TWO EXAMPLES IN NONLINEAR ELASTICITY

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This note is concerned with extremals for the integral

$$J(u) = \int_0^1 W(u_x)\, dx$$

with W a given smooth function of $u_x = \frac{du}{dx}$ and with u prescribed at $x = 0$ and $x = 1$; say

$$u(0) = 0 \ , \ u(1) = p_o \ .$$

In applications to one dimensional elasticity , W is the stored energy function. We will call $u_o(x) = p_o x$ the trivial solution .

Our examples point out the care needed in choosing function spaces when discussing the existence and stability of equilibrium solutions in elasticity , and they are indicative of difficulties for realistic models of nonlinear elastic materials in one and higher dimensions .

The purpose of these examples , more specifically , is as follows.

1. The trivial solution need not be isolated in any Sobolev space $W^{1,p} = W^{1,p}(0,1)$, $1 \leq p < \infty$ even though

   (a) the second variation of J is positive definite , and

   (b) it is isolated in $W^{2,p}$.

In particular, an implicit function theorem cannot be used to prove local existence and uniqueness in $W^{1,p}$ under assumption (a) alone .

2. Positivity of the second variation at the trivial solution implies u_o locally minimizes J in a topology as strong as $W^{1,\infty}$ although

(a) it <u>need not</u> imply u_o locally minimizes J in $W^{1,p}$ for any p , $1 \le p < \infty$.

(b) in any topology as strong as $W^{1,\infty}$ we <u>always have</u> for $\varepsilon > 0$ sufficiently small ,

$$\inf_{\|u-u_o\|=\varepsilon} J(u) = J(u_o) .$$

Before proceeding to these examples , we make some remarks .

(i) The space $W^{1,p}$ plays a basic role in the existence theory for minimizers in elasticity (Ball [1]) . In example 1 , however , W is not convex.

(ii) The second example shows that <u>in general</u> potential wells (the standard sufficient conditions for stability ; cf. references [5] , [6]) are impossible in topologies as strong as $W^{1,\infty}$. The above conclusions in example 2 were given by Knops [3] for the case $W(u_x) = \frac{1}{2}(u_x^2 - u_x^4)$ and by Knops and Payne [4] in some related three dimensional examples .

(iii) If convexity and polynomial growth conditions are imposed, conditions for a potential well may be met in $W^{1,p}$ by inspection . However it is unknown whether the equations of nonlinear elastodynamics are well posed for suitable weak solutions in $W^{1,p}$ (for any nontrivial choice of stored energy function) .

(iv) Koiter [6] has remarked that in practice the energy criterion is very successful . However this is consistent with the possibility that the energy criterion may fail for hyperelastodynamics . Indeed " in practice " one usually does not observe the very high frequency motions . Masking them may amount to replacing the quasilinear equations of elastodynamics by semilinear approximations . For the latter the proof of the validity of the energy criterion is basically trivial (cf. [7] , [8]) .

(v) The second example illustrates that the Morse lemma for the function J will fail in $W^{1,p}$, $1 \leq p < \infty$, but be valid in $W^{s,p}$, $s \geq 2$, $1 < p < \infty$. See Tromba [9] .

The First Example .

Let W be a smooth function of $\mathbb{R}$ to $\mathbb{R}$ and let $p_- < p_o < p_+$ be such that

$$W'(p_-) = W'(p_o) = W'(p_+)$$

and

$$W''(p_o) > 0 .$$

See figure 1 .

In $W^{2,p}$ (with the boundary conditions $u(0) = 0$, $u(1) = p_o$ as before) , the trivial solution is isolated because the map

$$u \longmapsto W(u_x)_x$$

from $W^{2,p}$ to L^p is smooth and its derivative at u_o is the linear operator

$$v \longmapsto W''(p_o)\, v_{xx},$$

which is an isomorphism . Therefore , by the inverse function theorem, u_o is an isolated zero of $W(u_x)_x$.

