The Absence of Killing Fields is Necessary for Linearization Stability of Einstein's Equations

JUDITH M. ARMS & JERROLD E. MARSDEN

1. Introduction. The purpose of this paper is to discuss the proofs of Theorems 1 and 2 below.

Theorem 1. *Let M be a compact, connected, oriented smooth manifold,* dim $M \geq 3$. *Let $\mathcal{M}$ be the space of C^∞[or $W^{s,p}$, $s > (n/p) + 1$] riemannian metrics on M, $\mathcal{F}$ the C^∞ [or $W^{s-2,p}$] scalar functions on $\mathcal{M}$ and $R : \mathcal{M} \to \mathcal{F}$ the scalar curvature map. Let $g_0 \in \mathcal{M}$ and $\rho_0 = R(g_0)$. Then the equation*

$$R(g) = \rho_0$$

is linearization stable at g_0 if and only if $DR(g_0)$ is surjective.

We recall (Fischer and Marsden [1973]) that a map $F : X \to Y$ is called *linearization stable* at $x_0 \in X$ if for every $v \in \ker DF(x_0)$, there is a C^1 curve $x(t)$ satisfying $F(x(t)) = y_0 = F(x_0)$, $x(0) = x_0$ and $x'(0) = v$.

The 'if' part of Theorem 1 is a consequence of the implicit function theorem. For this part of the theorem there is an important criterion of Bourguignon [1975], Fischer and Marsden [1975a] which states that $DR(g_0)$ is surjective if $\rho_0/(n-1)$ is not a constant in the spectrum of Δ_{g_0}, the Laplace-Beltrami operator of g_0.

It is suggested in Fischer and Marsden [1975a] and asserted in Bourguignon, Ebin and Marsden [1976] that the 'only if' part is true as well. The proof outlined is, however, incorrect. We shall point out the error and give the correct proof in §2.

To describe the analogous result in relativity we use the following notation. Let V_4 be a four-manifold and $M \subset V_4$ an embedded compact three-manifold. Let $\mathcal{L}$ denote the set of Lorentz metrics on V_4 of class C^∞ (or H^s, $s > (n/2) + 1$) for which M is a (spacelike) Cauchy surface, $S_2(V_4)$ the space of C^∞ (or H^{s-2}) symmetric two tensors on V_4 and

$$\text{Ein} : \mathcal{L} \to S_2$$

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be the Einstein tensor

$$\mathrm{Ein}({}^{(4)}g) = \mathrm{Ric}({}^{(4)}g) - \frac{1}{2} R({}^{(4)}g) \cdot {}^{(4)}g.$$

In the definition of linearization stability we use the C^∞ (or H^s) topology on compact subsets of V_4.

Theorem 2. Let $\mathrm{Ein}({}^{(4)}g_0) = 0$ *for* ${}^{(4)}g_0 \in \mathscr{L}$. *Then* ${}^{(4)}g_0$ *is linearization stable if and only if* ${}^{(4)}g_0$ *has no killing fields.*

The 'if' part of this theorem is due to Fischer and Marsden [1973] and Moncrief [1975]. The converse is suggested in Moncrief [1976] and Fischer and Marsden [1976]. We will prove the converse based on this suggestion in §3.

It is likely (Fischer, Marsden and Moncrief [1978]) that in the presence of Killing fields, the directions of linearization stability are precisely determined by a second order condition discussed below.

Theorem 2 can be generalized to gauge fields coupled to gravity. See Arms [1977 and 1979].

2. Linearization stability of the scalar curvature equation. The proofs are based on the following result of Bourguignon, Ebin and Marsden [1976].

