Automatica, Vol. 14, pp. 367–384
Pergamon Press Ltd. 1978. Printed in Great Britain

0005-1098/78/0701-0367$02.00/0

Bifurcation to Divergence and Flutter in Flow-induced Oscillations: An Infinite Dimensional Analysis*

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A qualitative dynamical technique, center manifold theory in particular, yields a useful, low dimensional, essential model of flow induced vibration which captures local bifurcation behavior under the action of control parameters.

Key Word Index—Bifurcation; center manifold; control parameters; limit cycles; nonlinear equations; stability; vibrations.

Summary—We outline the application of center manifold theory to a problem of flow-induced vibrations in which bifurcations occur under the action of control parameters. Using these techniques, the governing nonlinear partial differential equation (PDE) can be replaced locally by a vector field on a low dimensional manifold. The bifurcations thus detected, including 'global' bifurcations, yield a useful description of the qualitative dynamics of the original PDE.

1. INTRODUCTION

IN THIS paper we outline an application of some recent qualitative dynamical techniques; in particular those of center manifold and bifurcation theory. For our example we choose a problem of flow-induced oscillations, that of panel flutter[1, 2]. The present work follows an earlier study[3] in which a finite dimensional Galerkin approach was used. Here we show how the partial differential equation (PDE) governing panel motion can be recast as an ordinary differential equation (ODE) on a suitable function space X and outline the necessary existence, uniqueness and smoothness theory for the semiflow $F_t^{\mu}: X \to X$ induced by the ODE. The character of F_t^{μ} varies under the action of the control parameter(s) μ, and the fixed points $x_i = F_t^{\mu}(x_i)$ can appear, disappear or change their stability types in bifurcations. Under suitable hypotheses on the spectrum of the linearised semiflow $DF_t^{\mu}(x_0)$ at such a fixed point, the center manifold theory for flows[4] can be applied and the existence of a finite dimensional center manifold $\tilde{M}$ deduced. As a finite number of eigenvalues of $DF_t^{\mu}(x_0)$ pass through the unit circle, bifurcations occur which can be analysed locally without loss of information by studying the flow F_t^{μ} restricted to $\tilde{M}$. The dimension of $\tilde{M}$ is often only 1 or 2 and the resultant drastic reduction in dimension, while important in itself, also enables us to make interesting deductions on the qualitative structure of F_t^{μ}. In particular, we are able to use the Andronov–Takens classification of generic bifurcations of codimension 2 on two-manifolds[5]. The present approach is applicable to a wide range of continuum mechanical problems and a number of applications to hydrodynamic stability have already appeared[4].

It is interesting to note that engineers working on flow-induced oscillation problems have frequently resorted to the study of one or two degree of freedom nonlinear oscillators in cases where the full PDE is intractible. The present use of center manifold theory suggests that these approaches might indeed be justifiable and can be made completely rigorous. See [3] for more details.

The equation of motion of a thin panel, fixed at both ends and undergoing 'cylindrical' bending between $z=0$ and $z=1$ (Fig. 1) can be written in

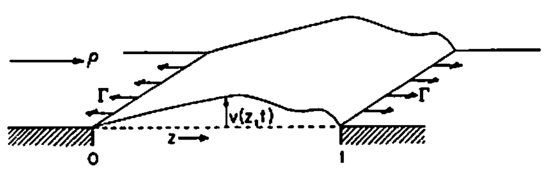

FIG. 1. The panel flutter problem.

terms of the lateral deflection $v = v(z, t)$ as

$$\alpha \dot{v}'''' + v'''' - \{\Gamma + \kappa \int_0^1 (v'(\zeta))^2 \, d\zeta + \sigma \int_0^1 (v'(\zeta)\dot{v}'(\zeta))\, d\zeta\} v'' + \rho v' + \sqrt{\rho}\, \delta \dot{v} + \ddot{v} = 0 \qquad (1)$$

see [1] and [3], section 2. Here $\cdot \equiv \partial/\partial t$ and $'$

*Received 22 August 1977; revised 18 January 1978. The original version of this paper was presented at the 2nd IFAC Symposium on the Control of Distributed Parameter Systems which was held in Coventry, England, during June–July 1977. The published Proceedings of this IFAC Meeting may be ordered from: Pergamon Press Ltd., Headington Hill Hall, Oxford OX3 0BW, England. This paper was recommended for publication in revised form by associate editor P. Parks.
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$\equiv \partial/\partial z$, and we have included viscoelastic structural damping terms α, σ as well as aerodynamic damping $\sqrt{\rho}\delta$; κ represents nonlinear (membrane) stiffness, ρ the dynamical pressure and Γ an in-plane tensile load. All quantities are nondimensionalised and associated with (1) we have boundary conditions at $z=0,1$ which might typically be simply supported ($v=(\dot{v}+\alpha v)''=0$) or clamped ($v=v'=0$). In the following we make the physically reasonable assumption that α, σ, δ, κ are fixed >0 and let the control parameter $\mu \in \{(\rho,\Gamma)|\rho \geqq 0\}$ vary. In contrast to previous studies[1, 2] in which (1) and similar equations were analysed for specific parameter values and initial conditions by numerical integration of a finite dimensional Galerkin approximation, here we study the qualitative behaviour of (1) under variations of μ.

2. EXISTENCE, UNIQUENESS AND SMOOTHNESS

Here we prove theorems showing that (1) defines a unique global semiflow F_t^μ on X and that F_t^μ is smooth. This program in a more general context will appear in a forthcoming paper[6]. We first formulate (1) as on ODE on a Banach space, choosing as our basic space

$$X = H_0^2([0,1]) \times L^2([0,1]),$$

where H_0^2 denotes the Sobolev space of twice differentiable functions which vanish at 0, 1 and L^2 is the usual Hilbert space of square integrable functions. We denote elements of X by $\{v,\dot{v}\}$ and select the norm

$$\|\{v,\dot{v}\}\|_X = (|\dot{v}|^2 + |v''|^2)^{1/2},$$

where $|\cdot|$ denotes the usual L^2 norm and define the linear operator

$$A_\mu = \begin{pmatrix} 0 & I \\ C_\mu & D_\mu \end{pmatrix}; \quad \begin{array}{l} C_\mu v = -v'''' + \Gamma v'' - \rho v' \\ D_\mu \dot{v} = -\alpha \dot{v}'''' - \sqrt{\rho}\,\delta \dot{v} \end{array} \tag{2}$$

The basic domain of A_μ, $D(A_\mu)$, consists of all $\{v,\dot{v}\} \in X$ such that $\dot{v} \in H_0^2$ and $v + \alpha \dot{v} \in H^4$, with the appropriate boundary conditions imposed. After defining the nonlinear operator

$$B\{v,\dot{v}\} = \{0, [\kappa|v'|^2 + \sigma(v',\dot{v}')]v''\},$$

where $(\cdot,\cdot)$ denotes the L^2 inner product, (1) can be written as

$$dx/dt = A_\mu x + B(x) \equiv G_\mu(x);$$
$$x = \{v,\dot{v}\} = x(t) \in D(A_\mu) \tag{3}$$

We define an energy function and some related Liapunov functions by

$$H_1(x(t)) = \frac{1}{2}\left[|\dot{v}|^2 + |v''|^2 + \frac{\kappa}{2}|v'|^4\right] \tag{4a}$$

$$H_2(x(t)) = \frac{1}{2}\left[|\dot{v}|^2 + \Gamma|v'|^2 + |v''|^2 + \frac{\kappa}{2}|v'|^4\right] \tag{4b}$$

$$H_3(x(t)) = \frac{1}{2}\left[\sqrt{\rho}\,\delta|v|^2 + \alpha|v''|^2 + 2(v,\dot{v}) + \frac{\sigma}{2}|v'|^4\right] \tag{5}$$

and let $H_a = H_1 + \nu H_3$ and $H_b = H_2 + \nu H_3$, where ν is a positive constant to be selected later.

We now prove a number of propositions related to the evolution problem (3). The first result gives some properties of the functions H_a and H_b and gives a parameter region in which all oscillatory motions die out and the panel does not flutter. Larger regions in the (ρ,Γ) plane for which this happens might be obtainable by use of a modified Liapunov function. The method here is adapted from Parks[7].

