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The Structure of the Space of Solutions of Einstein's Equations II: Several Killing Fields and the Einstein–Yang–Mills Equations

JUDITH M. ARMS

Department of Mathematics, University of Washington, Seattle, Washington 98195

JERROLD E. MARSDEN

Department of Mathematics, University of California, Berkeley, California 94720

AND

VINCENT MONCRIEF

Department of Physics, Yale University, P.O. Box 6666, New Haven, Connecticut 06511

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The space of solutions of Einstein's vacuum equations is shown to have conical singularities at each spacetime possessing a compact Cauchy surface of constant mean curvature and a nontrivial set of Killing fields. Similar results are shown for the coupled Einstein–Yang–Mills system. Combined with an appropriate slice theorem, the results show that the space of geometrically equivalent solutions is a stratified manifold with each stratum being a symplectic manifold characterized by the symmetry type of its members.

Contents.

INTRODUCTION

This paper is the second part of Fischer *et al.* (1980) and is a companion of Arms *et al.* (1981). In Fischer *et al.* (1980) we studied the space $\mathscr{E}$ of solutions to Einstein's vacuum equations near a spacetime $(V, {}^{(4)}g_0)$ which has a compact Cauchy surface of constant mean curvature and with $k = k({}^{(4)}g_0) =$ (dim of the space of Killing fields of ${}^{(4)}g_0) = 1$. Here we study the case $k \geqslant 1$ and extend the results to the coupled Einstein–Yang–Mills equations by utilizing the present methods together with known results for the pure Yang–Mills case from Moncrief (1977) and Arms (1981). In Arms *et al.* (1981) it was shown how to deal with the case $k \geqslant 1$ in case all the Killing fields are spacelike. The present paper builds on these techniques.

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If $k=0$, i.e., ${}^{(4)}g_0$ has only trivial Killing fields, then it has been known for some time that $\mathscr{E}$ is a smooth manifold near ${}^{(4)}g_0$. This was established in the period 1973–1975 through the work of Fischer, Marsden and Moncrief—see Fischer *et al.* (1980) for references. The analogous results for the pure Yang–Mills equations are due to Moncrief (1977) and, for the Einstein–Yang–Mills equations, to Arms (1979). Combined with an appropriate slice theorem (see Fischer *et al.* (1980), Isenberg and Marsden (1982) and references therein) and the reduction methods of Marsden and Weinstein (1974), one sees that $\mathscr{E}$/(gauge transformations) is a smooth symplectic manifold near such ${}^{(4)}g_0$. For $k \geqslant 1$, which is usually regarded as the most interesting case, $\mathscr{E}$ is no longer smooth and so $\mathscr{E}$/(gauge transformations) is considerably more complicated. In fact, it becomes a stratified symplectic manifold, as is explained in Isenberg and Marsden (1982).

Briefly, Fischer *et al.* (1980) proved that for $k=1$, $\mathscr{E}$ has a conical singularity at ${}^{(4)}g_0$; i.e., near ${}^{(4)}g_0$, $\mathscr{E}$ can be written in a suitable chart as the zero set of a homogeneous quadratic function. The generators of this cone consist of those symmetric two tensors ${}^{(4)}h$ such that

(i) ${}^{(4)}h$ satisfies the linearized Einstein equations

$$D\,\mathrm{Ein}({}^{(4)}g_0)\cdot {}^{(4)}h = 0,$$

where $\mathrm{Ein}({}^{(4)}g)$ is the Einstein tensor formed from a metric ${}^{(4)}g$ and D denotes the Fréchet derivative and

(ii) the Taub conserved quantities vanish:

$$\int_\Sigma {}^{(4)}X\cdot [D^2\,\mathrm{Ein}({}^{(4)}g_0)\cdot({}^{(4)}h, {}^{(4)}h)]\cdot {}^{(4)}Z_\Sigma\, d^3\Sigma = 0,$$

where ${}^{(4)}X$ is a Killing field for ${}^{(4)}g_0$, Σ is a compact Cauchy surface and ${}^{(4)}Z_\Sigma$ is its forward pointing unit normal.

It follows that *a necessary and sufficient condition for a solution ${}^{(4)}h$ of the linearized equations to be tangent to a curve of solutions to Einstein's equations passing through ${}^{(4)}g_0$ is that* (ii) *holds.*

The principal result of this paper is

THEOREM. *The preceding statement for $k({}^{(4)}g_0)=1$ also holds for $k({}^{(4)}g_0)\geqslant 1$ and similar results hold for the Einstein–Yang–Mills system.*

The precise function spaces employed in this theorem were detailed in Fischer *et al.* (1980) and will be assumed implicitly here.

The case $k({}^{(4)}g_0)\geqslant 1$ includes more interesting examples than the case $k({}^{(4)}g_0)=1$. In particular, it includes the flat spacetime $T^3\times\mathbb{R}$ (with $k=4$) which was the example which began the subject in the seminal paper of Brill and Deser (1973).

The proof of the above theorem occupies Sections 1–3 of the paper. A crucial tool required for $k > 1$ that was not needed for $k = 1$ is the "Kuranishi map," a mapping that was used originally by Kuranishi (1965) in studies of deformations of complex structures and applied by Atiyah *et al.* (1978) to the study of Euclidean Yang–Mills fields. As was shown by Arms *et al.* (1981), this technique may be used to prove the theorem in the case all the Killing fields are spacelike. The general plan of our proof in the general case is to combine the Kuranishi approach, which is able to deal with all the momentum constraints and a certain projection of the Hamiltonian constraint, with a special Morse lemma-type argument for the Hamiltonian constraint.

Section 1 reviews the definition and some basic properties of the Kuranishi map. In particular, we see that this map (restricted to a slice) is symplectic and that its inverse was implicitly defined in Fischer *et al.* (1980).

Section 2 shows that the Kuranishi map enables one to solve all of the momentum constraints and Section 3 then shows how to subsequently solve the Hamiltonian constraint by a special argument. This last step relies on a suitable infinite dimensional Morse lemma due to Tromba (1976) and Golubitsky and Marsden (1982).

Section 4 extends the results to the Einstein–Yang–Mills system, the Einstein–Maxwell system being a special case. There is a peculiar difficulty posed by this system. Namely, with the set up most natural from the bundle viewpoint, the supermomentum constraint is a cubic function of the fields, a situation which causes havoc with the Kuranishi method. This difficulty is overcome by a special parametrization of the shift and gauge shift (X, V), which, in effect, transforms the constraint to an equivalent quadratic one. When this is done, the methods proceed in a way similar to the vacuum case.

The final section discusses the singularities and how symmetry is broken by means of a collection of remarks and examples. The relationship to the work of Jantzen (1979) is briefly mentioned. Finally, a mechanism for symplectically desingularizing the solution space is discussed.

From our work in Fischer *et al.* (1980), it is enough to study the constraint equations. We now recall some of the notation that will be used in this connection. The reader should consult Fischer *et al.* (1980) for additional explanation and details.

Let M be a compact 3-manifold and $\mathscr{M}$ the space of ($W^{s,p}$, $s > 3/p + 1$) Riemannian metrics on M and let $P = T^*\mathscr{M}$ denote the "natural" cotangent bundle of $\mathscr{M}$; i.e., the fiber of $T^*\mathscr{M}$ over $g \in \mathscr{M}$ consists of all symmetric 2-contravariant tensor densities π (of class $W^{s-1,p}$). The constraint set of the vacuum Einstein equations on a 4-dimensional spacetime in which M is embedded as a compact hypersurface, is the set

$$\mathscr{C} = \Phi^{-1}(0),$$

where $\Phi: T^*\mathscr{M} \to$ (densities on M) $\times$ (one-form densities on M) is defined by $\Phi(g, \pi) = (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi))$ and $\mathscr{H}$ and $\mathscr{J}$ are given by

$$\mathscr{H}(g, \pi) = \{(\pi' \cdot \pi' - \tfrac{1}{2}(\operatorname{trace} \pi')^2) - R(g)\}\, \mu(g)$$

and

$$\mathscr{T}(g, \pi)_i = -2\pi_i{}^j{}_{|j}.$$

Here $\pi = \pi' \otimes \mu(g)$, $\mu(g)$ is the volume form of g, and $R(g)$ is the scalar curvature of g.

The vacuum Einstein equations for a metric ${}^{(4)}g$ are equivalent to the constraint equations $\Phi(g, \pi) = 0$ for the induced Cauchy data (g, π) on a hypersurface Σ and the evolution equations

$$\frac{\partial}{\partial t}\begin{pmatrix} g \\ \pi \end{pmatrix} = -\mathbb{J} \circ D\Phi(g, \pi)^* \begin{pmatrix} N \\ X \end{pmatrix}$$

relative to a given spacetime slicing. Here $\mathbb{J}$ is the (almost) complex structure on $T^*\mathscr{M}$ given by

$$\mathbb{J} = \begin{pmatrix} 0 & -\dfrac{I^\flat}{\mu(g)} \\ I^\# \mu(g) & 0 \end{pmatrix}$$

where $I^\flat$ and $I^\#$ are the index lowering and raising operators relative to g. The adjoint $D\Phi(g, \pi)^*$ is taken relative to the L^2 metric on $T^*\mathscr{M}$ given by

$$\langle\langle (h_1, \omega_1), (h_2, \omega_2) \rangle\rangle = \int \{h_1 \cdot h_2 + \omega_1 \cdot \omega_2\}\, \mu(g).$$

In this formula, $\cdot$ denotes contraction using the base point g and the natural pairing between (densities) $\times$ (one-form densities) and (functions) $\times$ (vector fields). Thus $D\Phi(g, \pi)^*$: (functions $\times$ vector fields) $\to T_{(g,\pi)}(T^*\mathscr{M})$, and one can compute it explicitly. (See Fischer *et al.* (1980) for the formula.)

The (weak) symplectic structure on $T^*\mathscr{M}$ is given by

$$\Omega((h_1, \omega_1), (h_2, \omega_2)) = \int \omega_2 \cdot h_1 - \omega_1 \cdot h_2 = \langle\langle \mathbb{J}(h_1, \omega_1)(h_2, \omega_2) \rangle\rangle$$

which is independent of (g, π). Note that $\mathbb{J}^2 = -I$, $\mathbb{J}$ is symplectic, and is orthogonal and skew adjoint with respect to $\langle\langle , \rangle\rangle$.