Lemma 1. Let P be a pseudodifferential operator of order m from sections of a vector bundle E to sections of a vector bundle F over a manifold M. Suppose the principal symbol of P is surjective but not injective. Then if $U \subset M$ *is open, the set of* C^∞ *sections of E with support in U and lying in the kernel of P is infinite dimensional.*

The proof is based on elliptic estimates described in Hormander [1966]. It is important to note that the term 'principal symbol' is used in the sense of Douglis and Nirenberg, *i.e.* if the operator is a product, the top order term in each factor is used, not necessarily the top order term for the whole operator. See Hormander's paper for details.

A key step in the proof of Theorem 1 is the following:

Lemma 2. If $\dim M \geq 3$ *and* $U \subset M$ *is open, then* $\{h \in \ker DR(g) | h$ *has support in* $U\}$ *is infinite dimensional.*

We recall that

$$DR(g) \cdot h = \Delta \mathrm{tr}\, h + \delta\delta h - \mathrm{Ric}(g) : h$$

where $(\delta h)_i = -h_i{}^j{}_{|j}$ is the negative convariant divergence (with respect to g), $\mathrm{tr}\, h = h^i_i$, Δ is the Laplace-Beltrami operator, $\mathrm{Ric}(g)$ the Ricci curvature and : denotes the double contraction.

In Bourguignon, Ebin and Marsden [1976], the operator $P = \delta$ on the bundle of traceless two tensors h satisfying $\mathrm{Ric}(g) : h = 0$ was considered. The difficulty is that the symbol of P need not be surjective on this space. For example, if $\mathrm{Ric}(g) = \zeta \otimes \zeta$, the symbol of P, $\sigma_\xi(h) = -h^\# \cdot \xi = -h^{ij}\xi_j$ is not surjective on the h satisfying $\mathrm{Ric}(g) : h = h(\zeta, \zeta) = 0$ and $\mathrm{tr}\, h = 0$, since σ_ζ maps into vectors orthogonal to ζ. This difficulty is relatively minor.

Proof of Lemma 2. Let E be the bundle of symmetric two tensors and $F = TM \times \mathbb{R}$. Let

$$P(h) = (\delta h, \Delta \mathrm{tr}\, h - \mathrm{Ric}(g) : h).$$

The symbol of P is

$$\sigma_\xi(h) = (-h^\# \cdot \xi, -(\mathrm{tr}\, h)|\xi|^2)$$

which is surjective for $\xi \neq 0$ and $\dim M \geq 2$. However, for $\dim M \geq 3$, $\dim E > \dim F$, so σ_ξ cannot be injective. Since $\ker P \subset \ker DR(g)$, Lemma 2 follows from Lemma 1. ∎

Using this choice of P, we now modify the proof of Theorem 1 suggested in Fischer and Marsden [1975a] and Bourguignon, Ebin and Marsden [1976] as follows:

Proof of Theorem 1. Suppose $R(g) = \rho_0$ is linearization stable at g_0 and $DR(g_0)$ is not surjective. We derive a contradiction. Since $DR(g_0)^*$, the adjoint operator, is always elliptic (Fischer and Marsden [1975a]), the Fredholm alternative implies the existence of a non-zero $f \in \ker DR(g_0)^*$. If $g(\lambda)$ is a curve of solutions of $R(g) = \rho_0$, then differentiating twice at $\lambda = 0$, multiplying by f and integrating yields the usual second order condition: for all $h \in \ker DR(g_0)$,

$$\int_M f\, D^2R(g_0)(h, h)\mu(g_0) = 0 \tag{1}$$

where $\mu(g_0)$ is the volume element of g_0. We can assume $f(x) > \delta > 0$ by restricting to a neighborhood U in M. A (long) calculation in Fischer and Marsden [1975a] for $D^2R(g_0)$ shows that (1) implies

$$\int_U \Big(f\Big\{ -\frac{1}{2}(\nabla h)^2 - \frac{1}{2}(\nabla(\mathrm{tr}\, h))^2 + R^{iajb}h_{ij}h_{ab}\Big\} - 2(h : \nabla f \otimes \nabla\, \mathrm{tr}\, h)\Big)\mu(g_0) = 0 \tag{2}$$