Proposition 2.1. (i) $H_a: X \to \mathfrak{R}$ *and* $H_b: X \to \mathfrak{R}$ *are* C^∞ *functions, bounded on bounded sets;*

(ii) *there are constants* $K>0$ *and* $B>0$ (*depending on the parameters* $\Gamma,\rho,\alpha,\ldots$) *such that if* $x(t) = \{v(t),\dot{v}(t)\}$ *is a solution of* (3), *then*

$$\frac{d}{dt} H_a(x(t)) \leqq K H_a(x(t))$$

and if $\|x(t)\| \geqq B$, *then*

$$\frac{d}{dt} H_a(x(t)) < 0;$$

(iii) *if* $\rho^2 < (\sqrt{\rho}\delta + \alpha\pi^4)^2(\Gamma + \pi^2)$, *then*

(a) $H_b(x) \geqq 0$ *and* $H_b(x) > 0$ *if* $x \neq 0$, $x \in X$;

(b) $(d/dt)H_b(x(t)) \leqq 0$ *and* $(d/dt)H_b(x(t)) < 0$ *if* $x(t) \neq \{0,0\}$ *where* $x(t)$ *is a solution of* (3); *and*

(c) $x(t) \to \{0,0\}$ *in* X *as* $t \to +\infty$; *i.e.,* $\{0,0\}$ *is the unique attracting fixed point of* (3). .

Proof. (i) It is well known, as indicated in [4] and references therein, that in one dimension, multiplication induces a continuous bilinear mapping of $H^1 \times H^1 \to H^1$. This, together with continuity of differentiation from H^2 to H^1 and from H^1 to L^2 and continuity of the inner product proves (i) since the functions H_1, H_2 and H_3 are made up of compositions of these operations.

(ii) Using a Fourier sine series we have the elementary inequality

$$|v'|^2 \geqq \pi^2 |v|^2$$

if v vanishes at $z=0,1$. If v' vanishes at $z=0,1$, we therefore get

$$|v''|^2 \geqq \pi^2 |v'|^2 \geqq \pi^4 |v|^2.$$

Without assuming v' vanishes at $z=0,1$ we can either prove $|v''|^2 \geqq \pi^4 |v|^2$ directly or as follows: integration by parts and the Schwarz inequality gives:

$$|v'|^2 = (v', v') = (v, v'') \leqq |v|\,|v''|$$

and hence

$$\pi^2 |v|^2 \leqq |v|\,|v''|, \text{ i.e., } \pi^4 |v|^2 \leqq |v''|^2.$$

It follows that

$$H_a \geqq \tfrac{1}{2}[|\dot v|^2 + 2\nu(v, \dot v) + \nu(\sqrt{\rho}\,\delta + \alpha\pi^4)|v|^2 + |v''|^2]$$

and, setting $\nu = (\sqrt{\rho}\,\delta + \alpha\pi^4)/2$ that

$$H_a \geqq \frac{1}{2}\left[\left|\dot v + \frac{(\sqrt{\rho}\,\delta + \alpha\pi^4)}{2} v\right|^2 + \frac{(\sqrt{\rho}\,\delta + \alpha\pi^4)}{4}|v|^2 + |v''|^2\right] \geqq \frac{1}{2}|v''|^2 + \frac{(\sqrt{\rho}\,\delta + \alpha\pi^4)}{8}|v|^2 \qquad (6)$$

Differentiating along solution curves of (3) we have

$$\begin{aligned}\frac{\mathrm{d}H_a}{\mathrm{d}t} &= -\rho(v', \dot v) - \Gamma(v', \dot v') - \sqrt{\rho}\,\delta|\dot v|^2 - \alpha|\dot v''|^2 \\ &\quad + \nu[|\dot v|^2 - \Gamma|v'|^2 - |v''|^2] - \sigma(v', \dot v')^2 - \nu\kappa|v'|^4 \\ &\leqq -\rho(v', \dot v) - \Gamma(v', \dot v') - [\sqrt{\rho}\,\delta - \nu]|\dot v|^2 - \alpha|\dot v''|^2 \\ &\quad - \nu|v''|^2 - \nu\Gamma|v'|^2 - \sigma(v', \dot v')^2 - \nu\kappa|v'|^4 \qquad (7)\end{aligned}$$

For $\nu < \sqrt{\rho}\,\delta$ we can use the Schwarz inequality and the elementary inequality

$$ab \leqq \frac{1}{2}\left(\varepsilon a^2 + \frac{1}{\varepsilon} b^2\right)$$

to give

$$|\rho(v', \dot v)| \leqq \frac{\rho}{2}\left[\gamma|v'|^2 + \frac{1}{\gamma}|\dot v|^2\right], \quad \text{for any } \gamma > 0$$

$$|\Gamma(v', \dot v')| \leqq \frac{\Gamma}{2}\left[\omega|v'|^2 + \frac{1}{\omega}|\dot v'|^2\right], \quad \text{for any } \omega > 0 \qquad (8)$$

Choosing γ and ω sufficiently large, $(\rho/2\gamma)|\dot v|^2$ and $(\Gamma/2\omega)|\dot v'|^2$ are absorbed by the $|\dot v|^2$ and $|\dot v''|^2$ terms of (7); we can then conclude that

$$\frac{\mathrm{d}H_a}{\mathrm{d}t} \leqq \left[\frac{\rho\gamma + \Gamma\omega}{2} - \nu\Gamma\right]|v'|^2 - \nu|v''|^2 - \sigma(v', \dot v')^2 - \nu\kappa|v'|^4. \qquad (9)$$

Clearly for small $|v'|$, H_a can increase along solution curves, but the $|v'|^4$ term guarantees that we can find a positive number C such that $\mathrm{d}H_a/\mathrm{d}t < 0$ for $|v'| \geqq C$. In fact, (9) implies that

$$\frac{\mathrm{d}H_a}{\mathrm{d}t} \leqq C_1|v'|^2 - C_2|v'|^4 = |v'|^2(C_1 - C_2|v'|^2)$$

for constants C_1, C_2. Hence, if

$$|v'| > \sqrt{C_1/C_2}, \quad \frac{\mathrm{d}H_a}{\mathrm{d}t} < 0.$$

It follows that $(\mathrm{d}H_a/\mathrm{d}t) < 0$ if $\|x\|_X \geqq B$ for a constant B.

Inequality (9) implies

$$\frac{\mathrm{d}H_a}{\mathrm{d}t} \leqq \left(\frac{\rho\gamma + \Gamma\omega}{2} - \nu\Gamma\right)|v'|^2.$$

Now we use the inequality

$$|v'|^2 \leqq |v|\,|v''|$$

noted above to conclude

$$|v'|^2 \leqq \tfrac{1}{2}|v|^2 + \tfrac{1}{2}|v''|^2$$

and hence,

$$\frac{\mathrm{d}H_a}{\mathrm{d}t} \leqq (\text{Constant})\,(|v|^2 + |v''|^2) \leqq (\text{Constant})H_a$$

by (6). Thus (ii) of Proposition 2.1 is proved.

(iii) For stability bounds on the fixed point $\{0, 0\} \in X$ we use the Liapunov function H_b. First note that by taking the 'natural' energy function (4b) and differentiating we obtain

$$\frac{\mathrm{d}H_2}{\mathrm{d}t} = -\rho(v', \dot v) - \sqrt{\rho}\,\delta|\dot v|^2 - \alpha|\dot v''|^2 - \sigma(v', \dot v')^2$$

for $\rho = 0$ we immediately have $\mathrm{d}H_2/\mathrm{d}t \leqq 0$ for

$\|x\| \geqq 0$. For $\rho \neq 0$ we work with H_b, obtaining an expression analogous to (7):

$$\frac{dH_b}{dt} \leqq -\rho(v',\dot{v}) - [\sqrt{\rho}\,\delta + \alpha\pi^4 - \nu]|\dot{v}|^2 - \nu[\Gamma + \pi^2]|v'|^2 - \sigma(v',\dot{v})^2 - \nu\kappa|v'|^4 \qquad (10)$$

For global asymptotic stability of $\{0,0\}$, (10) must be negative definite. This occurs for

$$\tfrac{1}{4}\rho^2 \leqq [\sqrt{\rho}\,\delta + \alpha\pi^4 - \nu]\nu[\Gamma + \pi^2].$$

