\text{mom}}(\boldsymbol{x}, \boldsymbol{p})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{p} + \frac{1}{2}\int (|\mathbf{Y}|^2 + |\text{curl } \mathbf{A}|^2)\, \mathrm{d}\boldsymbol{x}. \tag{5.2}$$

Notice that there is no coupling in the symplectic structure (5.1) between $\mathfrak{g}^*$ and $T^*\mathfrak{A}$, but there is coupling in the first term of (5.2).

5.1 *Theorem*. The evolution equations $\dot{\bar{F}} = \{\{\bar{F}, \bar{H}\}\}$ for a function $\bar{F}$ on $\mathfrak{g}^* \times T^*\mathfrak{A}$ with $\bar{H}$ given by (5.2) and $\{\{\ \}\}$ by (5.1) are the eqs. (1.1) and (1.2) with (1.2a) replaced by $\partial \mathbf{A}/\partial t = \mathbf{Y}$.

The proof of this theorem is a straightforward verification. The constraints (1.3) are, as in Morrison [25], subsidiary (constraint) equations which are consistent with the evolution equations. Eq. (1.3b) holds since $\boldsymbol{B} = \text{curl } \mathbf{A}$. Eq. (1.3a) expresses the fact that we are on the zero level of the momentum map generated by the gauge transformations. The corresponding reduced space *decouples* the energy, while *coupling* the symplectic structure. We turn to this in the next two sections.

6. A general construction for reduction of interacting systems

The work of Weinstein [36] on the equations of motion for a particle in a Yang–Mills field uses the following general set-up. Let $\pi: \mathcal{B} \to \mathcal{M}$ be a principal G-bundle and Q a Hamiltonian G-space (or a Poisson manifold which is a union of Hamiltonian G-spaces). Then G acts on $T^*\mathcal{B}$ and on Q, so it acts on $Q \times T^*\mathcal{B}$ (with the product symplectic structure). This action has a momentum map J and so may be reduced at 0:

$$(Q \times T^*\mathcal{B})_0 = J^{-1}(0)/G.$$

The reduced manifold carries a symplectic (or Poisson, if Q was a Poisson manifold) structure naturally induced from those of Q and $T^*\mathcal{B}$.

To obtain the phase space for an elementary particle in a Yang–Mills field one chooses $\mathcal{B}$ to be a G-bundle over 3-space $\mathcal{M}$ and Q a co-adjoint orbit for G (the internal variables). The Hamiltonian is constructed using a connection (i.e. a Yang–Mills field) for $\mathcal{B}$. In the special case of electromagnetism, $G = S^1$ and $Q = \{e\}$ is a point.

For the Vlasov–Maxwell system we choose our gauge bundle to be

$$\mathcal{B} = \mathfrak{A} \to \mathcal{M},$$

where $\mathcal{M} = \{\boldsymbol{B} \mid \text{div } \boldsymbol{B} = 0\}$, with $G = \mathfrak{G}$ the gauge group described in the previous section. As in Section 3, let $\mathfrak{S}$ denote the group of canonical transformations of $T^*M (= \mathbb{R}^6)$. We can let Q be either the symplectic manifold $T^*\mathfrak{S}$ or the Poisson manifold $\mathfrak{g}^*$. It is a little more direct to work with $\mathfrak{g}^*$, so we shall do this.

We wish to specify an action of $\mathfrak{G}$ on $\mathfrak{g}^*$

which, when combined with the action (4.7) on $T^*\mathfrak{A}$, will leave the Hamiltonian (5.2) invariant. The natural choice is to let $\psi \in \mathscr{G}$ act by the (linear) map

$$f_{\mathrm{mom}} \mapsto f_{\mathrm{mom}} \circ \tau_{-\nabla\psi}, \tag{6.1}$$

where $\tau_{-\nabla\psi}: \mathbb{R}^6 \to \mathbb{R}^6$ is the "momentum translation map" defined by

$$\tau_{-\nabla\psi}(\boldsymbol{x}, \boldsymbol{p}) = (\boldsymbol{x}, \boldsymbol{p} - \nabla\psi(\boldsymbol{x})). \tag{6.2}$$