for $h \in \ker P$, $\mathrm{supp}\, h \subset U$. Since $P(h) = 0$, $\Delta \mathrm{tr}\, h - \mathrm{Ric}(g) : h = 0$. Multiplying by $\mathrm{tr}\, h$ and integrating by parts gives the estimate

$$\|\mathrm{tr}\, h\|_{H^1} \leq C\|h\|_{H^0} \tag{3}$$

for a (generic) constant C and all $h \in \ker P$, $\mathrm{supp}\, h \subset U$. From the Schwarz inequality and (3),

$$\left|\int_U h : (\nabla f \otimes \nabla \operatorname{tr} h)\mu(g_0)\right| \leq C\|h\|_{H^0}\|\operatorname{tr} h\|_{H^1} \tag{4}$$
$$\leq C\|h\|^2_{H^0}$$

and clearly

$$\left|\int_U R^{iajb} h_{ij} h_{ab} \mu(g_0)\right| \leq C\|h\|^2_{H^0}. \tag{5}$$

Thus, substituting estimates (3), (4) and (5) into (2) gives

$$\|h\|^2_{H^1} \leq C\|h\|^2_{H^0} \tag{6}$$

for all $h \in \ker P$, supp $h \subset U$. Since the embedding of H^1 in H^0 is compact, the inequality (6) cannot hold on the infinite-dimensional space $\ker P \cap \{h | \operatorname{supp} h \subset U\}$. ■

3. Linearization stability of the Einstein equations. Let M be a compact 3-manifold, S_2 the symmetric covariant two tensors on M, S^2_d the symmetric contravariant two-tensor densities on M, $\mathcal{F}_d$ the scalar densities and Λ_d the one-form densities. Define

$$\mathcal{H} : S_2 \times S^2_d \to \mathcal{F}_d;\ (g, \pi) \mapsto \left(\pi' : \pi' - \frac{1}{2}(\operatorname{tr}\pi')^2 - R(g)\right)\mu(g)$$

and

$$\mathcal{J} : S_2 \times S^2_d \to \Lambda_d;\ (g, \pi) \mapsto 2\delta\pi$$

where π' denotes the tensor part of the tensor density $\pi \in S^2_d$. Let $\Phi = \mathcal{H} \times \mathcal{J}$.

From Fischer and Marsden [1973, 1975b or 1976] (which have minor differences in sign conventions) we know that linearization stability of the Einstein equations is equivalent to linearization stability of the constraint equations $\Phi(g, \pi) = 0$ on the Cauchy surface M. From Moncrief [1975], Killing fields for ${}^{(4)}g$ on V_4 are in one-to-one correspondence with elements of $\ker D\Phi(g, \pi)^*$ where (g, π) is the metric and canonical momentum induced on M from ${}^{(4)}g$. We have

$$\begin{aligned} D\Phi(g, \pi) \cdot (h, \omega) &= (D\mathcal{H}(g, \pi) \cdot (h, \omega), D\mathcal{J}(g, \pi) \cdot (h, \omega)) \\ &= \Bigg(\Bigg[\left\{2\pi' \cdot \pi' - (\operatorname{tr}\pi')\pi' + \left(\frac{1}{4}(\operatorname{tr}\pi')^2 - \frac{1}{2}\pi' : \pi'\right)g + \operatorname{Ein}(g)\right\} : \\ &\quad h - \Delta(\operatorname{tr} h) - \delta\delta h\Bigg]\mu(g) \\ &\quad + 2\left(\pi' - \frac{1}{2}(\operatorname{tr}\pi')g\right) : \omega,\ 2\left\{\delta\omega - \pi^{jk}\left(h_{ij|k} - \frac{1}{2}h_{jk|i}\right)\right\}\Bigg) \end{aligned}$$

where $(\pi' \cdot \pi')^{ij} = \pi^{ik}\pi_k{}^j$.