Taking the optimum choice of

$$\nu = (\sqrt{\rho}\,\delta + \alpha\pi^4)/2$$

we obtain (a) and (b) of part (iii). (Note that for $\rho^2 < (\sqrt{\rho}\,\delta + \alpha\pi^4)(\Gamma + \pi^2)$, $H_b \geqq 0$ automatically.) To prove (c), note that $H_b(x(t))$ decreases and is nonnegative, so converges to say, $H_\infty \geqq 0$ as $t \to \infty$. From

$$H_b(x(s)) - H_b(x(t)) = -\int_s^t \frac{d}{dt} H_b(x(\tau))\,d\tau, \quad t > s,$$
$$\geqq (\text{Const}) \textstyle\int_s^t |\dot{v}|^2\,d\tau$$

we find that v satisfies a Cauchy condition, so converges in L^2 as $t \to \infty$. Since $H_b \geqq 0$ and is decreasing, the limit must be zero; i.e., $\dot{v} \to 0$ in L^2. Similarly, $v' \to 0$ in $L^2 \cap L^4$ and hence $v \to 0$ in L^2. If we use these facts in the explicit expression for

$$H_b(x(s)) - H_b(x(t))$$

we find v'' converges in L^2. Since $v \to 0$, v'' must converge to zero. Thus $x(t) \to \{0,0\}$ in X. ■

Corollary 2.2. Let $x(t)$ be a solution of (3) for $\mu = (\rho, \Gamma)$ fixed. Then there is a constant $M > 0$ such that $\|x(t)\|_X \leqq M$ for all t for which $x(t)$ is defined.

Proof. Inequality (6) shows that

$$\|x(t)\|_X \leqq (\text{Constant}) H_a(x(t))$$

Let B be as in Proposition 2.1(ii), and

$$H_B = \sup\{H_a(x) \mid \|x\|_X \leqq B\} < \infty$$

Thus

$$H_a(x(t)) \leqq \max\{H(x_0), H_B\}$$

and so $\|x(t)\|_X \leqq M$, where

$$M = (\text{Constant}) \times (\max\{H(x_0), H_B\}) \qquad ■$$

A similar stability criterion for the linear problem was originally developed by Parks[7]. As he points out, such criteria are sufficient but not necessary; in fact our estimates are probably over conservative. This does not matter here since we are mainly interested in establishing that the nonlinear term $\kappa|v'|^2 v''$ in (3) leads to a global stability property. Thus part (ii) of Proposition 2.1, with Corollary 2.2, suggests global stability in the sense that as $t \to +\infty$ all solutions $x(t)$ of (3) approach some set $A \subset X$. For 'small' Γ and ρ, part (iii) guarantees that $A \equiv \{0,0\}$, the unique fixed point at the origin. The bulk of this paper is devoted to a study of the structure of A when $\{0,0\}$ first fails to be an attractor.

Note that the nonlinear damping term $\sigma(v',\dot{v}')v''$ does *not* guarantee stability alone, since $\sigma(v',\dot{v}')^2$ in (9) is zero in the event that v' and $\dot{v}'$ are orthogonal.

Another fact we shall need is the following.

Proposition 2.3. The map $B: X \to X$ is C^∞.

Proof. This follows directly from the fact that (v',v') and $(v'',\dot{v})$ come from continuous bilinear forms on X and the fact that $v \mapsto v''$ is bounded from H^2 to L^2. ■

The next proposition shows that the linear equation $dx/dt = A_\mu x$ is soluble in X. This is done in terms of semigroup theory (see, for example, Kato[8]).

Proposition 2.4. The linear operator A_μ is the generator of a C^0 semigroup in X.

Proof. Let A_0 denote the operator A_μ with the same domain, but with the terms Γ and ρ set to zero. Since $A_\mu - A_0$ is a bounded operator, it suffices to show that A_0 generates a C^0 semigroup. In fact, we shall show that A_0 generates a C^0 contraction semigroup. To do so, we establish two things:

(i) for $x \in D(A_0)$, $\langle A_0 x, x\rangle \leqq 0$

and

(ii) $\lambda - A_0$ is surjective for $\lambda > 0$.

From these it will follow that $-A_0$ is m-accretive, so A_0 generates a contraction semigroup.

The proof of (i) is easy, as in the proof of the energy inequality; if $x = (v, \dot{v})$, then, taking the inner product in X, we get

$$\langle A_0 x, x\rangle = \left\langle \begin{matrix} \dot{v} \\ -\alpha\dot{v}'''' - v'''' \end{matrix}, \begin{matrix} v \\ \dot{v} \end{matrix} \right\rangle$$
$$= (\dot{v}'', v'') + (-\alpha\dot{v}'''' - v'''', \dot{v})$$
$$= -\alpha|\dot{v}''|^2 \leqq 0.$$

Next, we prove that $\lambda - A_0$ has dense range. To do this, we suppose that for some

$$y \in X, \langle(\lambda - A_0)x, y\rangle = 0$$

for all $x \in D(A_0)$. We must show that $y=0$. Let $x = (v, \dot{v})$ and $y=(w, \dot{w})$. Then our supposition becomes

$$(\lambda v'' - \dot{v}'', w'')=0$$

and

$$(\lambda \dot{v} + \alpha \dot{v}'''' + v'''', \dot{w})=0$$

for all $v, \dot{v} \in D(A_0)$. Setting $\dot{v}''=0$ in the first equation and using the fact that v'' is arbitrary, we get $w''=0$, so $w=0$. The second equation, with $\dot{v}=0$ shows that $\dot{w}$ is a weak solution of $\dot{w}'''' =0$, so $\dot{w}$ is in fact smooth (it is a cubic polynomial). Setting $\dot{v}=\dot{w}$, $v=0$ and using $\lambda>0$ gives $\dot{w}=0$.

It remains to show that for $\lambda>0$, $\lambda - A_0$ has closed range. Let $x_n = (v_n, \dot{v}_n) \in D(A_0)$ and suppose

$$y_n = (\lambda - A_0)x_n \to y \in X.$$

Letting $y_n = (w_n, \dot{w}_n)$, we have

$$\lambda v_n - \dot{v}_n = w_n$$

and

$$\lambda \dot{v}_n + \alpha \dot{v}_n'''' + v_n'''' = \dot{w}_n$$

Multiplying the second equation by $\dot{v}_n$ and the first by v_n and integrating yields:

$$\lambda(\dot{v}_n, \dot{v}_n) + \alpha(\dot{v}_n'', \dot{v}_n'') + \lambda(v_n'', v_n'') = (\dot{w}_n, \dot{v}_n) + (w_n'', v_n'')$$

so

$$\|\dot{v}_n\|_{H^2} + \|v_n\|_{H^2} \leqq C(\|\dot{w}_n\|_{L^2} + \|w_n\|_{H^2})$$

Similarly,

$$\|\dot{v}_n - \dot{v}_m\|_{H^2} + \|v_n - v_m\|_{H^2} \leqq C(\|\dot{w}_n - \dot{w}_m\|_{L^2} + \|w_n - w_m\|)_{H^2}$$

Thus, $(v_n, \dot{v}_n) = x_n$ converges in X to, say, z and $\dot{v}_n$ converges in H^2. If we use this in the equation

$$\lambda \dot{v}_n + (\alpha \dot{v}_n + v_n)'''' = \dot{w}_n,$$

we get the estimate

$$\|(\alpha \dot{v}_n + v_n) - (\alpha \dot{v}_m + v_m)\|_{H^4} \leqq C\|\dot{w}_n - \dot{w}_m\|_{L^2}$$

so $\alpha \dot{v}_n + v_n$ converges in H^4. All this together implies that $z \in D(A_0)$ and thus $(\lambda - A_0)z = y$, so the range is closed and (ii) is proven. ∎

From computations in Section 4 one finds that the spectrum of A_0 lies in the half space $\mathrm{Re}\, z \leqq -\varepsilon$ for some $\varepsilon>0$. In fact, for the simply supported case

$$\varepsilon = \min(\alpha\pi^4/2, 1/\alpha)$$

It follows that in a suitable norm

$$\|\exp(tA_0)\| \leqq \exp(-\varepsilon' t)$$

where $0<\varepsilon'<\varepsilon$. (See [4, §2A].) In the original X norm, $\|\exp(tA_0)\| \leqq 1$ and so

$$\|\exp(tA_\mu)\| \leqq \exp(t\beta)$$

where $\beta = \|A_\mu - A_0\|$. In the contracting region of Proposition 2.1(iii), the semigroup $\exp(tA_\mu)$ will itself be a contraction. This may be proved using the norm associated to H_b in the proof of Proposition 2.4. We remark that the semigroup generated by A_0 is probably not analytic.