The Killing fields of ${}^{(4)}g$ are in one-to-one correspondence with elements of $\ker D\Phi(g, \pi)^*$ by means of perpendicular and parallel projection: $X \to (X_\perp, X_\|)$.

The following fundamental decomposition of Moncrief (1975) will be used (Fischer *et al.*, 1980, Theorem 2.5):

$$T_{(g,\pi)}(T^*\mathscr{M}) = \text{range}(-\mathbb{J} \circ D\Phi(g, \pi)^*) \oplus \text{range}(D\Phi(g, \pi)^*) \oplus [\ker(D\Phi(g, \pi) \circ \mathbb{J}) \cap \ker D\Phi(g, \pi)]. \tag{M}$$

Recall that range $(-\mathbb{J} \circ D\Phi(g,\pi)^*)$ represents the infinitesimal gauge transformations, range $(D\Phi(g,\pi)^*)$ is the orthogonal complement to the linearized constraints $\ker D\Phi(g,\pi)$, and $\ker D\Phi(g,\pi) \cap \ker(D\Phi(g,\pi) \circ \mathbb{J})$ is the space of linearized "true" degrees of freedom, a generalization of the usual "*TT*" component. The latter is a symplectic subspace of $T_{(g,\pi)}(T^*\mathscr{M})$; this is a basic and easily verified fact about Moncrief's decomposition; cf. Arms *et al.* (1981, Lemma 13).

The orthogonal complement of the gauges plays the role of the slice for the action of the diffeomorphism group of spacetime. Thus, we set $S_{(g,\pi)} = \{(g,\pi)\} + a$ neighborhood of zero in $\ker(D\Phi(g,\pi) \circ \mathbb{J})$. This lies in $T^*\mathscr{M}$ since $\mathscr{M}$ is open in $S_2(M)$, the covariant symmetric two tensors on M, and so $T^*\mathscr{M}$ is open in the linear space $S_2(M) \times S_d^2(M)$, where $S_d^2(M)$ is the space of contravariant symmetric two tensor densities. Note that $S_{(g,\pi)}$ corresponds to $S_{(g,\pi)} \cap \mathscr{C}_{\mathrm{tr}}$ in Fischer *et al.* (1980).

As was shown in Fischer *et al.* (1980) (see also Marsden and Tipler, 1980), if a nonstationary, vacuum spacetime has a compact Cauchy surface of constant mean curvature, then any spacelike Killing field is tangent to it. If the metric has a timelike Killing field, then the spacetime is flat and the Cauchy data are of the form $(g_0, 0)$, where g_0 is flat. In Arms *et al.* (1981) the case in which ${}^{(4)}g_0$ has only spacelike Killing fields was treated; this case will be spelled out in detail here, as well as the case in which one of the Killing fields is timelike.

1. The Kuranishi Map and Its Properties

We now construct a local diffeomorphism F of $T^*\mathscr{M}$ to itself which we will refer to as the "Kuranishi map." First, let $(g_0, \pi_0) \in \Phi^{-1}(0)$ be fixed and set

$$\Delta = D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^*.$$

Since $D\Phi(g_0, \pi_0)^*$ is an elliptic operator, Δ is an isomorphism of range$(D\Phi(g_0, \pi_0))$ to itself. Second, let $\mathbb{P}$ denote the orthogonal projection to range $(D\Phi(g_0, \pi_0))$ and set $G = \Delta^{-1} \circ \mathbb{P}$. Write $(h, \omega) = (g, \pi) - (g_0, \pi_0)$ and let the remainder be given by

$$\mathscr{R}(h, \omega) = \Phi(g, \pi) - D\Phi(g_0, \pi_0) \cdot (h, \omega).$$

Next, define F by

$$F(g, \pi) = (g, \pi) + D\Phi(g_0, \pi_0)^* \circ G \circ \mathscr{R}(h, \omega).$$

The basic properties of F are listed in a series of propositions. A number of these are similar to those in Arms *et al.* (1981) but are given here for completeness.

1.1 Proposition. *F is a diffeomorphism of a neighborhood of (g_0, π_0) onto a neighborhood of (g_0, π_0).*

Proof. Since Φ is smooth, so is F and $DF(g_0, \pi_0)$ is the identity. The result thus follows by the inverse function theorem. ∎

1.2 PROPOSITION. *F maps $S_{(g_0,\pi_0)}$ to itself.*

Proof. Let $(g,\pi)\in S_{(g_0,\pi_0)}$, so $(h,\omega)\in \ker(D\Phi(g_0,\pi_0)\circ \mathbb{J})$. Then from the identity $D\Phi(g_0,\pi_0)\circ \mathbb{J}\circ D\Phi(g_0,\pi_0)^*=0$ we get

$$\begin{aligned} &D\Phi(g_0,\pi_0)\circ \mathbb{J}\cdot(F(g,\pi)-(g_0,\pi_0)) \\ &\quad = D\Phi(g_0,\pi_0)\circ \mathbb{J}[(h,\omega)+D\Phi(g_0,\pi_0)^*\circ G\circ \mathscr{R}(h,\omega)]=0. \quad \blacksquare \end{aligned}$$

The *projected constraint set* is defined by

$$\mathscr{C}_{\mathbb{P}}=\{(g,\pi)\mid \mathbb{P}\Phi(g,\pi)=0\}$$

which is a smooth manifold in a neighborhood of (g_0,π_0) with tangent space at (g_0,π_0) given by $\ker D\Phi(g_0,\pi_0)$; see Fischer *et al.* (1980, Proposition 3.2).

1.3 PROPOSITION. *F maps a neighborhood of (g_0,π_0) in $\mathscr{C}_{\mathbb{P}}$ onto a neighborhood of (g_0,π_0) in $\{(g_0,\pi_0)\}+\ker D\Phi(g_0,\pi_0)$. Thus, F is a local chart for $\mathscr{C}_{\mathbb{P}}$.*

Proof. Since $DF(g_0,\pi_0)=I$, it suffices to show that $(g,\pi)\in\mathscr{C}_{\mathbb{P}}$ implies $F(g,\pi)\in\{(g_0,\pi_0)\}+\ker D\Phi(g_0,\pi_0)$. Assume then, that $(g,\pi)\in\mathscr{C}_{\mathbb{P}}$; i.e., $\mathbb{P}\Phi(g,\pi)=0$. Then

$$\begin{aligned} &D\Phi(g_0,\pi_0)\cdot[F(g,\pi)-(g_0,\pi_0)] \\ &\quad = D\Phi(g_0,\pi_0)\cdot[(h,\omega)+D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}\circ\mathscr{R}((h,\omega))] \\ &\quad = D\Phi(g_0,\pi_0)\cdot(h,\omega)+\mathbb{P}\circ\mathscr{R}((h,\omega)) \\ &\quad = \mathbb{P}[\Phi(g,\pi)]=0. \quad \blacksquare \end{aligned}$$

The manifolds $\mathscr{C}_{\mathbb{P}}$ and $S_{(g_0,\pi_0)}$ intersect transversally at (g_0,π_0) since their tangent spaces are

$$\ker D\Phi(g_0,\pi_0) \qquad \text{and} \qquad \ker D\Phi(g_0,\pi_0)\circ\mathbb{J}\supset \operatorname{range} D\Phi(g_0,\pi_0)^*$$

and $\ker D\Phi(g_0,\pi_0)\oplus \operatorname{range} D\Phi(g_0,\pi_0)^*=T_{(g_0,\pi_0)}(T^*\mathscr{M})$. Thus $\mathscr{C}_{\mathbb{P}}\cap S_{(g_0,\pi_0)}$ is a smooth manifold near (g_0,π_0) whose tangent space at (g_0,π_0) is the "TT" component. Thus, by our remarks in the introduction concerning Moncrief's decomposition, $\mathscr{C}_{\mathbb{P}}\cap S_{(g_0,\pi_0)}$ is a symplectic submanifold. We now prove that F is a *symplectic* chart for it.

1.4 PROPOSITION. *F is a local symplectic diffeomorphism of $\mathscr{C}_{\mathbb{P}}\cap S_{(g_0,\pi_0)}$ to $\{(g_0,\pi_0)\}+\ker D\Phi(g_0,\pi_0)\cap\ker D\Phi(g_0,\pi_0)\circ\mathbb{J}$.*

Proof. From 1.2 and 1.3 it follows that F is a local diffeomorphism between the stated spaces. To see that it is symplectic, note first that

$$DF(g,\pi)\cdot(h',\omega')=(h',\omega')+ D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}(D\Phi(g,\pi)\cdot(h',\omega')-D\Phi(g_0,\pi_0)\cdot(h',\omega')),$$

from the definition of F. Letting Ω be the symplectic form $\Omega((h_1,\omega_1),(h_2,\omega_2))=\langle\langle\mathbb{J}(h_1,\omega_1),(h_2,\omega_2)\rangle\rangle$, we have

$$\begin{aligned}&\Omega(DF(g,\pi)\cdot(h',\omega'),DF(g,\pi)\cdot(h'',\omega''))\\ &=\langle\langle\mathbb{J}(h',\omega'),(h'',\omega'')\rangle\rangle\\ &+\langle\langle\mathbb{J}(h',\omega'),D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}(D\Phi(g,\pi)\cdot(h'',\omega'')-D\Phi(g_0,\pi_0)\cdot(h'',\omega''))\rangle\rangle\\ &+\langle\langle\mathbb{J}\circ D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}(D\Phi(g,\pi)\cdot(h',\omega')-D\Phi(g_0,\pi_0)\cdot(h',\omega')),(h'',\omega'')\rangle\rangle\\ &+\langle\langle\mathbb{J}\circ D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}(D\Phi(g,\pi)(h',\omega')-D\Phi(g_0,\pi_0)\cdot(h',\omega')),\\ &\quad D\Phi(g_0,\pi_0)^*\circ\Delta^{-1}\circ\mathbb{P}(D\Phi(g,\pi)\cdot(h'',\omega'')-D\Phi(g_0,\pi_0)\cdot(h'',\omega''))\rangle\rangle.\end{aligned}$$

The last term vanishes by virtue of the identity $D\Phi(g_0,\pi_0)\circ\mathbb{J}\circ D\Phi(g_0,\pi_0)^*=0$ and the second and third terms vanish because $(h',\omega')\in\ker D\Phi(g_0,\pi_0)\circ\mathbb{J}$ and $(h'',\omega'')\in\ker D\Phi(g_0,\pi_0)\circ\mathbb{J}$, by construction of $S_{(g_0,\pi_0)}$. Thus

$$\Omega(DF(g,\pi)\cdot(h',\omega'),DF(g,\pi)\cdot(h'',\omega''))=\Omega((h',\omega'),(h'',\omega''))$$

so F is symplectic. ∎

This same proof works in the context of bifurcations of momentum mappings in Arms *et al.* (1981).