It is easy to verify that $\tau_{-\nabla\psi}$ is a canonical transformation, so it preserves the ordinary Poisson bracket on $\mathbb{R}^6$. It follows that the map (6.1) preserves the Poisson structure on $\mathfrak{F}^*$. A simple calculation gives:

6.1. *Lemma.* The action of $\mathfrak{G}$ on $\mathfrak{F}^*$ defined by (6.1) and (6.2) has a momentum map $J: \mathfrak{F}^* \to \mathfrak{g}^*$ given by

$$\langle J(f_{\mathrm{mom}}), \phi \rangle = -\int f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{p}) \phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{p}, \tag{6.3}$$

i.e.

$$J(f_{\mathrm{mom}}) = -\int f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{p})\, \mathrm{d}\boldsymbol{p}. \tag{6.4}$$

The right-hand side of (6.4) is a density on $\mathbb{R}^3$ which we may denote by $\rho_{f_{\mathrm{mom}}}$.

Now we define the action of $\mathfrak{G}$ on the product $\mathfrak{F}^* \times T^*\mathfrak{A}$ by combining (6.1) and (4.7), i.e. $\psi \in \mathfrak{G}$ maps

$$(f_{\mathrm{mom}}, \mathbf{A}, \boldsymbol{Y}) \mapsto (f_{\mathrm{mom}} \circ \tau_{-\nabla\psi}, \mathbf{A} + \nabla\psi, \boldsymbol{Y}). \tag{6.5}$$

Combining eq. (4.10) and (6.4) gives

6.2. *Lemma.* The momentum map $J: \mathfrak{F}^* \times T^*\mathfrak{A} \to \mathfrak{g}^*$ for the action (6.5) is given by

$$J(f_{\mathrm{mom}}, \mathbf{A}, \boldsymbol{Y}) = -\int f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{p})\, \mathrm{d}\boldsymbol{p} - \operatorname{div} \boldsymbol{Y}. \tag{6.6}$$

We may now describe the reduced Poisson manifold in terms of densities $f(\boldsymbol{x}, \boldsymbol{v})$ defined on position-*velocity* space.

6.3. *Proposition.* The reduced manifold $(\mathfrak{F}^* \times T^*\mathfrak{A})_0 = J^{-1}(0)/\mathfrak{G}$ may be identified with the *Maxwell–Vlasov phase space*

$$\mathcal{MV} = \Big\{ (f, \boldsymbol{B}, \boldsymbol{E}) \,\Big|\, \operatorname{div} \boldsymbol{B} = 0 \text{ and } \operatorname{div} \boldsymbol{E} = \int f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v}. \Big\}$$

Proof. To each $(f_{\mathrm{mom}}, \mathbf{A}, \boldsymbol{Y})$ in $J^{-1}(0)$ we associate the triple $(f, \boldsymbol{B}, \boldsymbol{E})$ in $\mathcal{MV}$, where

$$f(\boldsymbol{x}, \boldsymbol{v}) = f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x})),\ \boldsymbol{B} = \operatorname{curl} \mathbf{A}, \quad \text{and} \quad \boldsymbol{E} = -\boldsymbol{Y}. \tag{6.7}$$

The condition $J(f_{\mathrm{mom}}, \mathbf{A}, \boldsymbol{Y}) = 0$ is equivalent, by (6.6), to the Maxwell equation $\operatorname{div} \boldsymbol{E} = \int f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v}$ in the definition of $\mathcal{MV}$. It is easy to check that elements of $J^{-1}(0)$ are associated to the same $(f, \boldsymbol{B}, \boldsymbol{E})$ if and only if they are related by a gauge transformation (6.5), so our association gives a 1–1 correspondence between $J^{-1}(0)/\mathfrak{G}$ and $\mathcal{MV}$. ■

By the general theory of reduction, $\mathcal{MV}$ inherits a Poisson structure from the one on $\mathfrak{F}^* \times T^*\mathfrak{A}$. Since the Hamiltonian (5.2) is invariant under $\mathfrak{G}$, it follows from Theorem 5.1 that the Maxwell–Vlasov equations (1.1) and (1.2) are a Hamiltonian system on $\mathcal{MV}$ with respect to this structure. In the next section, we shall compute the explicit form of the inherited Poisson structure in the variables $(f, \boldsymbol{B}, \boldsymbol{E})$.