With these preliminaries we can now make use of a result originally due to Segal[9] concerning nonlinear evolution equations generated by operators of the form $A+B$, where A is a linear operator and B a C^k map, $k \geqq 1$. By modern standards, the result is rather elementary. Existence of solutions of

$$\frac{dx}{dt} = Ax + B(x)$$

can be deduced from the usual Picard iteration techniques. It is not quite so obvious that the solution depends in a C^k manner on the initial data. For the convenience of the reader we present a straightforward proof of this fact.

Proposition 2.5. Let X be a Banach space and U_t a linear semigroup on X with generator A and domain $D(A)$. Let $B: X \to X$ be C^k, $k \geqq 1$. Let $G = A + B$ on $D(A)$. Then

$$dx/dt = G(x); \quad x_0 = x(0) \in X \qquad (11)$$

defines a unique local semiflow $F_t(x_0)$: If $x_0 \in D(A)$, then $F_t(x_0)$ is in $D(A) = D(G)$, is X-differentiable and satisfies (11) with initial condition x_0, $F_t(x_0)$ is the unique such solution and moreover, F_t extends to a C^k map of an open set in X to X for each $t \geqq 0$.

Remarks. (a) In the terminology of Marsden and McCracken[4], G generates a *smooth semiflow*. The proof will show that if G_μ depends *continuously* on a parameter μ (with domain fixed), then so does its semiflow.

(b) If B is merely locally Lipschitz, one can still construct F_t, but B should be C^1 in order to show that F_t maps $D(A)$ to itself and for (11) to be satisfied in the strict sense.

Proof of Proposition 2.5. For $u_0 \in X$, we define $F_t(u_0) = u(t)$ by means of the Duhamel formula (variation of constants formula):

$$u(t) = U_t u_0 + \int_0^t U_{t-s} B(u(s))\,ds \qquad (12)$$

where $U_t = \exp(tA)$ is the semigroup generated by A. Since B is locally Lipschitz and $\|U_t\|_X \leqq M\exp(t\beta)$ for constants (M, β), the Picard iteration of ordinary differential equations shows that (12) defines a unique local semiflow $F_t(u_0)$. If B has a local Lipschitz constant K, then clearly

$$\|F_t(u_1) - F_t(u_2)\|_X \leqq M\exp((\beta + K)t)\|u_1 - u_2\|_X. \qquad (13)$$

Thus, $F_t(u_0)$ is continuous in t, u_0 and is locally Lipschitz in u_0.

We next show that *for fixed t, F_t is a C^k mapping*. For $x \in X$, let $\theta_t(x) \in B(X)$ (the bounded operators from X to X) satisfy the linearized equations:

$$\theta_t(x) = U_t + \int_0^t U_{t-s} DB(F_s(x)) \cdot \theta_s(x)\,ds. \qquad (14)$$

$\theta_t(x)$ is defined as long as $F_t(x)$ is defined. It is easy to check that $t \mapsto \theta_t(x)$ is continuous in the strong operator topology and that (for fixed t), $x \mapsto \theta_t(x)$ is norm continuous. We claim that $DF_t(x) = \theta_t(x)$ which will thus prove F_t is C^1. Let

$$\lambda_t(x, h) = \|F_t(x+h) - F_t(x) - \theta_t(x)\cdot h\|$$

Then

$$\begin{aligned}\lambda_t(x,h) &= \Big\|\int_0^t U_{t-s}\{B(F_s(x+h)) - B(F_s(x)) \\ &\quad - DB(F_s(x))\cdot\theta_s(x)\cdot h\}\,ds\Big\| \\ &\leqq M\exp(\beta|t|)\Big\{\int_0^t \|B(F_s(x+h)) - B(F_s(x)) \\ &\quad - DB(F_s(x))\cdot[F_s(x+h) - F_s(x)]\|\,ds \\ &\quad + \int_0^t \|DB(F_s(x))\cdot[F_s(x+h) \\ &\quad - F_s(x) - \theta_s(x)\cdot h]\|\,ds\Big\}\end{aligned}$$

Thus, given $\varepsilon > 0$, there is a $\delta > 0$ such that $\|h\| < \delta$ implies

$$\lambda_t(x,h) \leqq (\text{Const})\cdot\Big\{\|h\|\varepsilon + \int_0^t \lambda_s(x,h)\,ds\Big\}$$

Hence (by Gronwall's inequality),

$$\lambda_t(x,h) \leqq (\text{Const})\|h\|\varepsilon.$$

Hence, by definition of the derivative, $DF_t(x) = \theta_t(x)$. It is now a simple induction argument to show F_t is C^k.

Now we prove that *F_t maps $D(A)$ to $D(A)$ and*

$$\frac{d}{dt}F_t(u_0) = G(F_t(u_0))$$

is continuous in t. Let $u_0 \in D(A)$. Then, setting $u(t) = F_t(u_0)$, (12) gives

$$\begin{aligned}\frac{1}{h}[u(t+h) - u(t)] &= \frac{1}{h}[U_{t+h}u_0 - U_t u_0] \\ &\quad + \frac{1}{h}\int_0^t (U_{t+h-s} \\ &\quad - U_{t-s})B(u(s))\,ds \\ &\quad + \frac{1}{h}\int_t^{t+h} U_{t+h-s}B(u(s))\,ds \\ &= \frac{1}{h}[U_h(u(t)) - u(t)] \\ &\quad + \frac{1}{h}\int_t^{t+h} U_{t+h-s}B(u(s))\,ds \qquad (15)\end{aligned}$$

The second term $\to B(u(t))$ as $h \to 0$. Indeed,

$$\begin{aligned}&\left\|\frac{1}{h}\int_t^{t+h} U_{t+h-s}B(u(s))\,ds - B(u(t))\right\| \\ &\leqq \frac{1}{h}\int_t^{t+h}\|U_{t+h-s}B(u(s)) - B(u(t))\|\,ds \\ &\leqq \frac{1}{h}\int_t^{t+h}\|U_{t+h-s}B(u(s)) - U_{t+h-s}B(u(t))\|\,ds \\ &\quad + \frac{1}{h}\int_t^{t+h}\|U_{t+h-s}B(u(t)) - B(u(t))\|\,ds \\ &\leqq \frac{1}{h}\cdot(\text{Const})\cdot\int_t^{t+h}\|B(u(s)) - B(u(t))\|\,ds \\ &\quad + \frac{1}{h}\int_t^{t+h}\|U_{t+h-s}B(u(t)) - B(u(t))\|\,ds\end{aligned}$$

and each term $\to 0$ as $h \to 0$.

It follows that $F_t(u_0)$ is right differentiable at $t = 0$ and has derivative $G(u_0)$. To establish the formula at $t \neq 0$ we first prove that $u(t) \in D(A)$. But

$$\frac{1}{h}(F_{t+h}u_0 - F_t u_0) = \frac{1}{h}(F_t F_h u_0 - F_t u_0)$$

has a limit as $h \to 0$ since F_t is of class C^1. Hence, from (15),

$$\frac{1}{h}(U_h(u(t)) - u(t))$$

has a limit as $h \to 0$. Thus, $u(t) \in D(A)$. It follows that

$$\frac{d}{dt}F_t(u_0)=G(F_t(u_0))=DF_t(u_0)\cdot G(u_0).$$

Since this right derivative is continuous, the ordinary derivative exists as well. The proof of Proposition 2.5 is now complete. ■

For results of Dorroh and Marsden on the smoothness of nonlinear semiflows applicable to more delicate situations, see Marsden and McCracken[4].

There are some additional results of interest in connection with Proposition 2.5. The first concerns the regularity of solutions. For example, in equation (1), we wish to know that C^∞ initial data propagates to a C^∞ solution. In the abstract context, we show that solutions $u(t)$ of (11) which are initially in $D(A^n)$ are actually in $D(A^n)$ and not merely in $D(A)$; this implies the regularity of solutions of (1).