Next we study the relationship between the Kuranishi map and solutions of the projected constraints obtained via the inverse function theorem. Since $\mathscr{C}_{\mathrm{P}}$ is tangent to $\ker D\Phi(g_0,\pi_0)$ at (g_0,π_0), there is a unique smooth map

$$\Psi:\ker D\Phi(g_0,\pi_0)\to\operatorname{range} D\Phi(g_0,\pi_0)^*$$

defined on a neighborhood of zero such that $\Psi(0,0)=(0,0)$, $D\Psi(0,0)=0$ and such that $\mathscr{C}_{\mathrm{P}}$ is the graph of Ψ: i.e., locally,

$$\mathscr{C}_{\mathrm{P}}=\{(g,\pi)=(g_0,\pi_0)+(h,\omega)+\Psi(h,\omega)\mid(h,\omega)\in\ker D\Phi(g_0,\pi_0)\}.$$

If we write $\Psi(h,\omega)=D\Phi(g_0,\pi_0)^*\,(C(h,\omega),Y(h,\omega))$ then C and Y are determined by the nonlinear elliptic system

$$\mathbb{P}\Phi((g_0,\pi_0)+(h,\omega)+D\Phi(g_0,\pi_0)^*\,(C,Y))=0.$$

The derivative of the left hand side with respect to (C, Y) at $(h, \omega)=0$ and $(C, Y)=0$ and in the direction (C', Y') is

$$\Delta(C', Y') = D\Phi(g_0, \pi_0) \circ D\Phi(g_0, \pi_0)^* (C', Y')$$

and we know that Δ is an isomorphism of range $D\Phi(g_0, \pi_0)$ to itself. Thus, if we demand that $(C, Y) \in$ range $D\Phi(g_0, \pi_0)$, we can uniquely solve the above system for (C, Y) as functions of (h, ω) and thereby determine Ψ.

1.5 Proposition. *The map of* $\{(g_0, \pi_0)\} + \ker D\Phi(g_0, \pi_0)$ *to* $\mathscr{C}_{\mathbb{P}}$ *given by* $(g_0, \pi_0) + (h, \omega) \mapsto (g_0, \pi_0) + (h, \omega) + \Psi(h, \omega)$ *is the inverse of the Kuranishi map restricted to* $\mathscr{C}_{\mathbb{P}}$.

Proof. Let $(g, \pi) = (g_0, \pi_0) + (h, \omega) + \Psi(h, \omega)$ for $(h, \omega) \in \ker D\Phi(g_0, \pi_0)$, so (g, π) satisfies $\mathbb{P}\Phi(g, \pi) = 0$. By definition,

$$\begin{aligned} F(g, \pi) &= (g, \pi) + D\Phi(g_0, \pi_0)^* \circ \Delta^{-1} \\ &\quad \circ \mathbb{P}[\Phi(g, \pi) - D\Phi(g_0, \pi_0) \cdot ((h, \omega) + \Psi(h, \omega))] \\ &= (g, \pi) - D\Phi(g_0, \pi_0)^* \circ \Delta^{-1} \circ D\Phi(g_0, \pi_0) \cdot [(h, \omega) + \Psi(h, \omega)]. \end{aligned}$$

Now $D\Phi(g_0, \pi_0)^* \circ \Delta^{-1} \circ D\Phi(g_0, \pi_0)(h_1, \omega_1) = \mathbb{P}^*(h_1, \omega_1)$, where $\mathbb{P}^*$ is the projection to range $D\Phi(g_0, \pi_0)^*$. Thus

$$\begin{aligned} F(g, \pi) &= (g, \pi) - \mathbb{P}^*[(h, \omega) + \Psi(h, \omega)] \\ &= (g_0, \pi_0) + (h, \omega) + \Psi(h, \omega) - \mathbb{P}^*(h, \omega) - \Psi(h, \omega) \end{aligned}$$

since $\Psi(h, \omega)$ lies in range $D\Phi(g_0, \pi_0)^*$ by construction. It follows that

$$\begin{aligned} F(g, \pi) &= (g_0, \pi_0) + (I - \mathbb{P}^*)(h, \omega) \\ &= (g_0, \pi_0) + (h, \omega) \end{aligned}$$

since $\hat{\mathbb{P}} = I - \mathbb{P}^*$ is the projection to $\ker D\Phi(g_0, \pi_0)$ and (h, ω) lies in this space. ∎

Remark. In the original Kuranishi work, this inverse map was constructed by a power series. See Morrow and Kodaira (1971), p. 165–167.

For g_0 flat and $\pi_0 = 0$, we remark that the relationship between Ψ and (C, Y) is particularly simple:

$$\begin{aligned} Y &= D\Phi(g_0, 0)^* (C, Y) \\ &= (-\text{Hess } C - g_0 \Delta C, (L_Y g_0)^{\#} \mu(g_0)) \end{aligned}$$

(see Fischer *et al.*, 1980, p. 157 for the explicit formula for $D\Phi(g, \pi)^*$). This will be important in Section 3.

The above study of $\mathscr{C}_{\mathbb{P}}$ solves the part of the constraint equations that can be dealt with by the inverse function theorem, namely, $\mathbb{P}\Phi(g, \pi) = 0$. We now split the remaining equations $(I-\mathbb{P})\,\Phi(g,\pi)=0$ into timelike and spacelike ones. We recall from Fischer *et al.* (1980) that if M is embedded as a hypersurface of constant mean curvature in the spacetime generated by (g_0, π_0), then in a suitably chosen basis for the space of Killing fields any spacelike Killing field is tangent to M, so induces an element of $\ker D\Phi(g_0,\pi_0)^*$ of the form $(0, X)$. Let $(0, X_1), (0, X_2), \ldots, (0, X_l)$ be an L^2 orthonormal basis of elements of $\ker D\Phi(g_0,\pi_0)^*$. We have from Fischer *et al.* (1980):

$$L_{X_i} g_0 = 0 \qquad \text{and} \qquad L_{X_i}\pi_0 = 0, \qquad i = 1, \ldots, l.$$

If there are no timelike Killing fields, these span all of $\ker D\Phi(g_0,\pi_0)^*$. If there is a timelike Killing field then g_0 is flat, $\pi_0 = 0$ and $(1, 0)$ is the other basis element of this kernel.

Let $\mathbb{P}_{\mathscr{J}}$ be the L^2 orthogonal projection onto the span of $(0, X_i)$, $i = 1, \ldots, l$, and $\mathbb{P}_{\mathscr{H}}$ that onto $(1, 0)$ if there is a timelike Killing field. Thus,

$$I-\mathbb{P}=\mathbb{P}_{\mathscr{J}} \qquad \text{if there is no timelike Killing field}$$

and

$$I-\mathbb{P}=\mathbb{P}_{\mathscr{J}}\oplus\mathbb{P}_{\mathscr{H}} \qquad \text{if there is a timelike Killing field.}$$

Thus, identifying the span of $(0, X_i)$ with $\mathbb{R}^l$,

$$\begin{aligned}\mathbb{P}_{\mathscr{J}}\Phi(g,\pi) &= \left(\int_M X_1\cdot\mathscr{J}(g,\pi), \ldots, \int_M X_l\cdot\mathscr{J}(g,\pi)\right)\\ &= \left(\int_M (L_{X_1}g)\cdot\pi, \ldots, \int_M (L_{X_l}g)\cdot\pi\right)\end{aligned}$$

and

$$\mathbb{P}_{\mathscr{H}}\Phi(g,\pi)=\int_M \mathscr{H}(g,\pi).$$

Thus, we set $\mathscr{C}_{\mathscr{J}} = \{(g,\pi)\mid \mathbb{P}_{\mathscr{J}}\Phi(g,\pi)=0\}$ and $\mathscr{C}_{\mathscr{H}} = \{(g,\pi)\mid \mathbb{P}_{\mathscr{H}}\Phi(g,\pi)=0\}$ so

$$\mathscr{C}=\mathscr{C}_{\mathbb{P}}\cap\mathscr{C}_{\mathscr{J}} \qquad \text{if there are no timelike Killing fields}$$

and

$$\mathscr{C}=\mathscr{C}_{\mathbb{P}}\cap\mathscr{C}_{\mathscr{J}}\cap\mathscr{C}_{\mathscr{H}} \qquad \text{if there is a timelike Killing field.}$$

2. The Momentum Constraints

In this section we prove that the Kuranishi map takes $\mathscr{C}_{\mathbb{P}}\cap\mathscr{C}_{\mathscr{J}}\cap S_{(g_0,\pi_0)}$ to the cone

$$C_{\mathscr{J}} = \{(g_0, \pi_0)\} + \Big\{(h, \omega) \in \ker D\Phi(g_0, \pi_0) \cap \ker(D\Phi(g_0, \pi_0) \circ \mathbb{J}) \,\Big|$$

$$\int (L_{X_i} h) \cdot \omega = 0, i = 1, 2,..., l\Big\}.$$

If there are no timelike Killing fields, this gives the structure of $\mathscr{C} \cap S_{(g_0, \pi_0)}$; removing the gauges as described in Fischer *et al.* (1980, pp. 184, 191) and in Isenberg and Marsden (1982, Theorem 8.1), gives the desired structure of $\mathscr{C}$.

