vector of weight $\lambda + \alpha + \beta$ and hence orthogonal to v unless $\alpha = -\beta$. It follows from (1) that the spaces of tangent vectors

$$(E_\alpha - E_{-\alpha})_{[v]}, \qquad i(E_\alpha + E_{-\alpha})_{[v]}$$

are mutually orthogonal with respect to ω_v as α ranges over the set of positive roots. Some of these tangent vectors might be zero. To check whether the orbit is symplectic we need to know that if ω_v vanishes on this subspace then the tangent vectors are zero. Now $[E_\alpha, E_{-\alpha}] = r_\alpha \in t$ and $r_\alpha v = (\lambda \cdot \alpha)v$ where $\lambda \cdot \alpha$ denotes the value of λ on r_α. We have thus proved

Theorem. *An orbit $G \cdot [v]$ in $P(H)$ is symplectic if and only if v is a weight vector satisfying the following condition: If λ is the weight corresponding to v then $\lambda \cdot \alpha = 0$ implies that $E_\alpha v = 0$ for every root α.*

(In particular, regular weights, i.e., those λ for which $\lambda \cdot \alpha \neq 0$ for any α, give rise to symplectic orbits while the zero weight never gives rise to a symplectic orbit (unless the orbit is a point).)

The description of the Kaehler orbits is essentially a consequence of the Borel–Weil theorem. If the orbit were a complex submanifold of $P(H)$, its tangent space would be stable under multiplication by i and so we would get an action of $g^{\mathbb{C}}$ and hence of the complex group $G^{\mathbb{C}}$ on the orbit. The only compact Kaehler homogeneous spaces for $g^{\mathbb{C}}$ are of the form $G^{\mathbb{C}}/P$ where P contains a Borel subgroup. Thus $[v]$ is stabilized by a Borel subgroup and so v is a maximal weight vector. Thus

Proposition 2. *There is only one Kaehler orbit and it is the orbit of a projectivized maximal weight vector.*

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Four Applications of Nonlinear Analysis to Physics and Engineering

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Introduction

My goal is to describe, in as accessible terms as possible, four separate applications of nonlinear analysis to relativity, elasticity, chaotic dynamics and control theory that I have recently been involved with. The descriptions are in some sense superficial since many interesting technical points are glossed over. However, this is necessary to efficiently convey the flavor of the methods.

Most applications of mathematics to "real-life" problems of immediate need do not involve deep methods and ideas. For example, the force exerted on an aircraft frame by the landing gear when the vehicle lands is best computed, at least at first, by using undergraduate mathematics, engineering and experience. However applied mathematics in the broad sense ranges from such problems of urgency to "practical" problems involving deeper mathematics (compute the lift and flutter characteristics for a design modification of the 747) through to fundamental physical problems involving interactions with the frontier of mathematics that need not be of any immediate "need" (is turbulence predictable from the Navier Stokes equations alone?).

The applications I shall speak about are of the fundamental kind involving current research in mathematics and basic questions in physics and engineering that are normally not considered "practical." Most, if not all, of the other lectures I have heard at this conference fall into the same category.

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I have heard many arguments about what is and what is not "applied" mathematics, and have seen rifts between individuals and whole departments over this issue. For example, to some, general relativity is not applied mathematics, but quantum mechanics is. To others, even the most abstract continuum mechanics or control theory is applied mathematics while functional analysis or differential geometry used in any subject disqualifies that endeavor from being applied. This would all be very humerous if individuals did not take it so seriously. The results described below are "applied" if the term is used in its broad sense.

Space does not allow for the presentation of an accurate historical picture of each problem, nor for a thorough citation of other approaches. Most of this can, however, be tracked down by consultation of the literature which is cited at the end of the paper.

1. Spaces of Solutions in Relativistic Field Theories*

1.1 Vacuum Gravity

A *spacetime* is a four dimensional manifold V together with a pseudo-Riemannian tensor field g of signature $(+, +, +, -)$. Let $\mathrm{Riem}(g)$ denote the Riemannian–Christoffel curvature tensor computed from g. Relative to a chosen basis in the tangent space to V at a point $x \in V$, $\mathrm{Riem}(g)$ is given in terms of a four-index object denoted $R^{\alpha}{}_{\beta\gamma\delta}$. By contracting two indices, we construct the Ricci curvature $\mathrm{Ric}(g)$ (in coordinates $R_{\alpha\beta}$) and the scalar curvature $R(g)$ (in coordinates $R = R^{\alpha}{}_{\alpha}$. The Einstein tensor is defined by $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2}R(g)g$ (in coordinates, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$). The *Einstein equations* for vacuum gravity are simply that $\mathrm{Ein}(g) = 0$ (which is equivalent to $\mathrm{Ric}(g) = 0$).

Let V be fixed and let $\mathscr{E}$ be the set of *all* g's that satisfy the Einstein equations (plus some additional technical smoothness conditions). Let $g_0 \in \mathscr{E}$ be a given solution. We ask: what is the structure of $\mathscr{E}$ in the neighborhood of g_0?

There are two basic reasons why this question is asked. First of all, it is relevant to the problem of finding solutions to the Einstein equations in the form of a perturbation series:

$$g(\lambda) = g_0 + \lambda h_1 + \frac{\lambda^2}{2} h_2 + \cdots$$

where λ is a small parameter. If $g(\lambda)$ is to solve $\mathrm{Ein}(g(\lambda)) = 0$ identically in λ then clearly h_1 must satisfy the *linearized Einstein equations*:

$$D\,\mathrm{Ein}(g) \cdot h_1 = 0$$

* This section is based on joint work with J. Arms, A. Fischer and V. Moncrief.

where $D\,\mathrm{Ein}(g)$ is the derivative of the mapping $g \mapsto \mathrm{Ein}(g)$. For such a perturbation series to be possible, is it sufficient that h_1 satisfy the linearized Einstein equations? i.e., is h_1 necessarily a direction of *linearized stability*? We shall see that in general the answer is no, unless drastic additional conditions hold. The second reason why the structure of $\mathscr{E}$ is of interest is in the problem of quantization of the Einstein equations. Whether one quantizes by means of direct phase space techniques (due to Dirac, Segal, Souriau, and Kostant in various forms) or by Feynman path integrals, there will be difficulties near places where the space of classical solutions is such that the linearized theory is *not* a good approximation to the nonlinear theory.

For vacuum gravity, let us state the answer in a special case: suppose g_0 has a *compact* spacelike hypersurface $M \subset V$. (Technically, M should be a Cauchy surface and be deformable to a surface of constant mean curvature.) Let $\mathscr{S}_{g_0}$ be the Lie group of isometries of g_0 and let k be its dimension.

Theorem

(1) *If $k = 0$, then $\mathscr{E}$ is a smooth manifold in a neighborhood of g_0 with tangent space at g_0 given by the solutions of the linearized Einstein equations.*

(2) *If $k > 0$ then $\mathscr{E}$ is* not *a smooth manifold at g_0. A solution h_1 of the linearized equations is tangent to a curve in $\mathscr{E}$ if and only if h_1 is such that the Taub conserved quantities vanish; i.e., for every Killing field X for g_0,*

$$\int_M X \cdot [D^2\,\mathrm{Ein}(g_0) \cdot (h_1, h_1)] \cdot Z \, d\mu_M = 0$$

where Z is the unit normal to the hypersurface M, "$\cdot$" denotes contraction with respect to the metric g_0 and μ_M is the volume element on M.