Let $[D(A)]$ denote the domain of A with the graph norm:

$$|||x|||=||x||+||Ax||$$

Since generators are necessarily closed operators, $[D(A)]$ is a Banach space and inclusion $[D(A)]\subset X$ is continuous. Similarly,

$$[D(A^n)]\subset[D(A^{n-1})]\subset\cdots\subset X$$

Proposition 2.6. Let the conditions of Theorem 2.5. hold. Then,

(i) *if* $u_0\in D(A)$, *the map* $t\mapsto F_t(u_0)\in[D(A)]$ *is continuous, and generally,*

(ii) *if* $B:[D(A^l)]\to[D(A^l)]$ *and is* C^k, $l=1,\ldots,n-1$, *then* F_t: $[D(A^l)]\to[D(A^l)]$ *and is of class* C^k, $l=1,\ldots,n-1$ *and* F_t: $[D(A^n)]\to[D(A^n)]$ *is continuous.*

Proof. To prove (i), we note that for $u_0\in D(A)$,

$$AF_t(u_0)=\frac{d}{dt}F_t(u_0)-B(F_t(u_0)).$$

The right side is continuous in t, so

$$t\mapsto F_t(u_0)\in[D(A)]$$

is continuous.

One proves (ii) by induction on n. Consider the case $n=2$; let $u_0\in D(A^2)$ and $u(t)=F_t(u_0)$. Then, $u(t)\in D(A)$ and $u'(t)=Au(t)+B(u(t))$. As in the proof of 2.5, it follows that $u''(0)$ exists, and since F_t is C^2, $u''(t)$ exists. Then, using the identity

$$\frac{1}{h}[u'(t+h)-u'(t)]=A\frac{1}{h}[U_h(u(t))-u(t)]$$
$$+\frac{1}{h}\int_t^{t+h}U_{t+h-s}B(u(s))ds$$
$$+\frac{1}{h}[B(u(t+h))-B(u(t))]$$

it follows that

$$\lim_{h\to0}A\frac{1}{h}[U_h(u(t))-u(t)]$$

exists, so $u(t)\in D(A^2)$. This result, together with 2.5, gives (i) by induction. ■

The following global existence result is also of interest, since it guarantees that solutions continue to exist for *all* $t\geq0$.

Proposition 2.7. Let the conditions of Proposition 2.5. hold. Furthermore, assume that $||DB(x)||_X$ *is bounded for* x *in an* X*-bounded set. Let* $x(t)\in D(A)$ *be a maximal integral curve of* G *defined for* $t\in[0,b)$. *Suppose that for any finite* $T\leq b$, *there is a constant* M *such that* $||x\ t)||_X\leq M$, $0\leq t<T$. *Then* $b=\infty$ *and* $x(t)$ *is defined for all* $t\geq0$. *(Hence, if this is true for all integral curves,* F_t *is defined on all* X *for all* $t\geq0$).

Proof. The proof of Proposition 2.5 shows that for any ball $B\subset X$, there is an $\varepsilon>0$ such that any initial condition in B has an integral curve existing for time ε. If $b<\infty$, we can choose $T=b$ to conclude that $||u(t)||_X\leq M$ for $0\leq t\leq b$. Thus if $t-b<\varepsilon$ we can extend $u(t)$ beyond b, contradicting maximality of b. ■

Propositions 2.2, 2.3, and 2.4 show that the hypotheses of Propositions 2.5, 2.6, and 2.7 hold and we thus obtain our first main result:

Theorem 2.8. EXISTENCE, UNIQUENESS, SMOOTHNESS. Equation (1), (i.e. (3)), *defines a unique global semiflow* F_t^μ *on* $X=H_0^2\times L^2$. *If* $x_0\in D(A_\mu)$, *then* $F_t(x_0)=x(t)\in D(A_\mu)$ *is* X*-differentiable in* t *and satisfies* (3) *in the strong sense. Moreover,* $F_t^\mu:X\to X$ *is* C^∞ *for each* t *and* μ *and is jointly continuous in* (t,μ,x), *from* $\mathfrak{R}\times\mathfrak{R}^2\times X$ *to* X.

An existence theorem using weak methods was obtained for a related beam equation by Ball[10]. Since we require the stated differentiability results on F_t^μ which are not directly obtainable from Ball's results, it seems simpler for this example to proceed directly with strong solutions as we have done. However, note that Ball requires weak topologies for the more delicate Liapunov results he is concerned with.

In interpreting the qualitative behavior of the nonlinear semigroup $F_t^\mu: X \to X$, the following extension of the classical Liapunov linearization theorem is useful:

Proposition 2.9. Suppose $x_0 \in D(A)$ is a fixed point of (11), *i.e.,*

$$G(x_0) = A(x_0) + B(x_0) = 0$$

so $F_t(x_0) = x_0$ for all $t \geq 0$. Suppose that the spectrum of $\exp(DG(x_0))$ lies inside the unit circle, a positive distance from it. Then x_0 is locally exponentially stable; i.e., there is a neighborhood U of x_0 and a $\delta > 0$ such that if $x \in U$,

$$\|F_t x - x_0\| \leq C \exp(-\delta t)$$

Proof. (This relies only on the smoothness of F_t.)† From

$$\frac{d}{dt} F_t(u) = G(F_t(u))$$

we see that

$$\frac{d}{dt} DF_t(u) = DG(F_t(u)) \cdot DF_t(u)$$

and, in particular, the generator of the linear semigroup is

$$DG(x_0) = A + DB(x_0)$$

a bounded perturbation of the generator A. By the hypothesis on the spectrum, there is an $\varepsilon > 0$ and a suitable norm $|||\cdot|||$ such that

$$|||DF_t(x_0)||| \leq \exp(-\varepsilon t) \quad \text{for } t \geq 0$$

Thus, if $0 < \varepsilon' < \varepsilon$,

$$|||DF_t(x)||| \leq \exp(-\varepsilon' t) \quad \text{for } 0 \leq t \leq 1$$

and x in a neighborhood of x_0, say $U = \{x \mid |||x - x_0||| < r\}$. This is because F_t is C^1 with derivative continuous in t.

We claim that if $x \in U$, $0 \leq t \leq 1$, then $F_t x \in U$ and

$$|||F_t(x) - x_0||| \leq \exp(-\varepsilon' t) |||x - x_0|||.$$

Indeed, it is enough to prove this for small t since $\exp(-\varepsilon' t) < 1$. But it follows from this estimate:

$$\begin{aligned} &|||F_t(x) - x_0||| \\ &= |||F_t(x) - F_t(x_0)||| \\ &= |||\textstyle\int_0^1 DF_t(sx + (1-s)x_0) \cdot (x - x_0) ds||| \\ &\leq \textstyle\int_0^1 |||DF_t(sx + (1-s)x_0)|||\, |||(x - x_0)||| ds \\ &\leq \exp(-\varepsilon' t) |||x - x_0||| \end{aligned}$$

This result now holds for large t by using the facts that $F_t = F_{t/n}^n$ and $\exp(-\varepsilon' t) = [\exp(-\varepsilon' t/n)]^n$. Changing back to the original norm, the proposition is proved. ■

A similar proposition holds for the case in which part of the spectrum of $DG_\mu(x_0)$ lies in the right hand half plane. Here the linearization induces a (local) splitting of X into stable and unstable manifolds $W^s x_0$, $W^u x_0$ which are tangent at x_0 to the generalized eigenspaces associated with those parts of the spectrum in the left hand and right half planes. Intuitively $W^s x_0$ and $W^u x_0$ contain those 'directions' in which solutions flow 'towards' and 'away from' x_0 as $t \to \infty$. A similar set-up can be applied to more general critical elements such as closed orbits[4, 11].

We have now outlined some of the basic machinery for dealing with the qualitative analysis of a class of PDEs such as (1) with the parameter μ fixed. We now go on to study the case in which μ varies and the behavior of F_t^μ varies under its action. For the associated study of bifurcations we require the additional results discussed in Section 3.

3. CENTER MANIFOLD THEORY AND BIFURCATIONS

In this section we state the center manifold theorem for flows and an important associated result, and indicate their use in the panel flutter problem. In many bifurcation theorems, such as that of Hopf[4, 13], the nonlinear terms play a crucial role in providing (weak) attracting or repelling motions on the center manifold near the degenerate critical point, and hence the stability of the bifurcated orbits. This is often easy to guess, but not so simple to prove, as we shall discuss.