2.1 THEOREM. *The map F takes a neighborhood of (g_0, π_0) in $\mathscr{C}_{\mathrm{P}} \cap \mathscr{C}_{\mathscr{J}} \cap S_{(g_0, \pi_0)}$ 1–1 and onto a neighborhood of (g_0, π_0) in $C_{\mathscr{J}}$.*

Proof. By 1.4, it suffices to show that for $(g, \pi) \in \mathscr{C}_{\mathrm{P}} \cap S_{(g_0, \pi_0)}$,

$$\mathbb{P}_{\mathscr{J}} \Phi(g, \pi) = 0 \qquad \text{if and only if} \quad Q(F(g, \pi) - (g_0, \pi_0)) = 0,$$

where

$$Q(h, \omega) \in \mathbb{R}^l$$

is given by

$$Q(h, \omega) = \left(\int (L_{X_i} h) \cdot \omega\right)_{i=1,\dots,l}.$$

Letting, as above, $\hat{\mathbb{P}} = I - \mathbb{P}^*$ be the orthogonal projection onto $\ker D\Phi(g_0, \pi_0)$, we have

$$\begin{aligned} F(g, \pi) - (g_0, \pi_0) &= \hat{\mathbb{P}}(F(g, \pi) - (g_0, \pi_0)) \qquad \text{(by 1.3)} \\ &= \hat{\mathbb{P}}((h, \omega) + D\Phi(g_0, \pi_0)^* \circ \Delta^{-1} \circ \mathbb{P} \circ \mathscr{R}(h, \omega)), \end{aligned}$$

where $(h, \omega) = (g, \pi) - (g_0, \pi_0)$, by definition of F.

Thus, we have the identity

$$F(g, \pi) - (g_0, \pi_0) = \hat{\mathbb{P}}(h, \omega)$$

for $(g, \pi) \in \mathscr{C}_{\mathrm{P}}$; cf. 1.5. Therefore,

$$Q(F(g, \pi) - (g_0, \pi_0)) = Q(\hat{\mathbb{P}}(h, \omega)).$$

2.2 LEMMA. $Q(\hat{\mathbb{P}}(h, \omega)) = Q(h, \omega)$.

Proof. Let $\mathbb{P}^*(h, \omega) = (h^*, \omega^*)$ so $\hat{\mathbb{P}}(h, \omega) = (h, \omega) - (h^*, \omega^*)$. Thus

$$\begin{aligned} Q(\hat{\mathbb{P}}(h, \omega)) &= Q(h - h^*, \omega - \omega^*) \\ &= \int (L_{X_i}(h - h^*)) \cdot (\omega - \omega^*) \\ &= \int (L_{X_i} h) \cdot \omega - \int (L_{X_i} h) \cdot \omega^* - \int (L_{X_i} h^*) \cdot \omega + \int (L_{X_i} h^*) \cdot \omega^*. \end{aligned}$$

Since $(g, \pi) \in S_{(g_0, \pi_0)}$, $D\Phi(g_0, \pi_0) \circ \mathbb{J}(h, \omega) = 0$; i.e., $D\Phi(g_0, \pi_0) \cdot (-\omega^\flat/\mu_0, h^\# \mu_0) = 0$, where $\mu_0 = \mu(g_0)$ is the volume element of g_0. Also, $(h^*, \omega^*) \in$ range $D\Phi(g_0, \pi_0)^*$, so $(-\omega^{*\flat}/\mu_0, h^{*\#}\mu_0) \in$ range $\mathbb{J} \circ D\Phi(g_0, \pi_0)^*$; i.e., $(-\omega^{*\flat}/\mu_0, h^{*\#}\mu_0)$ is a gauge transformation. By gauge invariance of $D^2\Phi(g_0, \pi_0)$ (see Fischer *et al.*, 1980, Proposition 1.12),

$$\mathbb{P}_{\mathscr{T}} D^2\Phi(g_0 \cdot \pi_0) \left(\left(\frac{-\omega^\flat}{\mu_0}, h^\# \mu_0 \right), \left(\frac{-\omega^{*\flat}}{\mu_0}, h^{*\#} \mu_0 \right) \right) = 0,$$

i.e.,

$$-\int \left(L_{X_i} \frac{\omega^\flat}{\mu_0} \right) \cdot (h^{*\#} \mu_0) - \int \left(L_{X_i} \frac{\omega^{*\flat}}{\mu_0} \right) \cdot (h^\# \mu_0) = 0.$$

Since X_i is a killing field, $\flat$ and L_{X_i} commute, so we get

$$\int (L_{X_i} \omega) \cdot h^* + \int (L_{X_i} \omega^*) \cdot h = 0.$$

Integrating by parts,

$$\int (L_{X_i} h^*) \cdot \omega + \int (L_{X_i} h) \cdot \omega^* = 0.$$

Since range $\mathbb{J} \circ D\Phi(g_0, \pi_0)^* \subset \ker D\Phi(g_0, \pi_0)$, the same argument may be applied to (h^*, ω^*) in place of (h, ω) to give the identity

$$\int (L_{X_i} h^*) \cdot \omega^* = 0.$$

Therefore the last three terms in the identity for $Q(\hat{\mathbb{P}}(h, \omega))$ above drop out, leaving $Q(h, \omega)$. ∎

2.3 Lemma. $\mathbb{P}_{\mathscr{T}} \Phi(g, \pi) = Q(h, \omega)$.

Proof. $\mathbb{P}_{\mathscr{I}}\Phi(g, \pi) = \int (L_{X_i} g) \cdot \pi$. Since $L_{X_i} g_0 = 0$ and $L_{X_i} \pi_0 = 0$, we get

$$\begin{aligned}\mathbb{P}_{\mathscr{I}}\Phi(g, \pi) &= \int (L_{X_i}(g - g_0)) \cdot \pi \\ &= -\int (g - g_0) \cdot L_{X_i}\pi \\ &= -\int (g - g_0)\, L_{X_i}(\pi - \pi_0) \\ &= -\int h \cdot L_{X_i}\omega \\ &= \int (L_{X_i} h) \cdot \omega. \quad \blacksquare\end{aligned}$$

Thus we get the identity

$$Q(F(g, \pi) - (g_0, \pi_0)) = \mathbb{P}_{\mathscr{I}}\Phi(g, \pi)$$

which proves the theorem. ∎

There is another "bare hands" proof of Theorem 2.1 for the case g_0 flat and $\pi_0 = 0$ that is instructive. It uses the inverse of the Kuranishi map in terms of $Y(h, \omega)$ and $C(h, \omega)$ as was discussed in 1.5. This more computational proof will also be useful to us in the next section, so we give two lemmas relevant to this case. Thus, for the remainder of this section and the following one, we assume g_0 is flat and $\pi_0 = 0$.

2.4 Lemma. $\ker D\Phi(g_0, 0) \cap \ker(D\Phi(g_0, 0) \circ \mathbb{J})$ *consists of pairs* $(h^{\mathrm{tr\,tr}} + \frac{1}{3}\alpha g_0,\ \omega^{\mathrm{tr\,tr}} + \frac{1}{3}\beta g_0^{\#}\mu_0)$, *where* $h^{\mathrm{tr\,tr}}$ *and* $\omega^{\mathrm{tr\,tr}}$ *are arbitrary transverse traceless symmetric two tensors (i.e., divergence zero and trace zero)* α *and* β *are real constants and* $\mu_0 = \mu(g_0)$ *is the volume element of* g_0.

Proof. See Moncrief (1975). ∎

Recall from Section 1 that we can obtain elements of $\mathscr{C}_{\mathrm{P}} \cap S_{(g_0, 0)}$ by writing

$$g = g_0 + h^{\mathrm{tr\,tr}} + \tfrac{1}{3}\alpha g_0 + (-\mathrm{Hess}\, C - g_0 \Delta C)$$

and

$$\pi = \omega^{\mathrm{tr\,tr}} + \tfrac{1}{3}\beta g_0^{\#}\mu_0 + (L_Y g_0)^{\#}\mu_0,$$

where C and Y are solved for as functions of $h^{\mathrm{tr\,tr}}$, α, $\omega^{\mathrm{tr\,tr}}$ and β. This follows from 1.5. These equations parameterize solutions of the projected constraints (within $S_{(g_0, 0)}$) by the variables $h^{\mathrm{tr\,tr}}$, α, $\omega^{\mathrm{tr\,tr}}$ and β. The essence of Theorem 2.1 is that this correspondence also maps $\mathscr{C}_{\mathscr{I}}$ to $C_{\mathscr{I}}$. In the present case this is implied by the following identity.

2.5 LEMMA. *If* $L_X g_0 = 0$ *then*

$$\int (L_X g)\cdot \pi = \int (L_X h^{\mathrm{tr\,tr}})\cdot \omega^{\mathrm{tr\,tr}}.$$

Important remark. Notice that the quadratic form Q in (h, ω) in this case is independent of α and β.

Proof of 2.5.

$$\int L_X(g_0 + h^{\mathrm{tr\,tr}} + \tfrac{1}{3}\alpha g_0 - \mathrm{Hess}\, C - g_0 \Delta C)\cdot(\omega^{\mathrm{tr\,tr}} + \tfrac{1}{3}\beta g_0^{\#}\mu_0 + (L_Y g_0)^{\#}\mu_0)$$

$$= \int L_X(h^{\mathrm{tr\,tr}} - \mathrm{Hess}\, C - g_0 \Delta C)\cdot(\omega^{\mathrm{tr\,tr}} + \tfrac{1}{3}\beta g_0^{\#}\mu_0 + (L_Y g_0)^{\#}\mu_0)$$

$$= -\int (h^{\mathrm{tr\,tr}} - \mathrm{Hess}\, C - g_0 \Delta C)\cdot L_X(\omega^{\mathrm{tr\,tr}} + \tfrac{1}{3}\beta g_0^{\#}\mu_0 + (L_Y g_0)^{\#}\mu_0)$$

$$= -\int (h^{\mathrm{tr\,tr}} - \mathrm{Hess}\, C - g_0 \Delta C)\cdot L_X(\omega^{\mathrm{tr\,tr}} + (L_Y g_0)^{\#}\mu_0),$$

since $L_X g_0 = 0$. Expanding this, and integrating by parts, we get

$$\int (L_X h^{\mathrm{tr\,tr}})\,\omega^{\mathrm{tr\,tr}} + \int (\mathrm{Hess}\, C + g_0 \Delta C)\, L_X \omega^{\mathrm{tr\,tr}}$$

$$+ \int (L_X h^{\mathrm{tr\,tr}})_{ij}\,(Y^{i|j} + Y^{j|i})\,\mu_0 + \int L_X(-\mathrm{Hess}\, C - g_0 \Delta C)_{ij}\,(Y^{i|j} + Y^{j|i})\,\mu_0$$

$$= \int (L_X h^{\mathrm{tr\,tr}})\cdot \omega^{\mathrm{tr\,tr}} + \int -\nabla C\cdot \delta(L_X \omega^{\mathrm{tr\,tr}}) + \Delta C\cdot(L_X\,\mathrm{trace}\,\omega^{\mathrm{tr\,tr}})$$

$$-2\int Y\cdot \delta(L_X h^{\mathrm{tr\,tr}})\,\mu_0 + 2\int Y\cdot \delta L_X(\mathrm{Hess}\, C + g_0 \Delta C)\,\mu_0.$$

Now $\mathrm{tr}\,\omega^{\mathrm{tr\,tr}} = 0$ and $\delta(L_X k) = 0$ if $\delta k = 0$ and X is a Killing field, so all the terms drop out except the first. ∎

3. THE HAMILTONIAN CONSTRAINT

We now consider the case of g_0 flat and $\pi_0 = 0$. The Kuranishi map F takes $\mathscr{C}_P \cap \mathscr{C}_{\mathscr{J}} \cap S_{(g_0,0)}$ to the cone $C_{\mathscr{J}}$. To study the Hamiltonian constraint, i.e., the intersection

$$\mathscr{C} \cap S_{(g_0,0)} = \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_P \cap \mathscr{C}_{\mathscr{J}} \cap S_{(g_0,0)},$$

it will be necessary to do some explicit calculations and make use of the fact that the cone $C_{\mathscr{T}}$ does not depend on the variables α and β (see 2.4 and 2.5). To carry this out, it will be more convenient to use the inverse Kuranishi map.