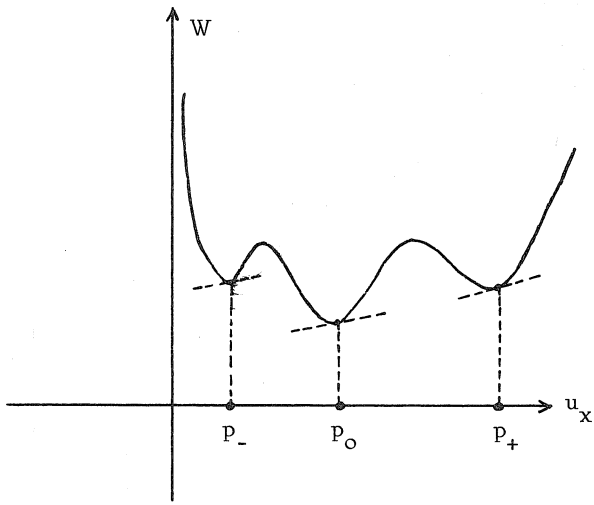

Figure 1

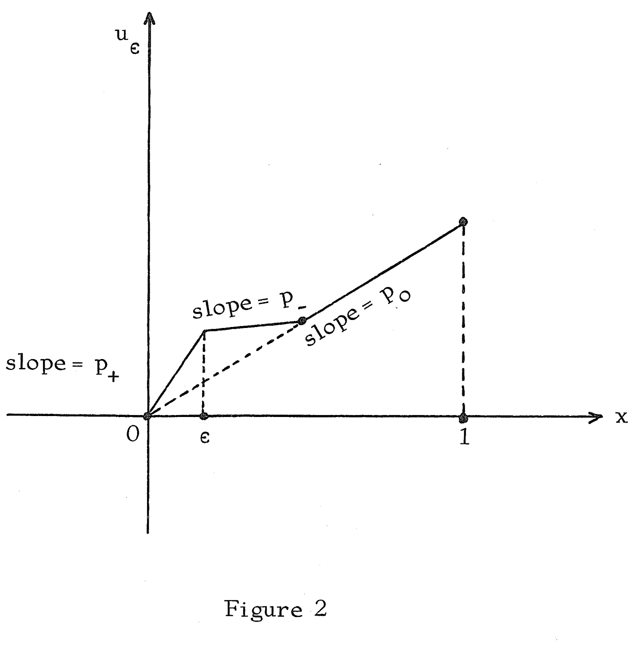

Figure 2

The second variation of J is positive definite (relative to the $W^{1,2}$ topology) at u_o because if v is in $W^{1,2}$ and vanishes at $x = 0, 1$,

$$\frac{d^2}{d\varepsilon^2} J(u_o + \varepsilon v)\Big|_{\varepsilon=0} = W''(p_o)\int_0^1 v_x^2 \, dx$$

$$\geq c \| v \|^2_{W^{1,2}} .$$

Now we show that u_o is not isolated in $W^{1,p}$.

Given $\varepsilon > 0$, let

$$u_\varepsilon(x) = \begin{cases} p_+ x \text{ for } 0 \leq x \leq \varepsilon \\ p_+\varepsilon + p_-(x-\varepsilon) \text{ for } \varepsilon \leq x \leq (p_+ - p_-)\varepsilon / (p_o - p_-) \\ p_o x \text{ for } (p_+ - p_-)\varepsilon/(p_o - p_-) \leq x \leq 1 \end{cases}$$

See fig. 2 . Since $W'(u_{\varepsilon x})$ is constant each u_ε is an extremal.

Also

$$\int_0^1 |u_{\varepsilon x} - u_{ox}|^p \, dx = \varepsilon |p_+ - p_o|^p + \left[\frac{p_+ - p_o}{p_o - p_-}\right] \varepsilon |p_- - p_o|^p$$

which tends to zero as $\varepsilon \to 0$. Thus u_o is not isolated in $W^{1,p}$.

Remarks . 1. If $W(p_-) = W(p_+) = W(p_o)$ and if $W(p) \geq W(p_o)$ for all p, the same argument shows that there are absolute minima of J arbitrarily close to u_o in $W^{1,p}$.

2. Phenomena like this seem to have first been noticed by Weierstrass . See Bolza [2] , footnote 1 , p. 40 .

The Second Example .

Let $W : \mathbb{R} \to \mathbb{R}$ be a smooth function with $W'(p_o) = 0$ and $W''(p_o) > 0$. As in the first example , $u_o(x) = p_o x$ is an extremal and the second variation of J at u_o is positive definite . Let X be a Banach space continuously included in $W^{1,\infty}$. Then there is an $\varepsilon > 0$ such that

$$\text{if } 0 < \| u - u_o \| < \varepsilon \quad \text{then} \quad J(u) > J(u_o)$$

i.e. u_o is a strict local minimum for J . This follows trivially from the fact that p_o is a local minimum of W and that the topology on X is a strong as that of $W^{1,\infty}$.

In $W^{1,p}$ one cannot conclude that u_o is a local minimum . Indeed the example $W(u_x) = \frac{1}{2}(u_x^2 - u_x^4)$ with $p_o = 0$ shows that in any $W^{1,p}$ neighbourhood , J(u) can be unbounded below , even though its second variation at u_o is positive definite .

Finally we show that

$$\inf_{\|u-u_o\|_X = \varepsilon} J(u) = J(u_o) \quad .$$

Indeed , by Taylor's theorem ,

$$J(u) - J(u_o) = \int_0^1 (W(u_x) - W(p_o))\, dx$$

$$= \int_0^1 \int_0^1 (1-s)\, W''(su_x + (1-s)p_o)\, (u_x - p_o)^2\, ds\, dx$$

$$\leq C \int_0^1 (u_x - p_o)^2\, dx$$

where $C > 0$, since $su_x + (1-s)\, p_o$ is essentially uniformly bounded (by the assumption $X \subset W^{1,\infty}$) and W" is continuous . However , the topology on X is strictly stronger than the $W^{1,2}$ topology , and so

$$\inf_{\|u - u_o\|_X = \varepsilon} \int_0^1 (u_x - p_o)^2 \, dx = 0 \ .$$

This proves our claim .

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