Theorem 3.1. Center Manifold Theorem for flows ([4]). *Let X be a Banach space admitting a C^∞ norm away from 0 and let F_t be a semiflow defined in a neighborhood of 0 for $0 \leq t \leq T$. Assume $F_t(0) = 0$ and that $F_t(x)$ is C^{k+1} in x with derivatives continuous in t. Assume that the spectrum of the linear semigroup $DF_t(0): X \to X$ is of*

†This theorem has been found in a number of special contexts by various authors, such as Prodi, Judovich, Sattinger, Gurtin, and McCamy. The version here is sketched in Marsden and McCracken[4].

the form $\exp(t(\sigma_1 \cup \sigma_2))$ *where* $\exp(t\sigma_1)$ *lies on the unit circle (i.e.* $\mathrm{Re}(\sigma_1)=0$*) and* $\exp(t\sigma_2)$ *lies inside the unit circle a nonzero distance from it, for* $t>0$ *(i.e.* $\mathrm{Re}(\sigma_2)<0$*). Let Y be the generalised eigenspace corresponding to* $\exp(t\sigma_1)$ *and assume* $\dim Y = d < \infty$.

Then, there exists a neighborhood V of 0 *in X and a* C^k *submanifold* $\tilde{M} \subset V$ *of dimension d passing through* 0 *and tangent to Y at* 0 *such that*

(a) *If* $x \in \tilde{M}$, $t>0$ *and* $F_t(x) \in V$, *then* $F_t(x) \in \tilde{M}$ *(local invariance)*

(b) *If* $t>0$ *and* $F_t(x)$ *remains defined and in V for all t, then* $F_t(x) \to \tilde{M}$ *as* $t \to \infty$ *(local attractivity).*

Remarks. If F_t is C^∞ then $\tilde{M}$ can be chosen to be C^l for any $l<\infty$. For the semigroup $F_t^\mu(x)$ with control parameter $\mu \in \mathfrak{R}^m$, if $F_t^\mu(x)$ is only assumed to be C^{k+1} in x and its x-derivatives depend continuously on t and μ and at $\mu=\mu_0$ part of the spectrum of $DF_t\mu_0(0)$ is on the unit circle, as in 3.1, then for μ near μ_0 we can choose a family of C^k invariant manifolds $\tilde{M}_\mu$ depending continuously on μ. *This family completely captures the bifurcational behaviour locally.* Note that in view of the Chernoff–Marsden results on separate and joint continuity, the 'continuity' in t and μ is in fact C^{k+1} smoothness [4, Thm 8A.7].

We note that D. Henry[15] has a version of the theorem to cover the case where the spectrum of $DF_t(0)$ has a component $\exp(t\sigma_3)$ comprising a *finite* number of eigenvalues *outside* the unit circle (i.e. $\mathrm{Re}(\sigma_3)>0$). Thus in addition to $\tilde{M}$ we also have invariant stable and unstable manifold W^s, W^u the dimensions of which are determined by the number of eigenvalues within and outside the unit circle; here $\dim W^u < \infty$. The theorem now provides a full infinite dimensional analogue of that for ODEs in $\mathfrak{R}^n$[3, 16, 17]. However, in the present case we need a further result, derived from the generalised Böchner–Montgomery Theorem:

Proposition 3.2. [4]. *Let* F_t *be a local* C^k *semiflow on a Banach manifold* $M, k \geq 2$ *and suppose that* F_t *leaves invariant a finite dimensional submanifold* $\tilde{M} \subset M$. *Then on* $\tilde{M}$, F_t *is locally reversible, jointly* C^k *in t and x and is generated by a* C^{k-1} *vector field on* $\tilde{M}$.

Provided F_t^μ satisfies the assumptions of 3.1 plus additional conditions related to specific bifurcations outlined below, 3.1 and 3.2 imply that we can find a $d+m$ dimensional subsystem $\tilde{M} \times U$, where U is a neighborhood of the critical parameter value $\mu=\mu_0$, such that $\tilde{M} \times U$ provides a local, finite dimensional, *essential model*. More details on the concept of essential models can be found in [18].

In the linear case the splitting into stable, center and unstable manifolds is closely related to the familiar concept of 'normal modes'. In finite dimensional problems such uncoupled modes are obtained by a suitable change of coordinates and the eigenspace associated with each mode is planar and isomorphic to $\mathfrak{R}^2$. Each mode behaves in a sense 'independently' and superpositional techniques may be used. The concept naturally generalizes to the infinite dimensional case and normal modes for the system

$$\frac{dx}{dt} = A_\mu x$$

could easily be defined by choosing a suitable (μ-dependent) basis for X (the basis of $H_0^2 \times L^2$ given by $\{\sin j\pi x\} \times \{\sin j\pi x\}$ applies for the case $\rho=0$). In the nonlinear case, then, it is natural to suppose that, at least locally, one can 'bend' the eigenspaces in such a manner that the nonlinear terms are also decoupled. This is exactly what the center manifold theorem allows us to do. See [18] for more discussion of this point.

Essentially we can say that *locally*, behavior on the stable and unstable manifolds does not change qualitatively as μ passes through its critical value and therefore that the bifurcations occurring in F_t^μ are restricted to the center manifold $\tilde{M}$. We need only study $\tilde{F}_t^\mu = (F_t^\mu$ restricted to $\tilde{M})$ to obtain a complete local characterization of bifurcational behavior, including information on the creation of new attracting and repelling solutions. In view of Proposition 3.2, $\tilde{F}_t^\mu$ is generated by a C^∞ vector field on $\tilde{M}$ and we are thus reduced to the study of a finite dimensional ODE on $\tilde{M}$:

$$\dot{\tilde{x}} = \tilde{G}(\tilde{x}); \tilde{x} \in \tilde{M}$$

Since $\tilde{M}$ is locally equivalent to $\mathfrak{R}^d$ we can make use of results on finite dimensional vector fields. In many examples, including that discussed here, $\dim(\tilde{M})=d \leq 2$ and we are thus able to use the special results relating to two dimensional vector fields[20–23]. Bifurcation theory for vector fields is not as well developed as that for mappings (elementary catastrophe theory[24]) but there are a number of very useful results. There is not space to discuss these in detail and here we merely outline two simple codimension one bifurcations: in a sense the most important since they occur most frequently in the absence of special symmetries or other *non-generic* conditions *when a single parameter changes.* For background see [5, 20–25].

Saddle-node. A single eigenvalue $\exp(t\sigma_1)$ of $DF_t^{\mu}(0)$ passes through 1 (σ_1 passes through 0) 'with non-zero speed':

$$d\sigma_1(\mu)/d\mu|_{\mu=0} \neq 0.$$

Here the situation is shown in Fig. 2a. In view of

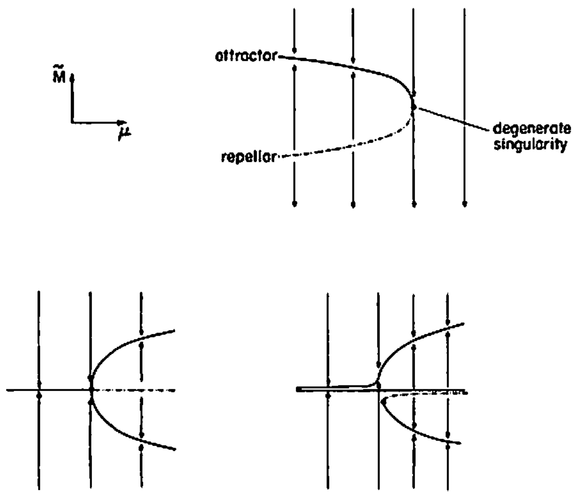

FIG. 2. Saddle-node bifurcations. (a) A 'simple' saddle node. (b) π-rotational symmetry. (c) A small imperfection.