Remark. The action of $\mathfrak{G}$ on $\mathfrak{F}^*$ preserves the co-adjoint orbits, which are symplectic manifolds. It follows that the Poisson manifold $\mathcal{MV}$ can be written as a union of symplectic manifolds, one for each co-adjoint orbit. These orbits are in the space of $f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{p})$'s rather than $f(\boldsymbol{x}, \boldsymbol{v})$'s, so the co-adjoint orbit decomposition is not readily visible in the space $\mathcal{MV}$.

It is possible to describe $J^{-1}(0)/\mathfrak{G}$ in terms of position-momentum densities, at the expense of having to choose a gauge. For example, the gauge condition $\operatorname{div} A = 0$ determines a slice for the action of $\mathfrak{G}$ on $\mathfrak{F}^* \times T^*\mathfrak{A}$ which then produces an identification of $J^{-1}(0)/\mathfrak{G}$ with

$$\mathcal{MV}_{\mathrm{mom}} = \{(f_{\mathrm{mom}}, \boldsymbol{B}, \boldsymbol{E}) \mid \operatorname{div} \boldsymbol{E} = \rho f_{\mathrm{mom}}\}. \tag{6.8}$$

Specifically, if we let $\psi_{\mathbf{A}} = -\Delta^{-1}(\operatorname{div} \mathbf{A})$, then the map $(f_{\mathrm{mom}}, \mathbf{A}, \mathbf{Y}) \mapsto (f_{\mathrm{mom}} \circ \tau_{-\nabla\psi_{\mathbf{A}}}, \mathbf{A} + \nabla\psi_{\mathbf{A}}, \mathbf{Y})$ projects onto the space satisfying the gauge condition, and so the map

$$(f_{\mathrm{mom}}, \mathbf{A}, \mathbf{Y}) \mapsto (f_{\mathrm{mom}} \circ \tau_{-\Delta\psi_{\mathbf{A}}}, \operatorname{curl} \mathbf{A}, -\mathbf{Y})$$

gives a 1–1 correspondence between $J^{-1}(0)/\mathfrak{G}$ and $\mathcal{MV}_{\mathrm{mom}}$.

7. Computation of the Poisson structure on $\mathcal{MV}$

7.1. *Theorem.* For two functionals F, G of the fields $(f, \boldsymbol{E}, \boldsymbol{B})$ we have

$$\begin{aligned}\{\{F, G\}\}(f, \boldsymbol{E}, \boldsymbol{B}) = &\int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v} + \int \left(\frac{\delta F}{\delta \boldsymbol{E}} \operatorname{curl} \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}} \operatorname{curl} \frac{\delta F}{\delta \boldsymbol{B}}\right) \mathrm{d}\boldsymbol{x} \\ &+ \int \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v} + \int f B \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}.\end{aligned} \tag{7.1}$$

Proof. Given $F(f, \boldsymbol{E}, \boldsymbol{B})$ on $\mathcal{MV}$, define $\bar{F}(f_{\mathrm{mom}}, \mathbf{A}, \mathbf{Y})$ by

$$\bar{F}(f_{\mathrm{mom}}, \mathbf{A}, \mathbf{Y}) = F(f, \boldsymbol{B}, \boldsymbol{E}), \tag{7.2}$$

where $(f_{\mathrm{mom}}, \mathbf{A}, \mathbf{Y})$ is determined by (6.7).

As in the proof of Proposition 4.1 $\{\{F, G\}\}$ is given by computing $\{\{\bar{F}, \bar{G}\}\}$ using (5.1) and expressing the answer in terms of $F, G, f, \boldsymbol{E}$, and $\boldsymbol{B}$. The first term of (5.1) is dealt with by a straightforward computation leading to the "minimal coupling formula":

7.2. *Lemma.* Let $\boldsymbol{B} = \operatorname{curl} \mathbf{A}$, $h(\boldsymbol{x}, \boldsymbol{v}) = h_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))$, and $k(\boldsymbol{x}, \boldsymbol{v}) = k_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))$. Then

$$\{h_{\mathrm{mom}}, k_{\mathrm{mom}}\} = \{h, k\} + \boldsymbol{B} \cdot \left(\frac{\partial h}{\partial \boldsymbol{v}} \times \frac{\partial k}{\partial \boldsymbol{v}}\right). \tag{7.3}$$

The first term of (5.1) now becomes

$$\begin{aligned}&\int f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v} \\ &\quad + \int f_{\mathrm{mom}}(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))\boldsymbol{B} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \boldsymbol{v}} \frac{\delta G}{\delta \boldsymbol{v}}\right)(\boldsymbol{x}, \boldsymbol{v} + \mathbf{A}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v},\end{aligned} \tag{7.4}$$

since $\delta\bar{F}/\delta f_{\mathrm{mom}} = \delta F/\delta f$.