All explicitly known solutions possess symmetries, so while (1) is "generic," (2) is what occurs in examples. This theorem gives a complete answer to the perturbation question: such a perturbation series is possible if and only if all the Taub quantities vanish.

Let us give a brief abstract indication of why such second order conditions should come in. Suppose X and Y are Banach spaces and $F: X \to Y$ is a smooth map. Suppose $F(X_0) = 0$ and $x(\lambda)$ is a curve with $x(0) = x_0$ and $F(x(\lambda)) \equiv 0$. Let $h_1 = x'(0)$ so by the chain rule $DF(x_0) \cdot h_1 = 0$. Now suppose $DF(x_0)$ is not surjective and in fact suppose there is a linear functional $l \in Y^*$ orthogonal to its range: $\langle l, DF(x_0) \cdot u \rangle = 0$ for all $u \in X$. By differentiating $F(x(\lambda)) = 0$ twice at $\lambda = 0$, we get

$$D^2F(x_0) \cdot (h_1, h_1) + DF(x_0) \cdot x''(0) = 0.$$

Applying l gives

$$\langle l, D^2F(x_0) \cdot (h_1, h_1) \rangle = 0$$

which are necessary second order conditions that must be satisfied by h_1.

It is by this general method that one arrives at the Taub conditions. The issue of whether or not these conditions are sufficient is much deeper, requiring extensive analysis and bifurcation theory (for $k = 1$, the Morse lemma is used, while for $k > 1$ the Kuranishi deformation theory is needed).

1.2 General Field Theories

Is the above phenomenon a peculiarity about vacuum gravity or is it part of a more general fact about relativistic field theories? The examples which have been and are being worked out suggest that the latter is the case. Good examples are the Yang–Mills equations for gauge theory, the Einstein–Dirac equations, the Einstein–Euler equations and super-gravity. In such examples there is a gauge group playing the role of the diffeomorphism group of spacetime for vacuum gravity. This gauge group acts on the fields; when it fixes a field, it is a *symmetry* for that field. The relationship between symmetries of a field and singularities in the space of solutions of the classical equations is then as it is for vacuum gravity.

For this program to carry through, one first writes the four dimensional equations as Hamiltonian evolution equations plus constraint equations by means of the $3 + 1$ procedures of Dirac. The constraint equations then must (1) be the Noether conserved quantities for the gauge group and (2) satisfy some technical ellipticity conditions. For (1) it may be necessary to shrink the gauge group somewhat, especially for spacetimes that are not spatially compact. (For example, the isometries of Monkowski space do not belong to the gauge group generating the constraints but rather they generate the total energy-momentum vector of the spacetime.)

1.3 Momentum Maps

The role of the constraint equations as the zero set of the Noether conserved quantity of the gauge group leads one to investigate zero sets of the conserved quantities associated with symmetry groups rather generally. This topic is of interest not only in relativistic field theories, but in classical mechanics too. For example the set of points in the phase space for n particles in $\mathbb{R}^3$ corresponding to zero total angular momentum in an interesting and complicated set, even for $n = 2$!

We shall present just a hint of the relationship between singularities and symmetries. The full story is a long one; one finally ends up with an answer similar to that in relativity.

First we need a bit of notation. Let M be a manifold and let a Lie group G act on M. Associated to each element ξ in the Lie algebra $\mathfrak{g}$ of G, we have a vector field ξ_M naturally induced on M. We shall denote the action by

$\Phi: G \times M \to M$ and we shall write $\Phi_g: M \to M$ for the transformation of M associated with the group element $g \in G$. Thus

$$\xi_M(x) = \frac{d}{dt}\Phi_{\exp(t\xi)}(x)\Big|_{t=0}.$$

Now let (P, ω) be a symplectic manifold, so ω is a closed nondegenerate two-form on P and let Φ be an action of a Lie group G on P. Assume the action is symplectic, i.e., $\Phi_g^*\omega = \omega$ for all $g \in G$. A *momentum mapping* is a smooth mapping $J: P \to \mathfrak{g}^*$ such that

$$\langle dJ(x) \cdot v_x, \xi \rangle = \omega_x(\xi_P(x), v_x)$$

for all $\xi \in \mathfrak{g}$, $v_x \in T_x P$ where $dJ(x)$ is the derivative of J at x, regarded as a linear map of T_xP to $\mathfrak{g}^*$ and $\langle\, ,\rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

A momentum map is *Ad^*-equivariant* when the following diagram commutes for each $g \in G$:

$$\begin{array}{ccc} P & \xrightarrow{\Phi_g} & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathfrak{g}^* & \xrightarrow[Ad^*_{g^{-1}}]{} & \mathfrak{g}^* \end{array}$$

where $Ad^*_{g^{-1}}$ denotes the co-adjoint action of G on $\mathfrak{g}^*$. If J is Ad^* equivariant, we call (P, ω, G, J) a *Hamiltonian G-space.*

Momentum maps represent the (Noether) conserved quantities associated with symmetry groups on phase space. This topic is of course a very old one, but it is only with more recent work of Souriau and Kostant that a deeper understanding has been achieved.

Let $\mathscr{S}_{x_0}$ = (the component of the identity of) $\{g \in G \mid gx_0 = x_0\}$, called the symmetry group of x_0. Its Lie algebra is denoted $\mathfrak{s}_x$, so

$$\mathfrak{s}_{x_0} = \{\xi \in \mathfrak{g} \mid \xi_P(x_0) = 0\}.$$

Let (P, ω, G, J) be a Hamiltonian G-space. If $x_0 \in P$, $\mu_0 = J(x_0)$ and if

$$dJ(x_0)\colon T_x P \to \mathfrak{g}^*$$

is surjective (with split kernel), then locally $J^{-1}(\mu_0)$ is a manifold and $\{J^{-1}(\mu) \mid \mu \in \mathfrak{g}^*\}$ forms a regular local foliation of a neighborhood of x_0. Thus, when $dJ(x_0)$ fails to be surjective, the set of solutions of $J(x) = 0$ could fail to be a manifold.

Theorem. *$dJ(x_0)$ is surjective if and only if* $\dim \mathscr{S}_{x_0} = 0$; *i.e.*, $\mathfrak{s}_{x_0} = \{0\}$.