Lemma 3. Given an open set $U \subset M$,

$$\ker D\Phi(g, \pi) \cap \{(h, \omega) \mid \operatorname{supp} h \subset U\}$$

is infinite dimensional.

Proof. Let

$$K = \left\{h \in S_2 \middle| \pi^{jk}\left(h_{ij|k} - \frac{1}{2} h_{jk|i}\right) \text{ is } L_2\text{-orthogonal to all the killing fields of } g\right\}.$$

Since the adjoint of 2δ is $\alpha : X \mapsto L_X g$ which is elliptic, by the Fredholm alternative we have a linear map

$$T : K \to S_d^2$$

such that

$$(1) \qquad (\delta T(h))_i = \pi^{jk}\left(h_{ij|k} - \frac{1}{2} h_{jk|i}\right)$$

and

$$\|T(h)\|_{H^0} \le C\|h\|_{H^0}.$$

(In fact, $\delta \circ \alpha$ is an isomorphism from the range of δ to itself and we let T be the zeroth order operator

$$T(h) = \alpha \circ (\delta \circ \alpha)^{-1}\left(\pi^{jk}\left(h_{ij|k} - \frac{1}{2} h_{jk|i}\right)\right).)$$

Define

$$P : K \to \mathfrak{X} \times \mathcal{F}$$

$$P(h) = \left(2\delta h, \left\{2\pi' \cdot \pi' - (\operatorname{tr}\pi')\pi' + \left(\frac{1}{4}(\operatorname{tr}\pi') - \frac{1}{2}\pi' : \pi'\right)g + \operatorname{Ein}(g)\right\} : \right.$$
$$\left. h - \Delta(\operatorname{tr} h) + 2\left(\pi' - \frac{1}{2}(\operatorname{tr}\pi')g\right) : T(h)'\right),$$

where $\mathfrak{X}$ is the set of vector fields on M, and extend P linearly to all of S_2 to give an operator $\bar{P}$; note that K has finite codimension.

The symbol of $\bar{P}$ is

$$\sigma_\xi(h) = (-2h^\# \cdot \xi, (\operatorname{tr} h)|\xi|^2)$$

which, as in Lemma 2, is surjective, but not injective. Thus $\ker \bar{P} \cap \{h \mid \operatorname{supp} h \subset U\}$ is infinite dimensional, by Lemma 1. Since K has finite codimension,

$$\ker P \cap \{h \mid \operatorname{supp} h \subset U\}$$

is also infinite dimensional. By definition of P and $T(h)$,

$$\ker D\Phi(g, \pi) \supset \{(h, \omega) | h \in \ker P, \text{ supp } h \subset U \text{ and } \omega = T(h)\}$$

and so Lemma 3 follows. ∎

By the work of Fischer, Marsden and Moncrief cited above, Theorem 2 reduces to the following.

Lemma 4. *If $D\Phi(g, \pi)$ is not surjective, then the constraint equations are not linearization stable.*

Proof. Since $D\Phi(g, \pi)^*$ is elliptic (Fischer and Marsden [1973]), $D\Phi(g, \pi)$ is not surjective if and only if there is a non-zero element $(N, X) \in \ker D\Phi(g, \pi)^*$. We know that (N, X) corresponds to a Killing field on V_4 by Moncrief [1975] and that linearization stability is hypersurface invariant. Therefore, by re-orienting the Cauchy surface M slightly, if necessary, we can assume that $N \not\equiv 0$ and, say, $N > \delta > 0$ on an open set U. The second order condition implied by linearization stability is

$$\int_M (N, X) \cdot D^2\Phi(g, \pi)((h, \omega), (h, \omega))\mu(g) = 0 \tag{3}$$

for all $(h, \omega) \in \ker D\Phi(g, \pi)$. (This condition is also hypersurface invariant, as has been shown by Moncrief [1976]).