From 1.5 and 2.4 we have smooth mappings

$$Y(h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}}, \beta) \qquad \text{and} \qquad C(h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}}, \beta)$$

which, together with their first derivatives, vanish at zero, and have the following property: for any α and β and any $(h^{\mathrm{tr\,tr}}, \omega^{\mathrm{tr\,tr}})$ satisfying

$$\int (L_{X_i} h^{\mathrm{tr\,tr}})\, \omega^{\mathrm{tr\,tr}} = 0, \qquad i = 1,\ldots, k, \tag{3.1}$$

where $X_1,\ldots, X_k$ are the Killing fields of g_0, the data

$$\begin{aligned} g &= g_0 + h^{\mathrm{tr\,tr}} + \tfrac{1}{3}\alpha g_0 - \operatorname{Hess} C - g_0 \Delta C, \\ \pi &= \omega^{\mathrm{tr\,tr}} + \tfrac{1}{3}\beta g_0^{\#}\mu_0 + (L_Y g_0)^{\#}\mu_0 \end{aligned} \tag{3.2}$$

lie in $\mathscr{C}_{\mathrm{D}} \cap \mathscr{C}_{\mathscr{T}} \cap S_{(g_0,0)}$. Thus, the mappings (Y, C) parametrize a full neighborhood of $(g_0, 0)$ in $\mathscr{C}_{\mathrm{D}} \cap \mathscr{C}_{\mathscr{T}} \cap S_{(g_0,0)}$ in terms of solutions $(h^{\mathrm{tr\,tr}}, \omega^{\mathrm{tr\,tr}})$ of (3.1) and (α, β). Note that cone (3.1) restricts $(h^{\mathrm{tr\,tr}}, \omega^{\mathrm{tr\,tr}})$ but leaves (α, β) unrestricted.

Consider now the following affine submanifold of $T^*\mathscr{M}$:

$$F = \{(g_0 + h^{cc}, 0) \in T^*\mathscr{M} \mid h^{cc} \text{ is covariant constant with respect to } g_0\}.$$

Basic properties of F are described in Fischer *et al.* (1980, p. 179ff).

Now regard (g, π) as functions of $h^{\mathrm{tr\,tr}}$, α, $\omega^{\mathrm{tr\,tr}}$, β by Eqs. (3.2). Without imposing Eqs. (3.1) yet, we substitute (3.2) into the expression for $\int \mathscr{H}(g, \pi)$. Thus $\int \mathscr{H}(g, \pi)$ is a smooth function of $h^{\mathrm{tr\,tr}}$, α, $\omega^{\mathrm{tr\,tr}}$, β and we can consider its Taylor expansion in these variables. Just as in Fischer *et al.* (1980, Sect. 6), we find that F is a nondegenerate critical manifold for $\int \mathscr{H}(g, \pi)$. Summarizing, we have

3.1 LEMMA. *In a neighborhood of* $(g_0, 0)$ *we have*

$$\begin{aligned} \int \mathscr{H}(g, \pi) = \int_M \frac{\omega^{\mathrm{tr\,tr}} \cdot \omega^{\mathrm{tr\,tr}}}{\mu_0} - \frac{1}{6}\beta^2 \operatorname{Vol}(M) + \frac{1}{4}\int_M \nabla h^{\mathrm{tr\,tr}} \cdot \nabla h^{\mathrm{tr\,tr}} \mu_0 \\ + G(h^{\mathrm{tr\,tr}}, \omega^{\mathrm{tr\,tr}}, \alpha, \beta), \end{aligned} \tag{3.3}$$

where the first and second derivatives of G vanish at zero. Moreover, each point of F is a critical point of $\int \mathscr{H}(g, \pi)$ *and* $\int \mathscr{H}(g, \pi)$ *vanishes on F; G vanishes on F as do its first and second derivatives. Finally, F is a nondegenerate critical manifold for* $\int \mathscr{H}(g, \pi)$ *in the sense of Fischer et al.* (1980, Sect. 6).

Leaving (3.1) unimposed, we make a further simplifying change of coordinates. We use the parametrized Morse lemma, as in Fischer *et al.* (1980, Sect. 6) (see also Golubitsky and Marsden, 1982) to eliminate the higher order terms. This proves the following:

3.2 LEMMA. *There is a smooth change of coordinates*

$$\varphi: (h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}}, \beta) \mapsto (\bar{h}^{\mathrm{tr\,tr}}, \bar{\alpha}, \bar{\omega}^{\mathrm{tr\,tr}}, \bar{\beta})$$

in a neighborhood of (0, 0, 0, 0) *which leaves F invariant, whose derivative at F is the identity and is such that* (3.3) *becomes*

$$\int_M \mathscr{H}(g, \pi) = \int_M \frac{\bar{\omega}^{\mathrm{tr\,tr}} \cdot \bar{\omega}^{\mathrm{tr\,tr}}}{\mu_0} - \frac{1}{6}\bar{\beta}^2 \operatorname{Vol}(M) + \frac{1}{4}\int_M \nabla\bar{h}^{\mathrm{tr\,tr}} \cdot \nabla\bar{h}^{\mathrm{tr\,tr}}\mu_0. \quad (3.4)$$

We shall use this change of coordinates to show that when (3.3) is set equal to zero, it can be solved for a double-valued function $\beta = \beta_\pm(h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}})$. This will show that the solution set is tangent and diffeomorphic to the cone on which the Taub conserved quantity associated to the timelike killing field vanishes; i.e., to the cone $C_{\mathscr{H}}$ defined to be the set of $(h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}}, \beta)$ such that

$$\int_M \frac{\omega^{\mathrm{tr\,tr}} \cdot \omega^{\mathrm{tr\,tr}}}{\mu_0} - \frac{1}{6}\beta^2 \operatorname{Vol}(M) + \frac{1}{4}\int_M \nabla h^{\mathrm{tr\,tr}} \cdot \nabla h^{\mathrm{tr\,tr}}\mu_0 = 0. \quad (3.5)$$

Thus, we can solve (3.1) for $h^{\mathrm{tr\,tr}}$ and $\omega^{\mathrm{tr\,tr}}$ and (3.3) for β independently, showing that the simultaneous solution set is diffeomorphic to the cone $C_{\mathscr{J}} \cap C_{\mathscr{H}}$.

The function β is constructed as follows: Let

$$\bar{\beta}_\pm = \pm\left(6\int_M \frac{\bar{\omega}^{\mathrm{tr\,tr}} \cdot \bar{\omega}^{\mathrm{tr\,tr}}}{\mu_0} + \frac{3}{2}\int_M \nabla\bar{h}^{\mathrm{tr\,tr}} \cdot \nabla\bar{h}^{\mathrm{tr\,tr}}\mu_0\right)^{1/2} \operatorname{Vol}(M)^{-1/2}. \quad (3.6)$$

Note that (3.6) defines two functions of $(\bar{h}^{\mathrm{tr\,tr}}, \bar{\omega}^{\mathrm{tr\,tr}})$, each of which is smooth away from (0, 0). Let

$$(h_\pm^{\mathrm{tr\,tr}}, \alpha_\pm, \omega_\pm^{\mathrm{tr\,tr}}, \beta_\pm) = \varphi^{-1}(\bar{h}^{\mathrm{tr\,tr}}, \bar{\alpha}, \bar{\omega}^{\mathrm{tr\,tr}}, \bar{\beta}_\pm). \quad (3.7)$$

This defines $\beta_\pm$ as a function of $(\bar{h}^{\mathrm{tr\,tr}}, \bar{\alpha}, \bar{\omega}^{\mathrm{tr\,tr}})$ and mappings $\psi_\pm$: $(\bar{h}^{\mathrm{tr\,tr}}, \bar{\alpha}, \bar{\omega}^{\mathrm{tr\,tr}}) \mapsto (h^{\mathrm{tr\,tr}}, \alpha_\pm, \omega_\pm^{\mathrm{tr\,tr}})$. From the fact that φ and φ^{-1} have derivative the identity on F, we see formally from the chain rule that $\psi_\pm$ are C^1 maps with derivative the identity on F. This is proved rigorously using some straightforward Sobolev estimates on the terms involving $\bar{\beta}_\pm$ which we omit. Thus, by the C^1 inverse function theorem, $\psi_\pm$ have local C^1 inverses, thereby defining $\beta_\pm$ as functions of $(h^{\mathrm{tr\,tr}}, \alpha, \omega^{\mathrm{tr\,tr}})$ such that $\int \mathscr{H}(g, \pi) = 0$. The following main theorem is now a consequence of this work.

3.3 THEOREM. *The association*

$$(h^{\text{tr tr}}, \alpha, \omega^{\text{tr tr}}, \beta) \mapsto (g, \pi),$$

where (g, π) *are given by* (3.2) *with* $\beta = \beta_\pm(h^{\text{tr tr}}, \alpha, \omega^{\text{tr tr}})$ *defined by* (3.6) *and Lemma* 3.2, *with* $\pm$ *depending on the sign of* β, *is a one-to-one correspondence between the cone* $C_{\mathscr{J}} \cap C_{\mathscr{F}}$ *defined by* (3.1) *and* (3.5) *and the nonlinear constraint set* $\mathscr{C} \cap S_{(g_0, 0)}$ *in a neighborhood of* $(g_0, 0)$. *This correspondence maps straight lines in the cone through* $(g_0, 0)$ (*i.e., a solution of the linearized equations satisfying the second order conditions*) *to smooth curves in* $\mathscr{C} \cap S_{(g_0, 0)}$ *with the same tangent at* $(g_0, 0)$.