Proof. $dJ(x_0)$ fails to be surjective iff there is a $\xi \neq 0$ such that $\langle dJ(x_0) \cdot v_{x_0}, \xi\rangle = 0$ for all $v_{x_0} \in T_{x_0} P$. From the definition of momentum map, this is equivalent to $\omega_{x_0}(\xi_P(x_0), v_{x_0}) = 0$ for all v_{x_0}. Since ω_{x_0} is non-degenerate, this is, in turn, equivalent to $\xi_P(x_0) = 0$; i.e., $\mathfrak{s}_{x_0} \neq \{0\}$. □

One then goes on to study the structure of $J^{-1}(\mu_0)$ when x_0 *has* symmetries, by investigating second order conditions and using methods of bifurcation theory. It turns out that, as in relativistic field theories, $J^{-1}(\mu_0)$ has quadratic singularities characterized by the vanishing of second order conditions. The connection is not an accident since the structure of the space of solutions of a relativistic field theory is determined by the vanishing of the momentum map associated with the gauge group of that theory.

2. The Traction Problem in Nonlinear Elasticity*

2.1 Terminology from Elasticity

Let $\mathscr{B} \subset \mathbb{R}^3$ be an open set with smooth boundary. We regard $\mathscr{B}$ as a reference state for an elastic body. A *configuration* or *deformation* of $\mathscr{B}$ is a (smooth) embedding $\phi: \mathscr{B} \to \mathbb{R}^3$. Let $\mathscr{C}$ denote all such ϕ's. The derivative of ϕ is denoted $F = D\phi$ and is called the *deformation gradient*. The body's elastic properties are characterized by a *stored energy function*, a function W of $X \in \mathscr{B}$ and 3×3 matrices. Thus, given $\phi \in \mathscr{C}$, we get a function of X by the composition $W(X, F(X))$. The (*first*) *Piola–Kirchhoff stress tensor* is defined by $T = \partial W/\partial F$, the derivative with respect to the second argument of W. We shall assume that the undeformed state is stress-free; i.e., $T = 0$ when $\phi =$ identity.

Let $B: \mathscr{B} \to \mathbb{R}^3$ denote a given *body force* (per unit volume) and $\tau: \partial\mathscr{B} \to \mathbb{R}^3$ a given *surface traction* (per unit area). The equilibrium equations for ϕ we shall study are

$$\begin{aligned} \mathrm{DIV}\, T + B &= 0 \quad \text{in } \mathscr{B} \\ T \cdot N &= \tau \quad \text{on } \partial\mathscr{B}. \end{aligned} \tag{E}$$

These equations are equivalent to finding the critical points in $\mathscr{C}$ of the energy:

$$V(\phi) = \int_{\mathscr{B}} W\, dV + \int_{\mathscr{B}} \phi \cdot B\, dV + \int_{\partial\mathscr{B}} \phi \cdot \tau\, dA.$$

Let $\mathscr{L}$ be the space of pairs $l = (B, \tau)$ of loads such that

$$\int_{\mathscr{B}} B(X)\, dV(X) + \int_{\partial\mathscr{B}} \tau(X)\, dA(X) = 0,$$

i.e., the total force is zero. By the divergence theorem, if l is a set of loads satisfying the equilibrium equations for some ϕ, then $l \in \mathscr{L}$.

* Based on joint work with D. R. J. Chillingworth and Y. H. Wan.

2.2 Discussion of the Traction Problem

If we were studying the displacement problem i.e., the boundary condition was ϕ prescribed on $\partial\mathscr{B}$, it would follow directly from the implicit function theorem that for any B near zero, there would be a unique ϕ near the identity satisfying the equilibrium equations. For the traction problem the kernel of the linearized equations consists of infinitesimal rigid body motions and the implicit function theorem fails. In fact, the solution set bifurcates near the identity and the geometry of the rotation group $SO(3)$ plays a crucial role. We can trivially remove the translations by specifying the image of a given point in $\mathscr{B}$, say $\phi(0) = 0$.

Our problem is to study the solutions of the equations (E) for various l. The methods by which we do this are those of bifurcation theory and singularity theory. Interestingly, the solutions, even for small l, can be as complex as those for the buckling of a plate with 9 or more nearby solutions.

That there are such difficulties with the traction problem was noticed in the 1930's by Signorini. The problem has been extensively studied by the Italian school, especially by Stoppelli. However, their analysis missed solutions because the methods used are not "robust;" i.e., they did not allow the loads to move in full neighborhoods (they did not include enough parameters). Moreover, others were missed because the global geometry of $SO(3)$ was not exploited. Finally, the stability of the various solutions was not obtained.

We shall give just a hint of our methods by sketching a new and much simplified proof of a theorem of Stoppelli in case there is no bifurcation.

Let $\Phi: \mathscr{C} \to \mathscr{L}$ be defined by

$$\Phi(\phi) = (-\mathrm{DIV}\, T, T \cdot N)$$

so the equilibrium equations are $\Phi(\phi) = l$.

Let

$$\mathscr{C}_l = \{u \in T_{id}\mathscr{C} \mid u(0) = 0 \quad \text{and} \quad Du(0) \text{ is symmetric}\}$$

and let the *equilibrated loads* be those whose torque in the reference configuration is zero, i.e.,

$$\mathscr{L}_e = \left\{ l \in \mathscr{L} \,\middle|\, \int_{\Omega} X \times B(X)\, dV(X) + \int_{\partial\Omega} X \times \tau(X)\, dA(X) = 0 \right\}.$$

Assuming the appropriate ellipticity conditions from linear elasticity, we know that

$$D\Phi(id)|_{\mathscr{C}_l}: \mathscr{C}_l \to \mathscr{L}_e$$

is an isomorphism.

Let $SO(3)$ act on $\mathscr{C}$ and $\mathscr{L}$ in the obvious way: For $Q \in SO(3)$, $\phi \in \mathscr{C}$ and $l \in \mathscr{L}$, let

$$(Q, \phi) \mapsto Q \circ \phi \qquad \text{and} \qquad (Q, l) \mapsto (Q \circ \beta, Q \circ \tau).$$

For $l \in \mathscr{L}$, let $\mathcal{O}_l$ denote the $SO(3)$ orbit of l:

$$\mathcal{O}_l = \{Ql \mid Q \in SO(3)\}.$$

Let $l \in \mathscr{L}_e$. Then l is said to have *no axis of equilibrium* if, for all $\xi \in SO(3)$, $\xi \neq 0$ we have

$$\xi l \notin \mathscr{L}_e,$$

i.e., any rotation of l destroys the equilibration. If l has an axis of equilibrium, then there is a vector $e \in \mathbb{R}^3$ such that rotations of l about e map l into $\mathscr{L}_e$, as is readily checked.

Lemma (Da Silva's Theorem). *Let $l \in \mathscr{L}$. Then $\mathcal{O}_l \cap \mathscr{L}_e \neq \varnothing$.*

PROOF. Define the *astatic load map* $k: \mathscr{L} \to M_3$(3×3 matrices) by

$$k(l) = K(B, \tau) = \int_\Omega X \otimes B(X)\, dV(X) + \int_{\partial\Omega} X \otimes \tau(X)\, dA(X)$$

so that $l \in \mathscr{L}_e$ iff $k(l)$ is symmetric. Now k is $SO(3)$ equivariant:

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{k} & M_3 \\ {\scriptstyle SO(3)}\downarrow & & \downarrow{\scriptstyle SO(3)} \\ \mathscr{L} & \xrightarrow{k} & M_3 \end{array}$$

where the action on M_3 is $(Q, A) \mapsto AQ^{-1}$, i.e.,

$$k(Ql) = k(l)Q^{-1}.$$

The result is now obvious from the polar decomposition. □

We also assume that Φ is equivariant (called *material frame indifference*):

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\Phi} & \mathscr{L} \\ {\scriptstyle SO(3)}\downarrow & & \downarrow{\scriptstyle SO(3)} \\ \mathscr{C} & \xrightarrow[\Phi]{} & \mathscr{L} \end{array}$$

Thus, to study the solutions of $\Phi(\phi) = l$ for a given l, we can assume that $l \in \mathscr{L}_e$.