The expression for $D^2\Phi(g, \pi)$ may be found in Moncrief [1976]; it contains and agrees with, the expression for $D^2R(g)$ used in §2. After minor convention changes, substitution of this formula in (3), integrating by parts and using $(N, X) \in \ker D\Phi(g, \pi)^*$ yield the following analogue of (2) in the previous section:

$$\begin{aligned}
0 = &\int_M N\left\{\frac{1}{2}(\nabla h)^2 - \frac{1}{2}(\nabla \mathrm{tr}\, h)^2 - (\delta h)^2 - (\mathrm{tr}\, h)(\delta^2 h)\right\}\mu(g) \\
&+ \int_M \{2(\nabla N \otimes \delta h) : h - (\nabla N \cdot \nabla \mathrm{tr}\, h)\mathrm{tr}\, h + 2(\nabla N \otimes \nabla \mathrm{tr}\, h) : h\}\mu(g) \\
&- 2\int_M (L_X h) : \omega + \int_M N\left\{-R^{iajb}h_{ij}h_{ab} - \mathrm{tr}\, h(\mathrm{Ric} : h)\right. \\
&- \frac{1}{4}R(\mathrm{tr}\, h)^2 + [(2\pi' \cdot \pi' - (\mathrm{tr}\, \pi')\pi')] : [(h \cdot h) - (\mathrm{tr}\, h)h] \\
&+ 2\pi'^{ij}\pi'^{k\ell}h_{ik}h_{j\ell} - (\pi' : h)^2 \\
&+ \frac{1}{4}(\mathrm{tr}\, h)^2\left(\pi' : \pi' - \frac{1}{2}(\mathrm{tr}\, \pi')^2\right) + 8(\pi' \cdot \omega' - 2(\mathrm{tr}\, \pi')\omega' \\
&- 2(\mathrm{tr}\, \omega')\pi') : h - \left(\pi' : \omega' - \frac{1}{2}(\mathrm{tr}\, \pi')(\mathrm{tr}\, \omega')\right)\mathrm{tr}\, h \\
&\left. + 2\,\omega' : \omega' - (\mathrm{tr}\, \omega')^2\right\}\mu(g) + \int_M (L_X \pi) : (h \cdot h).
\end{aligned}$$

For h such that $P(h) = 0$ and so $\delta h = 0$, this becomes

$$
\begin{aligned}
0 &= \int_M N\left\{\frac{1}{2}(\nabla h)^2 - \frac{1}{2}(\nabla \operatorname{tr} h)^2\right\}\mu(g) \\
&\quad + \int_M \{-(\nabla N \cdot \nabla \operatorname{tr} h)\operatorname{tr} h + 2(\nabla N \otimes \nabla \operatorname{tr} h) : h\}\mu(g) \qquad (4)\\
&\quad - 2\int_M (L_X h) : \omega + \int_M \{\text{algebraic terms quadratic in } (h, \omega)\}\mu(g).
\end{aligned}
$$

Also from $P(h) = 0$, we have, as in §2,

$$\|\operatorname{tr} h\|_{H^1} \le C\|h\|_{H^0}. \tag{5}$$

Using (4), $\omega = T(h)$, $\|T(h)\|_{H^0} \le C\|h\|_{H^0}$ and the Schwarz inequality we obtain,

$$\|h\|^2_{H^1} \le C(\|\operatorname{tr} h\|^2_{H^1} + \|h\|_{H^0}\|\operatorname{tr} h\|_{H^1} + \|h\|_{H^0}\|h\|_{H^1} + \|h\|^2_{H^0}).$$

By (5), we get

$$\|h\|_{H^1} \le C\|h\|_{H^0}$$

for all h such that $P(h) = 0$ and supp $h \subset U$. But, as before, such an inequality contradicts the infinite dimensionality of this set of h. ■

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Arms: University of Utah, Salt Lake City, UT 84112
Marsden: University of California, Berkeley, CA 94720