3.1 our 2 dimensional picture is justifiable. Note that the two fixed points annihilate one another. When special symmetries are present as in the example treated in this paper (equation (1) contains only 'cubic' terms) a 'symmetric' bifurcation can occur (Fig. 2b). A small perturbation or *imperfection* causes this to unfold into a simple non-bifurcating path and an isolated saddle-node (Fig. 2c), cf. [26] for an application to buckling. Thus the saddle node occurs in models of *divergence.*

Hopf. A complex conjugate pair of eigenvalues passes through the unit circle away from ± 1 with non-zero speed (σ_1, $\bar{\sigma}_1$ pass through $\pm ic$; $c > 0$). Then there exists in the neighborhood of 0 a one parameter family of closed orbits 'surrounding' 0 and lying in a 3 dimensional subsystem. The 'type' of the orbits, attracting or repelling in $\tilde{M}$, depends upon the nonlinear part of $F_t^{\mu}(0)$; Fig. 3 depicts the attracting case. The two situations are sometimes referred to as super- and sub-critical. See [4] for more details.

The two bifurcations outlined here are *local* in the sense that they can be analysed in terms of a linearised vector field, operator or semigroup. Bifurcations involving limit cycles must generally be treated in terms of their *Poincaré* maps[4, 19] and the center manifold theorem for maps can be used[4]. However, other condimension one bifurcations are considerably more difficult to detect

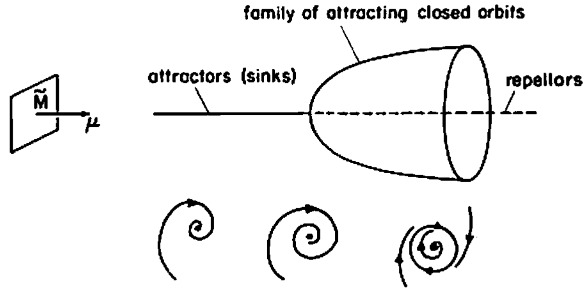

FIG. 3. The supercritical Hopf bifurcation (for the subcritical case reverse time, so that the attracting orbits become repelling).

since they involve the *global* behavior of trajectories joining saddle points. An example, in which a limit cycle is annihilated in a saddle connection is shown in Fig. 4[5, 22]. Recent

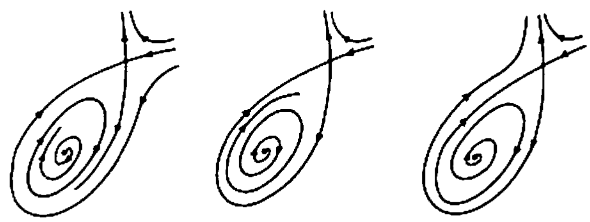

FIG. 4. A saddle connection (homoclinic orbit).

work of Takens[5], developing ideas of Andronov *et al.*[23] enables such bifurcations to be detected when they occur on two-manifolds. We make use of this in Section 5 (see also Kopell–Howard[17]).

We close this section with a discussion of stability criteria. In the Hopf bifurcation, for example, it is important to be able to compute whether the periodic orbit is stable or not, since computing the spectrum associated with the Poincaré map is generally impossible.

First consider the energy function H_2 of equation (4). We have seen that H_2 and associated Liapunov functions such as H_a do provide inner estimates of the size of stability regions. However, although improved choices of such functions might allow us to estimate the 'true' stability boundary reasonably well, they will not in general be able to determine stability exactly at the bifurcation point, unless the bifurcation set can be found analytically. As we see below the Hopf bifurcation set in the panel problem must be *estimated* from a finite dimensional numerical computation and it is in principle impossible to determine the sign of dH/dt on or very close to this set since we do not exactly know where it lies.

For determining the stability or 'direction' of the Hopf bifurcations we therefore turn to a criterion first discussed explicitly by Marsden and

McCracken[4] but implicit in Hopf's original finite dimensional theorem[13]. Details of this $V'''(\cdot)$ criterion are given in [4], chapter 4 and the criterion has since been extended and simplified by Hassard and Wan[14]. The method essentially consists in the computation of the nonlinear part of the vector field obtained by projection of the 'complete' vector field restricted to $\tilde{M}$ onto the eigenspace to which $\tilde{M}$ is tangent at the degenerate critical point. Although this is in theory possible when the vector field is infinite dimensional, the calculations seem formidable. In Section 5 we make use of two and four mode approximations and are therefore only dealing with vector fields on $\mathfrak{R}^4$ or $\mathfrak{R}^8$.

However, the energy expressions and consequent knowledge of global attractivity are of use in the following manner. Consider the evolution equation (3) for ρ, Γ 'small' and increasing. The Liapunov estimates ensure that for

$$\rho^2 < (\sqrt{\rho}\,\delta + \alpha\pi^4)^2(\Gamma + \pi^2)$$

there is a single sink at $\{0,0\} \in X$. We shall see below that as Γ or ρ increases, the linear operator A_μ eventually fails to generate a contraction semigroup, $\{0,0\}$ becomes non-hyperbolic and a bifurcation occurs. The nonlinear term $B(x)$ does not contain μ and the above bifurcation is thus the first to occur. If the origin is globally attracting below criticality and a Hopf bifurcation occurs first, it cannot be subcritical. It is thus likely that it is supercritical and hence the bifurcating closed orbits are stable. (The possibility that the closed orbits all occur at criticality, as in the undamped oscillator, seems unlikely because the vector field is strongly attracting for large amplitudes.)

In regions where other bifurcations or fixed points occur, Hopf bifurcations can be either sub- or super-critical. The $V'''(x_0)$ criterion is useful in such cases.

4. FINITE DIMENSIONAL APPROXIMATIONS AND CONVERGENCE

In order to obtain bifurcation results, it is first necessary to study the behavior of the spectrum of the equations linearized about a fixed point as $\mu = (\rho, \Gamma)$ varies. We shall discuss how this is done for the point $\{0,0\} \in X$ in the simply supported case. Other boundary conditions could be treated similarly.

We use the Galerkin averaging technique to obtain an n-mode approximate system. This system is on $\mathfrak{R}^{2n}$ and is given as follows (see [3]):

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{A}^n_\mu \mathbf{x} + \mathbf{B}^n(\mathbf{x}),$$

where $\mathbf{A}^n_\mu$ is the $2n \times 2n$ matrix given by

$$\mathbf{A}^n_\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & c_{12} & 0 & 0 & 0 & c_{14} & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ c_{21} & 0 & a_2 & b_2 & c_{23} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & c_{32} & 0 & a_3 & b_3 & c_{34} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ c_{41} & 0 & 0 & 0 & c_{43} & 0 & a_4 & b_4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{16a}$$

where

$$c_{ij} = 2ij(1-(-1)^{i+j})\rho/(j^2-i^2)$$

$$a_j = -\pi^2 j^2(\pi^2 j^2 + \Gamma) \quad \text{and} \quad b_j = -(\alpha\pi^4 j^4 + \sqrt{\rho}\,\delta)$$

and where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ \vdots \\ x_n \\ \dot{x}_n \end{bmatrix} \text{and } \mathbf{B}^n(x) = \begin{bmatrix} 0 \\ -\dfrac{\pi^4}{2}\left\{\sum_{i=1}^n i^2(kx_i^2 + \sigma x_i \dot{x}_i)\right\}x_1 \\ 0 \\ -\dfrac{\pi^4}{2}\left\{\sum_{i=1}^n i^2(kx_i^2 + \sigma x_i \dot{x}_i)\right\}x_2 \\ 0 \\ \vdots \end{bmatrix} \tag{16b}$$

The solution v^n we seek is

$$v^n = \sum_{i=1}^n x_i(t) e_i(z),$$

where $e_i = \sin i\pi z$ are a suitable set of orthonormal basis vectors. Proposition 2.13 (see the Appendix) shows that the Galerkin solution converges to the 'true' solution as $n \to \infty$.

Engineers frequently use techniques such as Galerkin's for the solution of nonlinear PDEs. In previous studies of equations such as (1) Dowell uses the method to obtain a finite set of ODEs such as (16) which are then solved by numerical integration for specific initial conditions and parameter values[1, 2]. In that case convergence is checked simply by carrying out the computations for systems with increasing numbers of modes and checking that solutions $v^n(x,t)$ converge as n increases (cf. [1], Figs 5 and 6). Here we are primarily concerned with *qualitative* properties of the system and hence with confirming that the bifurcations of the finite dimensional system (16), which we must use in eigenvalue estimations, do not differ from those of the full system in some parameter range of interest.