2.3 A Proof of Existence and Uniqueness in the Simplest Case

Suppose now that $l \in \mathscr{L}_e$ is given and has no axis of equilibrium. The main theorem in this case is due to Stoppelli which we now prove.

Lemma (a) $\dim \mathcal{O}_l = 3$ *and* (b) $T_l\mathcal{O}_l \oplus \mathscr{L}_e = \mathscr{L}$.

PROOF. If $\dim \mathcal{O}_l < 3$, there would be a $\xi \neq 0$, $\xi \in SO(3)$ such that $\xi l = 0$, which contradicts $\xi l \notin \mathscr{L}_e$. Thus (a) holds. Also, by the no axis of equilibrium assumption, $T_l\mathcal{O}_l \cap \mathscr{L}_e = \{0\}$. Since $\mathscr{L}_e$ has codimension 3 in $\mathscr{L}$ and (a) holds, we get (b). □

Let $\tilde{\Phi}$ be the restriction of Φ to $\mathscr{C}_l$, regarded as an affine subspace of $\mathscr{C}$ centered at the identity. As remarked before,

$$D\tilde{\Phi}(id): \mathscr{C}_l \to \mathscr{L}_e$$

is an isomorphism. In particular, it is one to one and so for ϕ in a neighborhood of the identity

$$\text{Range } \tilde{\Phi} \equiv N$$

is a submanifold of $\mathscr{L}$ tangent to $\mathscr{L}_e$ at the origin (see Figure 1). By the above lemma,

$$\{Ql \mid Q \in \text{a neighborhood } U \text{ of } Id \in SO(3)\}$$

is a neighborhood of l in the normal direction to $\mathscr{L}_e$. Thus

$$\{\lambda Ql \mid Q \in U, \lambda \in (-\varepsilon, \varepsilon)\}$$

is the cone in the normal bundle to $\mathscr{L}_e$.

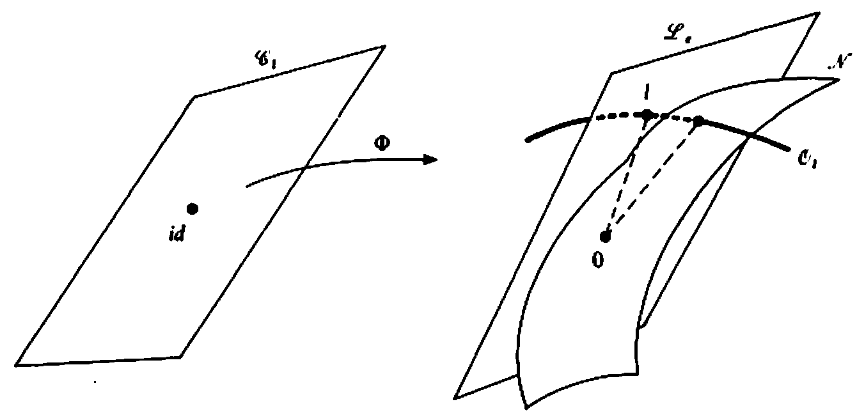

Figure 1 The Geometry of Stoppelli's Theorem

Since N is tangent to $\mathscr{L}_e$ at 0, for λ sufficiently small $\mathcal{O}_{\lambda l}$ will intersect N. Thus, for λ sufficiently small, there is a unique Q in a neighborhood of the Identity such that

$$\Phi(\bar{\phi}) = \lambda Ql$$

has a unique solution $\bar{\phi} \in \mathscr{C}_l$. Thus $\phi = Q^{-1}\bar{\phi}$ solves $\Phi(\phi) = \lambda l$. Thus we have proved:

Theorem (Stoppelli). *Suppose $l \in \mathscr{L}_e$ has no axis of equilibrium. Then for λ*

sufficiently small, there is a unique $\bar{\phi} \in \mathscr{C}_t$ and Q in a neighborhood of the identity such that $\phi = Q^{-1}\bar{\phi}$ solves the traction problem:

$$\Phi(\phi) = \lambda l.$$

2.4 Discussion of the General Case

The main problem is to study the situation when l is near a load l_0 with an axis of equilibrium. To do so one must first classify how degenerate the axis of equilibrium is. This is done by classifying how the orbits of the action of $SO(3)$ on M_3 meet Sym, the symmetric matrices. There are five such types. For example, if $A \in M_3$ has no axis of equilibrium and has distinct eigenvalues, then $\mathcal{O}_A$ meets Sym transversally in four points (Type 0). If A, however, has no axis of equilibrium and two equal non-zero eigenvalues, $\mathcal{O}_A$ meets Sym transversally in two points (with no axis of equilibrium) and a circle each point of which has an axis of equilibrium (Type 1). If A has a triple non-zero eigenvalue, $\mathcal{O}_A$ meets Sym transversally in one point (A itself) and in an $\mathbb{RP}^2$, each point of which has a circle of axes of equilibrium (Type 2). There are also the more degenerate types 3 and 4.

When the Liapunov–Schmidt procedure from bifurcation theory is applied to this situation, one ends up with a bifurcation problem of vector fields on S^1 for type 1 and of vector fields on $\mathbb{RP}^2$ for type 2. These can then be analyzed by singularity theory and one finds cusps and double cusps respectively. Previously, the best that was known was due to Stoppelli: he saw only particular sections of the cusps in type 1 and did not analyze type 2.

3. Chaotic Oscillations of a Forced Beam*

The study of chaotic motion in dynamical systems is now a burgeoning industry. The literature is currently in a state of explosion. We shall sketch an example from structural mechanics for which one can prove that the associated dynamical system has complex dynamics. Part of the interest is that methods of ordinary differential equations can be made to work for a certain class of partial differential equations.

We shall state the result for the main example first and then sketch the abstract theory which is used for the proof.

3.1 The Main Example

Consider a beam that is buckled by an external load Γ, so that there are two stable and one unstable equilibrium states (see Figure 2). The whole structure is shaken with a transverse periodic displacement, $f \cos \omega t$, and the

* Based on joint work with P. Holmes

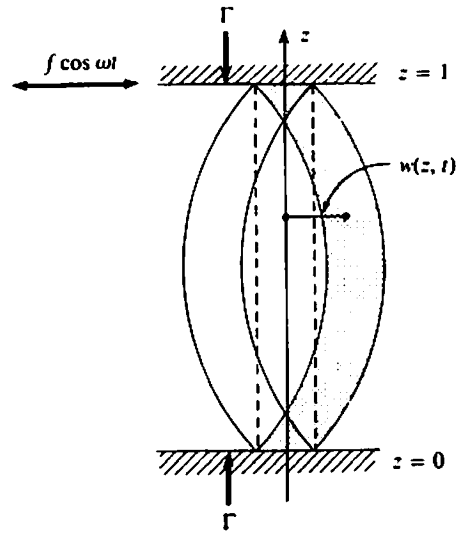

Figure 2 The forced, buckled beam

beam moves due to its inertia. One observes periodic motion about either of the two stable equilibria for small f, but as f is increased, the motion becomes aperiodic or chaotic.