Thus, changing variables, the first term of (5.1) becomes the first and last terms of (7.1). As for the second term of (5.1), we use

7.3. *Lemma.* We have

$$\frac{\delta \bar{F}}{\delta \mathbf{A}} = \operatorname{curl} \frac{\delta F}{\delta \boldsymbol{B}} + \frac{\delta F}{\delta f} \frac{\partial f}{\partial \boldsymbol{v}}, \qquad \frac{\delta \bar{F}}{\delta \mathbf{Y}} = -\frac{\delta F}{\delta \boldsymbol{E}}. \tag{7.5}$$

Proof. By the chain rule and definition of the functional derivative,

$$D_A\bar{F}(f_{\text{mom}}, \mathbf{A}, \mathbf{Y})\cdot \mathbf{A}' = \int \frac{\delta \bar{F}}{\delta \mathbf{A}} \mathbf{A}'\,\mathrm{d}\boldsymbol{x} =$$

$$D_fF(f, \mathbf{B}, \mathbf{E})\cdot D_Af\cdot \mathbf{A}' + D_{\mathbf{B}}F(f, \mathbf{B}, \mathbf{E})\cdot \operatorname{curl}\mathbf{A}' = \int \frac{\delta F}{\delta f}\frac{\partial f}{\partial \boldsymbol{v}}\cdot \mathbf{A}'\,\mathrm{d}\boldsymbol{x} + \int \frac{\delta F}{\delta \boldsymbol{B}}\cdot \operatorname{curl}\mathbf{A}'\,\mathrm{d}\boldsymbol{x}$$
$$= \int \frac{\delta F}{\delta f}\frac{\partial f}{\partial \boldsymbol{v}}\cdot \mathbf{A}'\,\mathrm{d}\boldsymbol{x} + \int \operatorname{curl}\frac{\delta F}{\delta \boldsymbol{B}}\cdot \mathbf{A}'\,\mathrm{d}\boldsymbol{x}. \quad ■$$

Substitution of (7.5) into the second term of (5.1) yields the second and third terms of (7.1). Thus, Theorem 7.1 is proved. ■

7.4. *Theorem.* The equations of motion 1.1 and 1.2 may be written

$$\dot{F} = \{\{F, H\}\}$$

where $\{\{\ ,\ \}\}$ is given by (7.1) and H by (1.6).

This follows directly from Theorem 5.1 by reduction, and can also be checked directly by a straightforward calculation.

One can also compute the Poisson structure on $\mathcal{MV}_{\text{mom}}$ (see (6.8)). We just state the result:

$$\{F, G\} = \int f_{\text{mom}}\left\{\frac{\delta F}{\delta f_{\text{mom}}}, \frac{\delta G}{\delta f_{\text{mom}}}\right\}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} + \int\left(\frac{\delta F}{\delta \boldsymbol{E}}\operatorname{curl}\frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}}\operatorname{curl}\frac{\delta F}{\delta \boldsymbol{B}}\right)\mathrm{d}\boldsymbol{x}$$
$$+ \left[\int f_{\text{mom}}\left(\frac{\partial}{\partial \boldsymbol{p}}\frac{\delta F}{\delta f}\right)\left(\frac{\partial}{\partial \boldsymbol{x}}\Delta^{-1}\operatorname{div}\frac{\delta G}{\delta \boldsymbol{E}}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} - \int f_{\text{mom}}\left(\frac{\partial}{\partial \boldsymbol{p}}\frac{\delta G}{\delta f}\right)\frac{\partial}{\partial \boldsymbol{x}}\left(\Delta^{-1}\operatorname{div}\frac{\delta F}{\delta \boldsymbol{E}}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p}\right]. \quad (7.6)$$