The theorem says that the second order conditions on linearized perturbations are sufficient for the existence of an exact perturbation curve. This, of course, was the main goal.

Finally, one needs to remove the gauge condition by eliminating $S_{(g_0, 0)}$. However, this can be done as described in Fischer *et al.* (1980) and Isenberg and Marsden (1982), so need not be repeated here.

4. THE EINSTEIN–YANG–MILLS EQUATIONS

The results proved above for gravity alone will now be generalized to the case of coupled gravitational and gauge fields. Similar results are expected to hold for other coupled systems such as the Einstein-scalar field equations (see Saraykar and Joshi, 1981 and 1982). The connection between linearization stability and symmetry holds for the Einstein–Dirac equations (see Bao *et al.*, 1982), but there could be difficulties with the analogue of Theorem 3.3 because of negative energy problems.

The program for proving that the solution set for coupled equations has conical singularities is as follows. The matter (additional field) equations must be Hamiltonian, possibly with additional (first class) constraints. (This requirement might preclude the study of certain fluid models.) Several technical details must be checked, such as ellipticity of the adjoint of the (generalized) constraint operator Φ and existence of a slice for the gauge transformations. The constraint map Φ must be split into an energy function (corresponding to timelike transformations) and a spacelike momentum, and for technical reasons it must be possible to parametrize the shift and gauge shift so the latter is quadratic and elliptic. Then the arguments of Section 2 may be generalized. If there is a timelike symmetry, a decomposition like Lemma 2.4 is needed so that the spacelike and Hamiltonian constraints may be separated as in Lemma 2.5. Then if the matter fields have positive energy in the sense of adding only positive quadratic terms to expansion (3.3) in Lemma 3.1, the quadratic structure of the singularities follows. Conservation laws, such as those proved for coupled gravitational and gauge fields by Anderson and Arms (1982), can be used to show that the second order conditions for linearization stability are independent of gauge and linearized gauge.

The first few steps of this program have been carried out in previous papers for gravity coupled to gauge fields (Arms, 1979) and to certain massless scalar fields (Saraykar and Joshi, 1981 and 1982). Thus it is already known in these cases that the solution set can have singularities only at symmetric fields. In this section we complete the program for the gauge field case.[1]

As in the case of gravity alone, it suffices to study the constraint equations. We follow the notation of Arms (1979) except for some modifications noted below; calculations that appear in that reference are omitted.

Let $\mathfrak{A}$ be the set of ($W^{s,p}$, $s > 3/p + 1$) gauge field potentials on a Cauchy surface M in spacetime, i.e., connections on a fixed principal fiber bundle B over M. An element of $\mathfrak{A}$ will be represented by a Lie algebra-valued pseudotensorial one-form A on a neighborhood in M; $A = \sigma^*\omega$, where σ is a local cross section of the bundle B and ω is the connection. We assume that the Lie algebra admits an adjoint action invariant, positive definite inner product, denoted below by $\dot{\cdot}$. Let $T^*\mathfrak{A}$ be the (L^2) cotangent bundle of $\mathfrak{A}$; thus $\eta \in T^*_A\mathfrak{A}$ is a tensorial dual Lie algebra-valued vector density, the "negative electric field density," and $T^*\mathfrak{A}$ is the phase space for the Yang–Mills field. Initial data for the coupled Einstein–Yang–Mills system are elements $(g, A, \pi, \eta) \in P = T^*(\mathscr{M} \times \mathfrak{A})$, where $\mathscr{M}$, g, and π are as for gravity alone.

The constraint equations as given in Arms (1979) will not work in the program outlined above. The spacelike momentum comes in two pieces, the total supermomentum $\mathscr{T}$ and a constraint $\mathscr{H}$ on the Yang–Mills initial data which generalizes Gauss' law. As stated in Arms (1979), $(\mathscr{T}, \mathscr{H})$ is not quadratic in (g, A, π, η), and this creates technical difficulties, as mentioned above. Fortunately, the situation is remedied by using, in essence, the old trick of adding a multiple of one constraint (in Dirac's language, a "weakly zero" quantity) to another. In terms of the momentum mapping, this corresponds to choosing a different embedding of $\mathscr{D}^3$, the diffeomorphism group on M, into the group $\mathscr{B}^3$ of bundle automorphisms (i.e., combined coordinate and gauge transformations) on B.

In the notation of Arms (1979) this procedure may be described as follows. The spacelike momentum $(\mathscr{T}, \mathscr{H})$ gives a contribution

$$H_S = \int_M X \cdot \mathscr{T} + V \dot{\cdot} \mathscr{H} = \int_M \{X^i(-2\pi^j_{i|j} + \eta^j_a(A^a_{j,i} - A^a_{i,j} + C^a_{bc}A^b_i A^c_j)) + V^a(\eta^j_{a|j} + C^c_{ab}A^b_j\eta^j_c)\}$$

to the total superhamiltonian. Here C^a_{bc} are the structure constant of the Lie algebra in a suitable basis. Integrating by parts one can re-express H_S as

$$H_S = \int_M X \cdot \tilde{\mathscr{T}} + \tilde{V} \cdot \mathscr{H},$$

[1] The case of the massless scalar field seems to be somewhat simpler than the present case, since there are no additional constraint equations. (The *massive* scalar field violates the strong energy condition, and several important steps in the program fail; see Saraykar and Joshi (1981) and especially (1982).)

where

$$\tilde{\mathscr{J}}_i = -2\pi^j_{i|j} + \eta^j_a A^a_{j,i} - (\eta^j_a A^a_i)_{,j}$$
$$= -2\pi^j_{i|j} + \eta^j_a A^a_{j|i} - (\eta^j_a A^a_i)_{|j}$$

and

$$\tilde{V}^a = V^a + X^i A^a_i.$$

The new supermomentum $(\tilde{\mathscr{J}}, \mathscr{K})$ is now quadratic and the constraints $(\tilde{\mathscr{J}}, \mathscr{K}) = (0, 0)$ are equivalent to $(\mathscr{J}, \mathscr{K}) = (0, 0)$. We shall give a more intrinsic description of this procedure below. To simplify the notation we write $(\mathscr{J}, \mathscr{K})$ for the new (quadratic) spacelike momentum and (X, V) for the new generalized shift.

The pair $\Theta = (\mathscr{J}, \mathscr{K})$ is the momentum map for the action of $\mathscr{B}^3$ on P (via pullback). Now $\mathscr{B}^3$ is a semi-direct product of the diffeomorphism group $\mathscr{D}^3$ and the gauge transformation group $\mathscr{G}$ (bundle automorphisms that cover the identity). The group $\mathscr{G}$ sits naturally in $\mathscr{B}^3$, and has the momentum map $\mathscr{K} = \hat{\nabla} \cdot \eta$, where $\hat{\nabla}\cdot$ indicates the doubly covariant divergence. On the other hand $\mathscr{D}^3$ has no natural copy in $\mathscr{B}^3$. There is an action of $\mathscr{D}^3$ on P, used in Arms (1979), which is most easily described in terms of its infinitesimal generators. An element of the Lie algebra of $\mathscr{D}^3$ is a vector field X on M. For each point $(g, A, \pi, \eta) \in P$, lift X horizontally to B, using the connection A. Let $\bar{X}$ indicate this lifted vector field, and let $\bar{\eta}$ be the tensorial object on B corresponding to η. The infinitesimal generator at (g, A, π, η) of the $\mathscr{B}^3$ action on P is then given by $(L_X g, \sigma^* L_{\bar{X}}\omega, L_X\pi, \sigma^* L_{\bar{X}}\bar{\eta})$. (For a more concrete description, see Arms (1979), at the end of Section IIIB.) However, the momentum map for this action ($\mathscr{J}$ in the reference), is cubic in A and η.

To eliminate the cubic term, we proceed as follows. Choose a point $(g_0, A_0, \pi_0, \eta_0) \in P$, and lift X horizontally with respect to A_0. This lifted field generates an action on B which in turn gives rise via pullback to an action on P. The momentum map $\mathscr{J}$ for this action is the new $\tilde{\mathscr{J}}$ given above; in invariant notation, it is given by

$$\mathscr{J} = -2\delta\pi + \eta \times \because \beta + (\hat{\nabla} \cdot \eta)(A - A_0),$$

where β is the "magnetic field density" and $\times$ indicates the ordinary cross product in an orthonormal frame. (For more details on notation, see Arms (1979)).

The Hamiltonian $\mathscr{H}$ for the coupled fields is given by

$$\mathscr{H} = \{\pi' \cdot \pi' - \tfrac{1}{2}(\operatorname{tr} \pi')^2 - R + \tfrac{1}{2}(\eta' \because \eta' + \beta' \because \beta')\}\, \mu(g),$$

where π' is the tensor part of π, and $\because$ indicates contraction using both metrics (g and the Lie algebra metric). Let $\Phi = (\mathscr{H}, \mathscr{J}, \mathscr{K})$. This modification does not change the principal part of the operator $D\Phi$, so the calculations in Arms (1979) show that $D\Phi^*$ is elliptic. The coupled constraint equations are given by

$$\Phi = 0$$

and the evolution equations are

$$\frac{\partial}{\partial t}\begin{bmatrix} g \\ A \\ \pi \\ \eta \end{bmatrix} = -\mathbb{J} \circ D\Phi^* \begin{bmatrix} N \\ X \\ V \end{bmatrix}, \tag{4.1}$$

where N and X are the lapse and shift of the spacelike slicing and V specifies the evolution of the Yang–Mills gauge. (Here $\mathbb{J}$ again represents the almost complex structure associated with the natural symplectic structure on the cotangent bundle P and the obvious L^2metric.) From Eq. (4.1) one sees that a simultaneous symmetry of the fields is an element $(N, X, V) \in \ker D\Phi^*$, where N and X are the components of a Killing field (as for gravity alone) and V is the (infinitesimal) gauge transformation needed to preserve the gauge field potential under the Killing field flow.