A specific model for the transverse deflection $w(z, t)$ of the centerline of the beam is the following partial differential equation:

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 [w']^2 \, d\zeta\right) w'' = \varepsilon(f \cos \omega t - \delta \dot{w}) \qquad (1)$$

where $\cdot = \partial/\partial t$, $' = \partial/\partial z$, Γ = external load, κ = stiffness due to "membrane" effects, δ = damping, and ε is a parameter used to measure the size of f and δ. Amongst many possible boundary conditions we shall choose $w = w'' = 0$ at $z = 0, 1$, i.e., simply supported, or hinged ends. With these boundary conditions, the eigenvalues of the linearized, unforced equations, i.e., complex numbers λ such that

$$\lambda^2 w + w'''' + \Gamma w'' = 0$$

for some non-zero w satisfying $w = w'' = 0$ at $z = 0, 1$, form a countable set

$$\lambda_j = \pm \pi j \sqrt{\Gamma - \pi^2 j^2}, \qquad j = 1, 2, \ldots.$$

Assume that

$$\pi^2 < \Gamma < 4\pi^2,$$

in which case the solution $w = 0$ is unstable with one positive and one negative eigenvalue and the nonlinear equation (1) with $\varepsilon = 0$, $\kappa > 0$ has two nontrivial stable buckled equilibrium states.

A simplified model for the dynamics of (1) is obtained by seeking lowest mode solutions of the form

$$w(z, t) = x(t)\sin(\pi z).$$

Substitution into (1) and taking the inner product with the basis function $\sin(\pi z)$ gives a Duffing type equation for the modal displacement $x(t)$:

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma \cos \omega t - \delta \dot{x}), \tag{2}$$

where $\beta = \pi^2(\Gamma - \pi^2) > 0$, $\alpha = \kappa\pi^4/2$ and $\gamma = 4f/\pi$.

Further assumptions we make on (1) are as follows:

(1) (No resonance): $j^2\pi^2(j^2\pi^2 - \Gamma) \neq \omega^2, j = 2, 3, 4, \ldots$.
(2) (Large forcing to damping ratio):

$$\frac{f}{\delta} > \left(\frac{\pi}{3}\frac{\Gamma - \pi^2}{\omega\sqrt{k}}\right) \cosh\left(\frac{\omega}{2\sqrt{\Gamma - \pi^2}}\right).$$

(3) (Small forcing and damping): ε is sufficiently small.

On an appropriate function space X, one shows that (1) has well-defined dynamics; elements of X are certain pairs $(w, \dot{w})$. In particular, there is a time $2\pi/\omega$ map $P: X \to X$ that takes initial data and advances it in time by one period of the forcing function.

Theorem. *Under the above hypotheses, there is some power P^N of P that has an invariant set $\Lambda \subset X$ on which P^N is conjugate to a shift on two symbols. In particular,* (1) *has infinitely many periodic orbits with arbitrarily high period.*

This set Λ arises in a way similar to Smale's famous "horseshoe."

3.2 Abstract Hypotheses

We consider an evolution equation in a Banach space X of the form

$$\dot{x} = f_0(x) + \varepsilon f_1(x, t) \tag{3}$$

where f_1 is periodic of period T in t. Our hypotheses on (3) are as follows.

(H1) (a) *Assume $f_0(x) = Ax + B(x)$ where A is an (unbounded) linear operator which generates a C^0 one parameter group of transformations on X and where $B: X \to X$ is C^∞. Assume that $B(0) = 0$ and $DB(0) = 0$.*

(b) *Assume $f_1: X \times S^1 \to X$ is C^∞ where $S^1 = \mathbb{R}/(T)$, the circle of length T.*

Assumption 1 implies that the associated suspended autonomous system on $X \times S^1$,

$$\begin{aligned} \dot{x} &= f_0(x) + \varepsilon f_1(x, \theta) \\ \dot{\theta} &= 1, \end{aligned} \tag{4}$$

has a smooth local flow, F_t^ε. This means that $F_t^\varepsilon: X \times S^1 \to X \times S^1$ is a smooth map defined for small $|t|$ which is jointly continuous in all variables $\varepsilon, t, x \in X, \theta \in S^1$ and for x_0 in the domain of A, $t \mapsto F_t^\varepsilon(x_0, \theta_0)$ is the unique solution of (4) with initial condition x_0, θ_0.

The final part of assumption 1 follows:

(c) *Assume that F_t^ε is defined for all $t \in \mathbb{R}$ for $\varepsilon > 0$ sufficiently small.*

Our second assumption is that the unperturbed system is Hamiltonian. This means that X carries a skew symmetric continuous bilinear map Ω: $X \times X \to \mathbb{R}$ which is weakly non-degenerate (i.e., $\Omega(u, v) = 0$ for all v implies $u = 0$) called the *symplectic form* and there is a smooth function $H_0: X \to \mathbb{R}$ such that

$$\Omega(f_0(x), u) = dH_0(x) \cdot u$$

for all x in D_A, the domain of A.

(H2) (a) *Assume that the unperturbed system $\dot{x} = f_0(x)$ is Hamiltonian with energy $H_0: X \to \mathbb{R}$.*

(b) *Assume there is a symplectic 2-manifold $\Sigma \subset X$ invariant under the flow F_t^0 and that on Σ the fixed point $p_0 = 0$ has a homoclinic orbit $x_0(t)$, i.e.,*

$$\dot{x}_0(t) = f_0(x_0(t))$$

and

$$\lim_{t \to +\infty} x_0(t) = \lim_{t \to -\infty} x_0(t) = 0.$$

Next we introduce a non-resonance hypothesis.

(H3) (a) *Assume that the forcing term $f_1(x, t)$ in* (3) *has the form*

$$f_1(x, t) = A_1 x + f(t) + g(x, t) \tag{5}$$

where $A_1: X \to X$ is a bounded linear operator, f is periodic with period T, $g(x, t)$ is t-periodic with period T and satisfies $g(0, t) = 0$, $D_x g(0, t) = 0$, so g admits the estimate

$$\|g(x, t)\| \leq (\text{Const})\|x\|^2 \tag{6}$$

for x in a neighborhood of 0.

(b) *Suppose that the "linearized" system*

$$\dot{x}_L = Ax_L + \varepsilon A_1 x_L + \varepsilon f(t) \tag{7}$$

has a T-periodic solution $x_L(t, \varepsilon)$ such that $x_L(t, \varepsilon) = O(\varepsilon)$.