Recall that mv_{mom} is defined relative to the gauge condition $\operatorname{div}\mathbf{A} = 0$. The Hamiltonian becomes

$$H(f_{\text{mom}}, \boldsymbol{B}, \boldsymbol{E}) = \int \frac{|\boldsymbol{v}|^2}{2} f_{\text{mom}}(\boldsymbol{x}, \boldsymbol{p})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} + \frac{1}{2}\int (\boldsymbol{E}^2 + \boldsymbol{B}^2)\,\mathrm{d}\boldsymbol{x}, \quad (7.7)$$

where $\boldsymbol{v} = \boldsymbol{p} + \Delta^{-1}\operatorname{curl}\boldsymbol{B}$, and again the equations of motion read $\dot{F} = \{\{F, H\}\}$.

8. Additional remarks

(A) Poisson structures may be viewed as bundle maps taking covectors to vectors. (This is the form most convenient for determining equations of motion.†) Viewed this way, Morrison's bracket is the map

$$(f^*, \boldsymbol{B}^*, \boldsymbol{E}^*) \mapsto (\delta f, \delta \boldsymbol{B}, \delta \boldsymbol{E}),$$

given by

$$\delta f = -\{f, f^*\} - \frac{\partial f}{\partial v}\cdot \boldsymbol{E}^* - \frac{\partial f}{\partial v}\cdot(\boldsymbol{v}\times \boldsymbol{B}^*),$$

$$\delta \boldsymbol{B} = \int \left(\frac{\partial f}{\partial \boldsymbol{v}}\times \boldsymbol{v}\right) f^*\,\mathrm{d}\boldsymbol{v} - \operatorname{curl}\boldsymbol{E}^*, \quad (8.1)$$

$$\delta \boldsymbol{E} = \int \frac{\partial f}{\partial \boldsymbol{v}} f^*\,\mathrm{d}\boldsymbol{v} + \operatorname{curl}\boldsymbol{B}^*,$$

while our cosymplectic structure (7.1) is given by

$$\delta f = -\{f, f^*\} - \frac{\partial f}{\partial v}\cdot \boldsymbol{E}^* + \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(f\boldsymbol{B}\times\frac{\partial f^*}{\partial \boldsymbol{v}}\right),$$

$$\delta \boldsymbol{B} = -\operatorname{curl}\boldsymbol{E}^*, \quad (8.2)$$

$$\delta \boldsymbol{E} = \int \frac{\partial f}{\partial \boldsymbol{v}} f^*\,\mathrm{d}\boldsymbol{v} + \operatorname{curl}\boldsymbol{B}^*.$$

(B) A "cold plasma" may be defined as one

†Sign conventions are such that on symplectic manifolds, $\mathrm{d}H \mapsto X_H$, the Hamiltonian vector field of H.

for which f is a δ measure supported on the graph of a vector field $p = \theta(x)$. This property persists as f evolves by composition with a canonical transformation. In fact, the property that θ is curl-free is also maintained, since this corresponds to the graph's being a Lagrangian submanifold. After a long time, the submanifold may no longer be a graph. This is the "shock" phenomenon, leading to *multiple streaming* (Davidson [11].) We remark that Maslov ([24], p. 44) has already noticed this evolution of Lagrangian submanifolds for the Poisson–Vlasov equation.

(C) We would like to understand in general terms the contraction of one Hamiltonian system to another. Examples are the passage to the restricted three body problem from the full three body problem, the limit $c \to \infty$ to get the Poisson–Vlasov equation, and the limit of infinite conductivity (among other things) to get the equations of magnetohydrodynamics from the Euler–Maxwell equations. It would also be of interest to realize both the Maxwell–Vlasov and the Euler–Maxwell equations as limiting cases of a grand Hamiltonian system related to the Boltzmann equation.

(D) We have remarked that our formalism generalizes to give a Poisson structure for the Yang–Mills–Vlasov equations. Is such a structure useful in nuclear physics for Yang–Mills plasmas?

(E) Holm [16] and Morrison [26] have shown, formally, how to introduce "Clebsch variables" (called "Schutz potentials" relativistically) which bring the bracket into canonical form. It would be interesting to prove rigorously that these provide Darboux coordinates for the symplectic leaves of the reduced spaces.

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Note added in proof. By using methods analogous to those of this paper, the Poisson structure for two-fluid plasma dynamics has been derived in [42].

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