Many of the arguments in the introduction and section one follow verbatim with the new Φ and obvious notational changes such as replacing (g, π), by (g, A, π, η) and (N, X) by (N, X, V). The fundamental decomposition, its interpretation, the orthogonal slice, the construction of the Kuranishi map F, and Propositions 1.1 through 1.5 all hold for the coupled case with no essential change in the proof.

The arguments of sections 2 and 3, dealing with spacelike and timelike constraints corresponding to particular symmetries, require a fairly explicit characterization of those symmetries on a constant mean curvature hypersurface M. By the arguments of Arms (1979, Sect. IVB) it follows that on M a symmetry $(N, X, V) \in \ker D\Phi^*$ satisfies either (a) $N = 0$ (i.e., all symmetries are tangent to M), or (b) N is constant and the initial data is trivial (i.e., g and A are flat and π and η are zero). In case (a), the "spacelike" case, there is a basis of $\ker D\Phi^*$ of the form $\{(0, X_i, V_i) \mid i = 1,..., l$, $L_{X_i} g = 0$, $L_{X_i}\pi = 0$, $L_{X_i}\eta = [\eta, V_i]$ and $L_{X_i}A = DV_i\}$; in case (b), there is a basis with $(1, 0, 0)$ as one element and the rest of the basis like that in case (a).

Let $\mathbb{P}_\Theta$ be the L^2 orthogonal projection onto the span of $\{(0, X_i, V_i)\}$ and $\mathbb{P}_{\mathscr{H}}$ the projection onto $(1, 0, 0)$. Thus the ith component of $\mathbb{P}_\Theta \circ \Phi$ is given by

$$\int_M [X_i \cdot \mathscr{J} + V_i \,\dot{:}\, \mathscr{K}]$$

and

$$\mathbb{P}_{\mathscr{H}} \circ \Phi = \int_M \mathscr{H}.$$

It follows that if $\mathscr{C}_{\mathrm{P}} = \{(g, A, \pi, \eta) \mid \mathbb{P} \circ \Phi = 0\}$, $\mathscr{C}_\Theta = \{(g, A, \pi, \eta) \colon \mathbb{P}_\Theta \circ \Phi = 0\}$, and $\mathscr{C}_{\mathscr{H}} = \{(g, A, \pi, \eta) \mid \mathbb{P}_{\mathscr{H}} \circ \Phi = 0\}$, then

$$\begin{aligned} \mathscr{C} &= \mathscr{C}_{\mathrm{P}} \cap \mathscr{C}_\Theta && \text{in case (a)} \\ &= \mathscr{C}_{\mathrm{P}} \cap \mathscr{C}_\Theta \cap \mathscr{C}_{\mathscr{H}} && \text{in case (b),} \end{aligned}$$

where $I - \mathbb{P} = \mathbb{P}_\Theta \oplus \mathbb{P}_{\mathscr{H}}$.

We first consider $\mathscr{C}_P \cap \mathscr{C}_\Theta$, analogous to $\mathscr{C}_P \cap \mathscr{C}_{\mathscr{T}}$ as in Section 2. For the Einstein–Yang–Mills case, we need something playing the role of a slice for the $\mathscr{B}^4$ action. The mapping playing the role of the momentum map for this action is $\Phi = (\mathscr{H}, \Theta)$, where $\Theta = (\mathscr{T}, \mathscr{K})$. Thus, range $\mathbb{J} \circ D\Phi^*$ plays the role of the tangent space to the $\mathscr{B}^4$ orbit at $(g_0, A_0, \pi_0, \eta_0)$). As in Moncrief's decomposition ((M) in the introduction),

$$\ker D\Phi \circ \mathbb{J} = \ker(D\mathscr{T} \circ \mathbb{J}) \cap \ker(D\mathscr{K} \circ \mathbb{J}) \cap \ker(D\mathscr{H} \circ \mathbb{J})$$

is the orthogonal complement to the orbit. Thus, as in Fischer *et al.* (1980, Sect. 5.5),

$$S_0 = \{(g_0, A_0, \pi_0, \eta_0)\} + \mathscr{U},$$

where $\mathscr{U}$ is a suitably small ball in $\ker D\Phi \circ \mathbb{J}$ (in a suitable $W^{s,p}$ metric), plays the role of a slice for the $\mathscr{B}^4$ action.

Now we have, analogous to Theorem 2.1,

4.1 THEOREM. *The map F maps $\mathscr{C}_P \cap \mathscr{C}_\Theta \cap S_0$ locally* 1–1 *onto the cone*

$$C_\Theta = \{(g_0, \pi_0, A_0, \eta_0)\} + \{(h, b, \omega, \theta) \in \ker D\Phi \cap \ker D\Phi \circ \mathbb{J} \mid Q(h, b, \omega, \theta) = 0\},$$

where $Q = \mathbb{P}_\Theta(\Theta - D\Theta) = \mathbb{P}_\Theta(\Phi - D\Phi)$.

The proof is a straightforward generalization (by modification of notation) of the proof of Theorem 2.1.

In case (a), removing the gauges as in Fischer *et al.* (1980) completes the main results. For case (b), an analog to the decomposition in Lemma 2.4 is needed to separate the timelike (Hamiltonian) and spacelike constraints. Such a decomposition follows from Proposition 1.5 generalized to the coupled case and computations in Arms (1979): points $(g, A, \pi, \eta) \in \mathscr{C}_P \cap S_0$ near $(g_0, A_0, \pi_0, \eta_0)$ satisfying case (b) may be expressed as

$$\begin{aligned} g &= g_0 + h^{\mathrm{tr\,tr}} + \tfrac{1}{3}\alpha g_0 - \mathrm{Hess}\, C - \Delta C g_0, \\ A &= A_0 + b^{\mathrm{tr}}, \\ \pi &= \omega^{\mathrm{tr\,tr}} + (\tfrac{1}{3}\beta g_0^{\#} + (L_Y g)^{\#})\mu_0, \\ \eta &= \theta^{\mathrm{tr}} - (DU)\mu_0, \end{aligned} \tag{4.2}$$

where $(C, Y, U) \in$ domain of $D\Phi^*$, is a function of $h^{\mathrm{tr\,tr}}$, b^{tr}, $\omega^{\mathrm{tr\,tr}}$, θ^{tr}, α, and β given by the (generalized) Ψ map of Proposition 1.5, and b^{tr} and θ^{tr} have vanishing gauge covariant divergence.

Note that g and π in Eq. (4.2) are unchanged from Section 2. Thus Lemma 2.5 remains valid in the coupled case, and Lemma 3.1 is unchanged except for the addition of a *positive* quadratic term

$$\tfrac{1}{2}\int_M (\eta \,\vdots\, \eta + \beta \,\vdots\, \beta)/\mu_0.$$

Then the rest of the proof of the conical structure of $\mathscr{C}_P \cap \mathscr{C}_\Theta \cap \mathscr{C}_{\mathscr{Y}}$ follows exactly as before.

Thus we obtain the following when g_0 and A_0 are flat and $\pi_0 = 0$, $\eta_0 = 0$.

4.2 THEOREM. *There is a 1–1 correspondence between a neighborhood of* $(g_0, A_0, 0, 0)$ *in the cone* $C_\Theta \cap C_{\mathscr{Y}} \cap S_0$ *and a neighborhood of* $(g_0, A_0, 0, 0)$ *in the nonlinear constraint set* $\mathscr{C} \cap S_0$ *which maps straight lines in the cone through* $(g_0, A_0, 0, 0)$ *to a smooth curve in* $\mathscr{C} \cap S_0$ *with the same tangent at* $(g_0, A_0, 0, 0)$.

This result has the same interpretation as for gravity: solutions of the linearized constraint equations are nonspurious, i.e., are linearization stable, if and only if the second order conditions $0 = \int_M (N, X, V) \cdot D^2\Phi(g_0, A_0, \pi_0, \eta_0)((h, b, \omega, \theta), (h, b, \omega, \theta))$ are satisfied on the hypersurface. As we have already mentioned, these second order conditions are actually hypersurface and gauge invariant, using the conservation laws of Anderson and Arms (1982).

5. DISCUSSION

This paper completes our study of linearization stability and the local structure of the space of solutions for the Einstein and Einstein Yang–Mills equations on spacetimes possessing a compact Cauchy surface with constant mean curvature. We have shown that the space of solutions has a quadratic singularity at every solution possessing a group of symmetries of dimension at least one. Using second order conditions, we have identified those linearized solutions which can be used in an honest perturbation expansion. This section discusses a few miscellaneous issues relevant to our results.

The Second Order Conditions

For vacuum gravity, the necessary and sufficient conditions that a linearized solution ${}^{(4)}h$ at ${}^{(4)}g_0$ be tangent to a curve ${}^{(4)}g(\lambda)$ of exact vacuum solutions satisfying ${}^{(4)}g_0 = {}^{(4)}g(0)$ is that the Taub conserved quantities vanish identically,

$$\int_\Sigma {}^{(4)}X \cdot [D^2 \operatorname{Ein}({}^{(4)}g_0) \cdot ({}^{(4)}h, {}^{(4)}h)] \cdot {}^{(4)}Z_\Sigma \, d^3\Sigma = 0$$

for all killing fields ${}^{(4)}X$ of the metric ${}^{(4)}g_0$. These quantities comprise the momentum map for the isometry group of ${}^{(4)}g_0$ acting on the linearized theory. (For purely spacelike symmetries one can show this directly. For a timelike symmetry this is the known fact that $\frac{1}{2}\int_\Sigma (1, 0) \cdot D^2\Phi(g, 0)((h, \omega), (h, \omega))\, d^3\Sigma$ is a Hamiltonian for the perturbations—this is the second variation method.)

The proofs in Fischer *et al.* (1980) and the present paper on the space of solutions worked with the constraint equations directly. The role of the Taub quantities is to show that the order conditions are hypersurface independent and gauge invariant.

In Section 4 we treated the Einstein–Yang–Mills equations by studying the constraints on a given hypersurface. To show analogously that the conditions are hypersurface independent and gauge invariant, and hence an intrinsic property of the spacetime solution set, one needs the Einstein–Yang–Mills analogue of the conserved Taub quantities treated from a covariant point of view. This aspect may be found in Anderson and Arms (1982).