For finite dimensional systems, (H3) can be replaced by the assumption that 1 does not lie in the spectrum of e^{TA}; i.e., none of the eigenvalues of A resonate with the forcing frequency.

Next, we need an assumption that A_1 contributes positive damping and that $p_0 = 0$ is a saddle.

(H4) (a) *For $\varepsilon = 0$, e^{TA} has a spectrum consisting of two simple real eigenvalues $e^{\pm\lambda T}$, $\lambda \neq 0$, with the rest of the spectrum on the unit circle.*

(b) *For $\varepsilon > 0$, $e^{T(A+\varepsilon A_1)}$ has a spectrum consisting of two simple real eigenvalues $e^{T\lambda_\varepsilon^\pm}$ (varying continuously in ε from perturbation theory of spectra) with the rest of the spectrum, σ_R^ε, inside the unit circle $|z| = 1$ and obeying the estimates*

$$C_2\varepsilon \leq \text{distance } (\sigma_R^\varepsilon, |z| = 1) \leq C_1\varepsilon \tag{8}$$

for C_1, C_2 positive constants.

Finally, we need an extra hypothesis on the nonlinear term. We have already assumed B vanishes at least quadratically as does g. Now we assume B vanishes cubically.

(H5) $B(0) = 0$, $DB(0) = 0$, *and* $D^2B(0) = 0$.

This means that in a neighborhood of 0,

$$\|B(x)\| \leq \text{Const } \|x\|^3$$

(actually, $B(x) = o(\|x\|^2)$ would do).

3.3 Some Technical Lemmas

Consider the suspended system (4) with its flow $F_t^\varepsilon: X \times S^1 \to X \times S^1$. Let $P^\varepsilon: X \to X$ be defined by

$$P^\varepsilon(x) = \pi_1 \cdot (F_T^\varepsilon(x, 0))$$

where $\pi_1: X \times S^1 \to X$ is the projection onto the first factor. The map P^ε is just the Poincaré map for the flow F_t^ε. Note that $P^0(p_0) = p_0$, and that fixed points of P^ε correspond to periodic orbits of F_t^ε.

Lemma 1. *For $\varepsilon > 0$ small, there is a unique fixed point p_ε of P^ε near $p_0 = 0$; moreover $p_\varepsilon - p_0 = O(\varepsilon)$, i.e., there is a constant K such that $\|p_\varepsilon\| \leq K\varepsilon$ (for all (small) ε).*

For ordinary differential equations, Lemma 1 is a standard fact about persistence of fixed points, assuming 1 does not lie in the spectrum of e^{TA} (i.e., p_0 is hyperbolic). For general partial differential equations, the proof is similar in spirit but is more delicate, requiring our assumptions. An analysis of the spectrum yields the following.

Lemma 2. *For $\varepsilon > 0$ sufficiently small, the spectrum of $DP^\varepsilon(p_\varepsilon)$ lies strictly inside the unit circle with the exception of the single real eigenvalue $e^{T\lambda_\varepsilon^+} > 1$.*

The next lemma deals with invariant manifolds.

Lemma 3. *Corresponding to the eigenvalues $e^{T\lambda_\varepsilon^\pm}$ there are unique invariant manifolds $W^{ss}(p_\varepsilon)$ (the strong stable manifold) and $W^u(p_\varepsilon)$ (the unstable manifold) of p_ε for the map p_ε such that*

(i) *$W^{ss}(p_\varepsilon)$ and $W^u(p_\varepsilon)$ are tangent to the eigenspaces of $e^{T\lambda_\varepsilon^\pm}$, respectively, at p_ε;*
(ii) *they are invariant under P^ε;*
(iii) *if $x \in W^{ss}(p_\varepsilon)$ then*

$$\lim_{n\to\infty}(P^\varepsilon)^n(x) = p_\varepsilon$$

and if $x \in W^u(p_\varepsilon)$ then

$$\lim_{n\to-\infty}(P^\varepsilon)^n(n) = p_\varepsilon;$$

(iv) *for any finite t^*, $W^{ss}(p_\varepsilon)$ is C^r close as $\varepsilon \to 0$ to the homoclinic orbit $x_0(t)$, $t^* \leq t < \infty$ and for any finite t_*, $W^u(p_\varepsilon)$ is C^r close to $x_0(t)$, $-\infty < t \leq t_*$ as $\varepsilon \to 0$ (here, r is any fixed integer, $0 \leq r < \infty$).*

The Poincaré map P^ε was associated to the section $X \times \{0\}$ in $X \times S^1$. Equally well, we can take the section $X \times \{t_0\}$ to get Poincaré maps $P_{t_0}^\varepsilon$. By definition,

$$P_{t_0}^\varepsilon(x) = \pi_1(F_T^\varepsilon(x, t_0)).$$

There is an analogue of Lemmas 1, 2, and 3 for $P_{t_0}^\varepsilon$. Let $p_\varepsilon(t_0)$ denote its unique fixed point and $W_\varepsilon^{ss}(p_\varepsilon(t_0))$ and $W_\varepsilon^u(p_\varepsilon(t_0))$ be its strong stable and unstable manifolds. Lemma 2 implies that the stable manifold $W^s(p_\varepsilon)$ of p_ε has codimension 1 in X. The same is then true of $W^s(p_\varepsilon(t_0))$ as well.

Let $\gamma_\varepsilon(t)$ denote the periodic orbit of the (suspended) system (4) with $\gamma_\varepsilon(0) = (p_\varepsilon, 0)$. We have

$$\gamma_\varepsilon(t) = (p_\varepsilon(t), t).$$

The invariant manifolds for the periodic orbit γ_ε are denoted $W_\varepsilon^{ss}(\gamma_\varepsilon)$, $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$. We have

$$W_\varepsilon^s(p_\varepsilon(t_0)) = W_\varepsilon^s(\gamma_\varepsilon) \cap (X \times \{t_0\})$$
$$W_\varepsilon^{ss}(p_\varepsilon(t_0)) = W_\varepsilon^{ss}(\gamma_\varepsilon) \cap (X \times \{t_0\})$$

and

$$W_\varepsilon^u(p_\varepsilon(t_0)) = W_\varepsilon^u(\gamma_\varepsilon) \cap (X \times \{t_0\}).$$

We wish to study the structure of $W_\varepsilon^u(p_\varepsilon(t_0))$ and $W_\varepsilon^s(p_\varepsilon(t_0))$ and their intersections. To do this, we first study the perturbation of solution curves in $W_\varepsilon^{ss}(\gamma_\varepsilon)$, $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$.

Choose a point, say $x_0(0)$, on the homoclinic orbit for the unperturbed system. Choose a codimension 1 hyperplane H transverse to the homoclinic orbit at $x_0(0)$. Since $W_\varepsilon^{ss}(p_\varepsilon(t_0))$ is C^r close to $x_0(0)$, it intersects H in a unique point, say $x_\varepsilon^s(t_0, t_0)$. Define $(x_\varepsilon^s(t, t_0), t)$ to be the unique integral curve of the suspended system (4) with initial condition $x_\varepsilon^s(t_0, t_0)$. Define $x_\varepsilon^u(t, t_0)$ in a similar way. We have

$$x_\varepsilon^s(t_0, t_0) = x_0(0) + \varepsilon v^s + O(\varepsilon^2)$$

and

$$x_\varepsilon^u(t_0, t_0) = x_0(0) + \varepsilon v^u + O(\varepsilon^2)$$

by construction, where $\|0(\varepsilon^2)\| \leq \text{Constant} \cdot \varepsilon^2$ and v^s and v^u are fixed vectors. Notice that

$$(P_{t_0}^\varepsilon)^n x_\varepsilon^s(t_0, t_0) = x_\varepsilon^s(t_0 + nT, t_0) \to p_\varepsilon(t_0) \quad \text{as} \quad n \to \infty.$$

Since $x_\varepsilon^s(t, t_0)$ is an integral curve of a perturbation, we can write

$$x_\varepsilon^s(t, t_0) = x_0(t - t_0) + \varepsilon x_1^s(t, t_0) + O(\varepsilon^2),$$

where $x_1^s(t, t_0)$ is the solution of the first variation equation

$$\frac{d}{dt} x_1^s(t, t_0) = Df_0(x_0(t - t_0)) \cdot x_1^s(t, t_0) + f_1(x_0(t - t_0), t), \tag{9}$$

with $x_1^s(t_0, t_0) = v^s$.

3.4 The Melnikov Function

Define the *Melnikov function* by

$$\Delta_\varepsilon(t, t_0) = \Omega(f_0(x_0(t - t_0)), x_\varepsilon^s(t, t_0) - x_\varepsilon^u(t, t_0))$$

and set

$$\Delta_\varepsilon(t_0) = \Delta_\varepsilon(t_0, t_0).$$

Lemma 4. *If ε is sufficiently small and $\Delta_\varepsilon(t_0)$ has a simple zero at some t_0 and maxima and minima that are at least $O(\varepsilon)$, then $W_\varepsilon^u(p_\varepsilon(t_0))$ and $W_\varepsilon^s(p_\varepsilon(t_0))$ intersect transversally near $x_0(0)$.*

The idea is that if $\Delta_\varepsilon(t_0)$ changes sign, then $x_\varepsilon^s(t_0, t_0) - x_\varepsilon^u(t_0, t_0)$ changes orientation relative to $f_0(x_0(0))$. Indeed, this is what symplectic forms measure. If this is the case, then as t_0 increases, $x_\varepsilon^s(t_0, t_0)$ and $x_\varepsilon^u(t_0, t_0)$ "cross," producing the transversal intersection.

The next lemma gives a remarkable formula that enables one to explicitly compute the leading order terms in $\Delta_\varepsilon(t_0)$ in examples.

Lemma 5. *The following formula holds:*

$$\Delta_\varepsilon(t_0) = -\varepsilon \int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt + O(\varepsilon^2).$$

PROOF. Write $\Delta_\varepsilon(t, t_0) = \Delta_\varepsilon^+(t, t_0) - \Delta_\varepsilon^-(t, t_0) + O(\varepsilon^2)$, where

$$\Delta_\varepsilon^+(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0))$$

and

$$\Delta_\varepsilon^-(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon x_1^u(t, t_0)).$$

Using (9), we get

$$\frac{d}{dt} \Delta_\varepsilon^+(t, t_0) = \Omega(Df_0(x_0(t, t_0)) \cdot f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0))$$
$$+ \Omega(f_0(x_0(t - t_0)), \varepsilon\{Df_0(x_0(t - t_0)) \cdot x_1^s(t, t_0) + f_1(x_0(t - t_0), t)\}).$$

Since f_0 is Hamiltonian, Df_0 is Ω-skew. Therefore the terms involving x_1^s drop out, leaving

$$\frac{d}{dt} \Delta_\varepsilon^+(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon f_1(x_0(t - t_0), t)).$$

Integrating, we have

$$-\Delta_\varepsilon^+(t_0, t_0) = \varepsilon \int_{t_0}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt,$$

since

$$\Delta_\varepsilon^+(\infty, t_0) = \Omega(f_0(p_0), \varepsilon f_1(p_0, \infty)) = 0, \quad \text{because } f_0(p_0) = 0.$$

Similarly, we obtain

$$\Delta_\varepsilon^-(t_0, t_0) = \varepsilon \int_{-\infty}^{t_0} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt$$

and adding gives the stated formula. □

We summarize the situation as follows.

Theorem. *Let hypotheses* (H1)–(H5) *hold. Let*

$$M(t_0) = \int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt.$$

Suppose that $M(t_0)$ has a simple zero as a function of t_0. Then for $\varepsilon > 0$ sufficiently small, the stable manifold $W_\varepsilon^s(p_\varepsilon(t_0))$ of p_ε for $P_{t_0}^\varepsilon$ and the unstable manifold $W_\varepsilon^u(p_\varepsilon(t_0))$ intersect transversally.

Having established the transversal intersection of the stable and unstable manifolds, one can now plug into known results in dynamical systems (going back to Poincaré) to deduce that the dynamics must indeed be complex. In particular, the previous theorem concerning equation (1) may be deduced.

4. A Control Problem for a Beam*

We wish to point out some unexpected peculiarities in a seemingly straight forward control problem. In particular, the naive methods used for ordinary differential equations do not work for the partial differential equation we discuss. The difficulty has to do with controlling all the modes at once. If the energy norm is used, controllability is impossible. However, if a different asymptotic condition on the modes is used, control is possible.

4.1 The General Scheme for Controllability

Things will run smoothest if we treat the abstract situation first. We consider an evolution equation of the form

$$\dot{u}(t) = \mathscr{A}u(t) + p(t)\mathscr{B}(u(t)) \qquad (1)$$

where $\mathscr{A}$ generates a C^0 semigroup on a Banach space X, $p(t)$ is a real value function of t that is locally L^1, and $\mathscr{B}: X \to X$ is C^k, $k \geq 1$. The control question we ask is: let u_0 be given initial data for u and let $T > 0$ be given; does there exist a neighborhood U of $e^{\mathscr{A}T}u_0$ in X such that for any $v \in V$ there exists a p such that the solution of (1) with initial data u_0 reaches v after time T? If the answer is yes, we say (1) is *locally controllable* around the free solution $e^{\mathscr{A}t}u_0$.

The obvious way to tackle this problem is to use the implicit function theorem. Write (1) in integrated form:

$$u(t) = e^{\mathscr{A}t}u_0 + \int_0^t e^{(t-s)\mathscr{A}}p(x)\mathscr{B}(u(s))\,ds. \qquad (2)$$

Let p belong to a specified Banach space $Z \subset L^1([0, T], \mathbb{R})$. Standard techniques using the contraction mapping theorem show that for short time, (2) has a unique solution $u(t, p, u_0)$ that is C^k in p and u_0. If we assume $\|\mathscr{B}(x)\| \leq C + K\|x\|$ (for example, $\mathscr{B}$ linear will be of interest to us), then solutions are globally defined, so we do not need to worry about taking short time intervals. The choice $p = 0$ corresponds to the free solution $e^{t\mathscr{A}}u_0$. The derivative $L: Z \to X$ of $u(T, p, u_0)$ with respect to p at $p = 0$ is found by implicitly differentiating (2). One gets

$$Lp = \int_0^T e^{(t-s)\mathscr{A}}p(s)\mathscr{B}(e^{s\mathscr{A}}u_0)\,ds. \qquad (3)$$

The implicit function theorem then gives:

Theorem. *If $L: Z \to X$ is a surjective linear map, then (1) is locally controllable around the free solution.*

* Based on joint work with J. Ball and M. Slemrod.