The Symplectic Space of True Degrees of Freedom

If one wishes to divide out by the gauge group (diffeomorphisms of spacetime for gravity, bundle automorphisms over the identity for Yang–Mills and bundle automorphisms over diffeomorphisms for the Einstein–Yang–Mills equations), then a suitable slice theorem is needed. For the pure Yang–Mills equations (with a compact group), such a slice theorem is relatively routine (see, for example, Singer, 1978, Babelon and Vialett, 1981, and Kondracki and Rogulski, 1981). For vacuum gravity, see Isenberg and Marsden (1982). Using a combination of the methods from these papers should enable one to prove a slice theorem for the Einstein–Yang–Mills equations.

Using such a slice theorem, the results of Fischer *et al.* (1980), this paper and Arms *et al.* (1981) (especially Lemma 18), one can show that the space of solutions modulo gauge transformations is a stratified symplectic manifold, i.e., a stratified manifold, each stratum of which is symplectic. The procedure for vacuum gravity is spelled out in Isenberg and Marsden (1982). It is also shown there, using the slice theorem and York parametrization, that the generic points consisting of spacetimes with no symmetries is an open dense set. Thus the generic symplectic stratum in the reduced space is also open and dense.

For pure Yang–Mills fields, the results of Arms (1981) can similarly be used to establish the analogous results in that case. For the Einstein–Yang–Mills equations the symplectic stratification follows from the results of Section 4, this paper, and an Einstein–Yang–Mills slice theorem.

Respecting Symmetry Types

In the context of general momentum mappings, Arms *et al.* (1981) Theorem 4′ showed that the Kuranishi map preserves the symmetry type for any Lie subalgebra $\mathfrak{H} \subset s_{x_0}$, the symmetry (= isotropy) algebra of x_0. That is, solutions of the nonlinear problem with symmetry type $\mathfrak{H}$ are mapped to those elements in the cone consisting of the linearized solutions satisfying the second order conditions and having the same symmetry type $\mathfrak{H}$.

For the momentum constraints and for the Yang–Mills constraints, this remains true by essentially the same methods. This is also basically true for the full constraints as well, but the Hamiltonian constraint must be treated by a separate argument. If ${}^{(4)}g_0$ has a timelike Killing field, then ${}^{(4)}g_0$ is flat and we may choose Σ so that $\operatorname{tr} k = 0$. If we are looking for nearby solutions which also have a timelike Killing field, then this amounts to studying the space of flat 3-metrics which can be done as in Fischer and Marsden (1975). If $\mathfrak{H}$ does not include time translations then the result can be dealt with by the Kuranishi method.

We note that with Σ fixed, $\pi = 0$ and g flat, g can change its spacelike isometry group; i.e., there can be branching within the flat metrics. However, this cannot happen for $\Sigma = T^3$; this follows from Proposition 8.1 of Fischer *et al.* (1980).

Bianchi Types and Symmetry Breaking

Jantzen (1979) studied the space of solutions within various Bianchi types. He noted that one could find conical singularities near, for example, the Taub solution within Bianchi IX. For the full Einstein equations one can break the symmetry, encountering a bifurcation at each step, by several paths in the lattice of subgroups for a given symmetry group. For example, one can branch from the Taub solution on S^3 as follows:

		S^3 Gowdy ↗	→ S^3 Moncrief ↘	
		T^2	T_1	
Spacetime:	Taub			→ Generic solution
Symmetry group:	$SU(2) \times S^1$			$\varnothing$
		↘ Generic Bianchi IX	→ ? ↗	
		$SU(2)$	T_1	

For the three torus (T^3) topology one can branch as follows:

Spacetime:	T^3 flat →	Kasner →	Gowdy →	Moncrief →	Generic solution
Symmetry group:	$T^3 \times \mathbb{R}$	T^3	T^2	T^1	$\varnothing$

For a discussion of Gowdy spacetimes with T^2 symmetry see Moncrief (1981). Recently Moncrief has used similar methods to construct Gowdy-like solutions with T^1 symmetry (Moncrief (1982)).

Analysis of the Second Order Conditions

To analyze, in detail, the second order condition for a given background solution would involve considerable computation using tensorial harmonics, as in Jantzen (1980). To illustrate what is involved, we show how to treat a scalar model equation which captures the essence of the quadratic conditions for the T^3 Kasner solution.

On T^3 consider the space of pairs (ϕ, π) where ϕ is a scalar field and π is a scalar density. Let the coordinate basis fields $\partial/\partial x^j$ play the role of Killing fields and consider the "second order conditions" $\int_{T^3} \pi \mathscr{L}_{\partial/\partial x^j} \phi \, d^3x = 0$. Now Fourier transform ϕ and π by writing

$$\phi = \sum_k (q_k^{(r)} + i q_k^{(i)})\, e^{ik\cdot x},$$

and

$$\pi = i \sum_k (p_k^{(r)} + i p_k^{(i)}) e^{ik \cdot x},$$

where k ranges over the appropriate discrete lattice for T^3. The linear reality conditions $q_k^{(r)} = q_{-k}^{(r)}$ and $p_k^{(r)} = -p_{-k}^{(r)}$, etc., that restrict the q's and p's are assumed satisfied.

The second order conditions take the form

$$0 = \int_{T^3} \pi \mathscr{L}_{\partial/\partial x^j} \phi = \sum_k k_j (q_k^{(r)} p_k^{(r)} + q_k^{(i)} p_k^{(i)}).$$

Changing variables according to $q_k^{(r)} = (1/\sqrt{2})(Q_k^{(r)} + P_k^{(r)})$, $p_k^{(r)} = (1/\sqrt{2})(P_k^{(r)} - Q_k^{(r)})$, etc., and restricting the sum to range only over the independent k's (note that k and $-k$ give the same contribution), we get

$$0 = \sum_{|k|} k_j [(P_k^{(r)2} - Q_k^{(r)2} + P_k^{(i)2} - Q_k^{(i)2})].$$

Now choose three independent lattice vectors such as $e^{(1)} = (1, 0, 0)$, $e^{(2)} = (0, 1, 0)$ and $e^{(3)} = (0, 0, 1)$ and rewrite the second order conditions as

$$(P_{e^{(j)}}^{(r)})^2 - (Q_{e^{(j)}}^{(r)})^2 = R_j(Q, P), \qquad j = 1, 2, 3.$$

The crucial point is that R_1 does not depend upon $\{P_{e^{(1)}}^{(r)}, Q_{e^{(1)}}^{(r)}, P_{e^{(2)}}^{(r)}, Q_{e^{(2)}}^{(r)}, P_{e^{(3)}}^{(r)}, Q_{e^{(3)}}^{(r)}\}$, and similarly for R_2 and R_3. This can be seen by inspection from the above expressions. Thus we can choose the variables in the set complementary to $\{Q_{e^{(j)}}^{(r)}, P_{e^{(j)}}^{(r)}\}$ arbitrarily and then solve the three independent "cone" equations

$$(P_{e^{(j)}}^{(r)})^2 - (Q_{e^{(j)}}^{(r)})^2 = R_j(Q, P) = r_j.$$

Thus the singularity structure is that of $P^2 - Q^2 = r$ with r a variable parameter. See Fig. 1.

The divergence constraint in relativity for the T^3 Kasner solution is similar; indeed the tensor indices decouple and do not materially affect the above arguments.

Resolving Singularities[2]

The question of resolving the singularities in the space of solutions may be important for quantization or other purposes (cf. Sniatycki, 1982). What we seek is to include our solution space into the solution space for a modified problem with no singularities. We describe the idea in a general symplectic context, as in Arms *et al.* (1981).

[2] We thank R. Bryant for remarks on "marked surfaces" which motivated the discussion here.

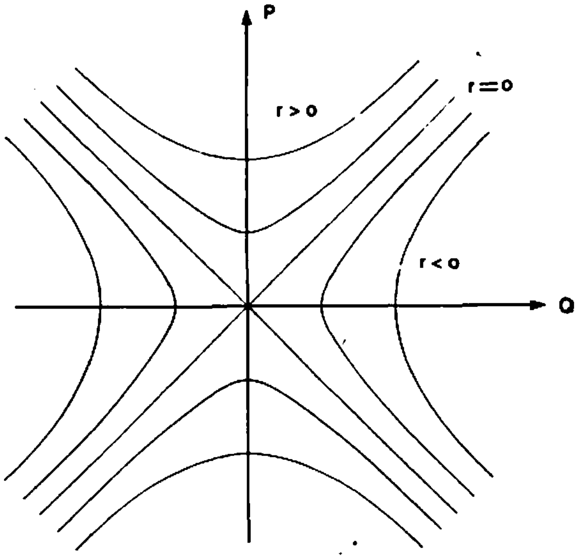

FIGURE 1

Let P be a symplectic manifold, G a Lie group acting on P by canonical transformations and $J: P \to \mathfrak{G}^*$ an Ad*-equivariant momentum map. Let $x_0 \in P$, $J(x_0) = 0$ and G_{x_0} be the isotropy group of x_0. We know from Arms *et al.* (1981) that when $\dim G_{x_0} \geqslant 1$ and certain technical conditions are satisfied $J^{-1}(0)$ has a conical singularity at x_0.

We want to embed the set $J = 0$ naturally into a set $\tilde{J}^{-1}(0)$ without singularities. We can enlarge P to $P \times T^*G$ with the obvious action of G, but this uses more extra variables than is necessary.

Instead, assume the G-action admits a slice S_{x_0} at x_0 and let $\mathscr{C}_{\mathrm{P}}$ be the projected constraints (obtained by the Liapunov–Schmidt procedure). By Arms *et al.* (1981), $S_{x_0} \cap \mathscr{C}_{\mathrm{P}}$ is a symplectic manifold. Now G_{x_0} acts on it with a momentum map $j_{x_0} = J$ restricted to $S_{x_0} \cap \mathscr{C}_{\mathrm{P}}$. Now consider $(\mathscr{C}_{\mathrm{P}} \cap S_{x_0}) \times T^*G_{x_0}$ and let G_{x_0} act on it by the obvious product action. The momentum map now is $\tilde{J}(x, \alpha_g) = J(x) + R_g^* \alpha_g$, where R_g is right translation by $g \in G_{x_0}$. Then $\tilde{J}^{-1}(0)$ has no singularity at $(x_0, 0)$ and $J^{-1}(0) \cap (\mathscr{C}_{\mathrm{P}} \cap S_{x_0})$ naturally embeds in $\tilde{J}^{-1}(0)$. These extra symplectic variables $T^*G_{x_0}$(dim $= 2 \times$ dim of symmetry group) thus are exactly the right number to symplectically "resolve" the singularity.

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