For example, if $X = \mathbb{R}^n$ and $\mathscr{B}$ is linear, we can expand

$$e^{-s\mathscr{A}}\mathscr{B}e^{s\mathscr{A}} = \mathscr{B} + s[\mathscr{A}, \mathscr{B}] + \frac{s^2}{2}[\mathscr{A}, [\mathscr{A}, \mathscr{B}]] + \cdots$$

to recover the standard controllability criterion:

$$\dim \operatorname{span}\{\mathscr{B}u_0, [\mathscr{A}, \mathscr{B}]u_0, [\mathscr{A}, [\mathscr{A}, \mathscr{B}]]u_0, \ldots\} = n.$$

If one wishes to only observe a finite dimensional piece of u, the above method is effective in examples. (By this we mean to control Gu, where $G: X \to \mathbb{R}^n$ is a surjective linear map ... this means we control n "modes" of u.) However, even in the simplest examples, L may have dense range but not be onto. We give such an example below.

4.2 Hyperbolic Systems

Let A be a positive self-adjoint operator on a real Hilbert space H with inner product $\langle\ ,\ \rangle_H$. Let A have a spectrum consisting of eigenvalues λ_n^2, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ with corresponding orthonormalized eigenfunctions ϕ_n. Let $B: D(A^{1/2}) \to H$ be bounded. We consider the equation

$$\ddot{w} + Aw + pBw = 0.$$

This is in the form (1) with

$$u = \begin{pmatrix} w \\ \dot{w} \end{pmatrix}$$

and

$$\mathscr{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 0 & 0 \\ -B & 0 \end{pmatrix}.$$

Here $X = D(A^{1/2}) \times H$ and $\mathscr{A}$ generates a C^0 group of isometries on X. The inner product on X is given by the "energy inner product:"

$$\langle (y_1, z_1), (y_2, z_2) \rangle_X = \langle A^{1/2}y_1, A^{1/2}y_2 \rangle_H + \langle z_1, z_2 \rangle_H.$$

Write

$$u_0 = \begin{pmatrix} \sum_{m=1}^{\infty} b_m \phi_m \\ \sum_{m=1}^{\infty} -\lambda_m c_m \phi_m \end{pmatrix} \in X$$

where

$$\sum_{m=1}^{\infty} \lambda_m^2 (b_m^2 + c_m^2) < \infty.$$

If we set $a_m = \frac{1}{2}(b_m + ic_m)$ we have

$$e^{\mathscr{A}s}u_0 = \begin{pmatrix} \sum_{m=1}^{\infty} [a_m \exp(i\lambda_m s) + \bar{a}_m \exp(-i\lambda_m s)]\phi_m \\ \sum_{m=1}^{\infty} i\lambda_m[a_m \exp(i\lambda_m s) - \bar{a}_m \exp(-i\lambda_m s)]\phi_m \end{pmatrix}$$

and

$$\mathscr{B}e^{\mathscr{A}s}u_0 = \begin{pmatrix} 0 \\ \sum_{m=1}^{\infty} [a_m \exp(i\lambda_m s) + \bar{a}_m \exp(-i\lambda_m s)]B\phi_m \end{pmatrix}.$$

To simplify matters, let us assume that $\langle B\phi_m, \phi_n\rangle_H = d_m \delta_{mm}$. Then

$$e^{-s\mathscr{A}}\mathscr{B}e^{s\mathscr{A}}u_0 = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{-id_n}{2\lambda_n}\{a_n \exp(2i\lambda_n s) - a_n \exp(-2i\lambda_n s) - (a_n - \bar{a}_n)\}\phi_n \\ \sum_{n=1}^{\infty} -\frac{d_n}{2}\{a_n \exp(2i\lambda_n s) + \bar{a}_n \exp(-2i\lambda_n s) + (a_n + \bar{a}_n)\}\phi_n \end{pmatrix}. \tag{4}$$

This formula can now be inserted into (3) to give Lp in terms of the basis ϕ_n. Since it generates a group, surjectivity of L comes down to the solvability of

$$\hat{L}p = \int_0^T p(s)e^{-s\mathscr{A}}\mathscr{B}(e^{s\mathscr{A}}u_0)\, ds = h \tag{5}$$

for $p(s)$ given $h \in X$.

4.3 An Example

We consider a vibrating beam with hinged ends and an axial load $p(t)$ as a control:

$$\begin{aligned} w_{tt} + w_{xxxx} + p(t)w_{xx} &= 0, \qquad 0 \le x \le 1 \\ w = w_{xx} &= 0 \quad \text{at} \quad x = 0, 1. \end{aligned} \tag{6}$$

Here $\lambda_n = n^2\pi^2$, $\phi_n = (1/\sqrt{2})\sin(n\pi x)$ and $d_n = -n^2\pi^2$. We can seek to solve (5) for p by expanding p in a Fourier series. For example, take $T \ge 1/\pi$ and attempt to find p's on $[0, 1/\pi]$ by writing

$$p(s) = p_0 + \Sigma\{p_{n^2} \exp(2in^2\pi^2 s) + \bar{p}_{n^2} \exp(-2in^2\pi^2 s) \tag{7}$$

and suppressing the remaining coefficients. To do this it is natural to try choosing p's in L^2. Inserting (4) and (7) into (5), we can determine h. Note that $d_n/\lambda_n = -1$ and $\{a_n\lambda_n\} \in l_2$. If we write

$$h = \begin{pmatrix} \Sigma\alpha_m\phi_m \\ \Sigma - \lambda_m\beta_m\phi_m \end{pmatrix},$$

the condition for h to be in X is $\Sigma\lambda_m^2(\alpha_m^2 + \beta_m^2) < \infty$. But the condition for h to be in the range of $\hat{L}$ with an L^2 p is that $\{a_n d_n p_{n^2}\} \in l_2$. This is, however, a stronger condition than $h \in X$. Thus, we conclude that $\hat{L}$ and hence L has range that is dense in but not equal to X.

In fact, one can show that not only is L not surjective, but that (6) is *not* locally controllable in the energy norm.

To overcome this difficulty one can contemplate more sophisticated inverse function theorems, and indeed these may be necessary in general. However, for a class of equations that includes this example, a more naive trick works. Namely, instead of the X norm, make up a new space namely the range of $\hat{L}$ and use the graph norm. Miraculously, the solution $u(t, p, u_0)$ stays in this space and is still smooth in the new topology. In this stronger norm then, the implicit function theorem can still be used. The verification of these statements is somewhat lengthy, but in principle the method is straightforward.

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