We consider the fixed point $\{0,0\}$. Here the linearised operator is simply A_μ and its approximation the $2n\times 2n$ matrix $\mathbf{A}_\mu^n$. If $\rho=0$ the spectrum is easy to determine since the matrix becomes block diagonal (i.e. the modes decouple), and one gets

$$\lambda_j = 1/2(b_j \pm \sqrt{b_j^2+4a_j}$$
$$= -(j^4\pi^4\alpha/2)\{1\pm\sqrt{1-4(j^2\pi^2+\Gamma)/\alpha^2\pi^6 j^6}\} \quad (17)$$

In this case $\mathbf{A}_\mu^n$ for $n=\infty$ is identical to A_μ and the eigenvalues given by (17) are exact. In particular for $\rho=\Gamma\equiv 0$, $\mathbf{A}_\mu^\infty$ is identical to A_0 of Proposition 2.4. For $j\leq\sqrt{2}/\pi\sqrt{\alpha}$ the eigenvalues are complex conjugate with real parts $=-j^4\pi^4\alpha/2$ and as $j\to\infty$ the (purely real) eigenvalues $\to-\infty$ and $-1/\alpha$. This yields an estimate of

$$\varepsilon = \min\left(\frac{1}{\alpha}, j^4\pi^4\alpha/2\right)$$

and hence an estimate for the decay of the semigroup generated by A_μ; see the remarks following Proposition 2.4.

Turning to the case $\rho>0$ we first note that a single mode approximation $v=x_1\sin\pi z$ does not exhibit flutter, as can be seen from the ODE

$$\frac{d}{dt}\begin{pmatrix}x_1\\ \dot{x}_1\end{pmatrix} = \begin{pmatrix}\dot{x}_1\\ -\pi^2(\pi^2+\Gamma)x_1-(\alpha\pi^4+\sqrt{\rho}\,\delta)\dot{x}_1 - \frac{\pi^4}{2}(\kappa x_1^2+\sigma x_1\dot{x}_1)x_1\end{pmatrix}$$

although a symmetric saddle-node as in Fig. 2(b) does occur at $\Gamma=-\pi^2$. However, the two mode model, which was discussed in detail in [3], does exhibit flutter and moreover appears to exhibit the qualitative behavior of four, six and higher mode models and of the full infinite dimensional system (3). In particular, the dimension of the center manifold $\tilde{M}$ appears not to increase and thus the essential model should remain (qualitatively) identical. To check this we must check that the behavior of the eigenvalues given by (17) does not differ qualitatively from that of the eigenvalues in the full system. As shown in previous studies[3] and in Section 5 below, we are particularly interested in a parameter range in the neighborhood of $\mu=\mu_0$; $(\Gamma,\rho)\approx(-2.3\pi^2, 108)$ for which $\mathbf{A}_\mu^n$ has a double zero eigenvalue when $\delta=0.1$ and $\alpha=0.005$. We wish to confirm that A_μ has a double zero eigenvalue close to μ_0 and that the remaining eigenvalues are in the negative half plane, as is the case for $\mathbf{A}_\mu^n$.

To illustrate the delicacy of the estimates, we first compare the finite dimensional two and four mode models. In the latter case the eigenvalues of $\mathbf{A}_\mu^4$ are the eight roots of the polynomial

$$m_{12}m_{34}+rem=0,$$

where

$$m_{12}=(\lambda^2-b_1\lambda-a_1)(\lambda^2-b_2\lambda-a_2)+c_{12}^2,$$
$$m_{34}=(\lambda^2-b_3\lambda-a_3)(\lambda^2-b_4\lambda-a_4)+c_{34}^2,$$

and

$$rem=(\lambda^2-b_1\lambda-a_1)(\lambda^2-b_4\lambda-a_4)c_{23}^2 + (\lambda^2-b_2\lambda-a_2)(\lambda^2-b_3\lambda-a_3)c_{14}^2+c_{14}^2c_{23}^2=0$$

Here m_{12} and m_{34} are the quartics obtained from two mode models taking modes (1 and 2) and (3 and 4) in pairs. It is not immediately clear that *rem* is small in comparison with this product but lengthy estimates and numerical work indicates that for μ near μ_0 the eigenvalue evolutions as μ varies are *qualitatively* identical. The double zero occurs in the two mode case for $(\Gamma,\rho)\simeq(-2.23\pi^2, 107.8)$ (remaining eigenvalues $\simeq -5.18 \pm 23.59i$) and in the four mode case for $(\Gamma,\rho)\simeq(-2.29\pi^2, 112.5)$ (remaining eigenvalues $\simeq -5.01 \pm 24.02i$; $-20.32\pm 73.82i$; $-62.99\pm 131.80i$). Increasing to six or eight modes appears to make little further difference to the top 4 eigenvalues, as Dowell's results suggest[1]. Studies of the operator $\mathbf{DG}_\mu^n(\pm\mathbf{x}_1^n)$ ($\mathbf{G}_\mu^n$ linearised at the two fixed points $\pm\mathbf{x}_1^n$ which bifurcate from $\{0\}$ as μ crosses a curve in the neighborhood of μ_0) indicate that its convergence properties are similar. We now return to the infinite dimensional case.

Strictly we should consider the general convergence situation for the operator

$$DG_\mu(x_0);\ G_\mu(\cdot)=A_\mu\cdot+B(\cdot)$$

linearised at a general fixed point $x_0\neq\{0,0\}$. However, since our analyses are local and we are primarily concerned with the behavior and stability of solutions splitting off $\{0,0\}$, the component $DB(x_0)$ of $DG_\mu(x_0)$ will be small compared to A_μ, since x_0 is close to $\{0\}$. Hence we must first establish that the eigenvalue behavior of an approximate operator $A_\mu^{(0)}$ (defined below) provides an acceptable estimate for that of A_μ. Then we can use $DG_\mu^{(0)}(x_0)$ (where $x_0=\mathbf{x}_0^n$), the fixed point of the Galerkin system on $\mathfrak{R}^{2n}$, with all other coordinates set to zero, and $DG_\mu^{(0)}(\cdot)$ is defined similarly to $A_\mu^{(0)}$) to provide estimates for $DG_\mu(x_0)$.

This prompts us to consider the linear operator

$$A_\mu^{(0)}=\begin{pmatrix} \mathbf{A}_\mu^n & 0 \\ 0 & A_\mu^r \end{pmatrix}:X\to X$$

where $\mathbf{A}_\mu^n$ is n-mode approximation (for example, the four mode one studied above) and A_μ^r is obtained by setting all the c_{ij}'s in (4.1) to zero for i or $j\geq n$, i.e., the '$(\infty-2n)\times(\infty-2n)$' matrix

$$A_\mu^r=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdot \\ a_5 & b_5 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 1 & \cdot \\ 0 & 0 & a_6 & b_6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad (18)$$

We now consider the full operator A_μ as a perturbation of $A_\mu^{(0)}$. Note that the eigenvalues of $A_\mu^{(0)}$ can be determined exactly: the first $2n$ by numerical computation of $\det|A_\mu^n-\lambda I|$ and the remaining (infinite) spectrum from (17) with $j\geq n+1$. It is easy to check that the c_{ij}; $i,j\geq n+1$ terms ignored in A_μ^r become increasingly negligible as n increases, *even for large* ρ. Thus for a given μ and with n chosen sufficiently large, A_μ is a *small* perturbation of $A_\mu^{(0)}$. Lengthy algebraic estimates then ensure that as n increases so the accuracy of the eigenvalues estimated from $A_\mu^{(0)}$ increases *for* μ *fixed.* The general case of convergence of qualitative behavior as μ varies is discussed in [6].

In particular for $\mu\approx\mu_0$ and for $4,6,8,\ldots$ modes the eigenvalues approximate the top $4,6,8,\ldots$ eigenvalues of the exact system well. (We work in even numbers of modes so the c_{ij}'s appear regularly). The location of the bifurcation curves for a 2-mode model may be inaccurate (near the bifurcation point of concern to us, the error in Γ is of the order of 0.5 and in ρ about 5). This phenomenon is also borne out in the work of Dowell[1], where he notes that a two mode model exhibits flutter at a value of ρ approximately 20% lower than that for the four, six and higher mode models, when $\Gamma=0$. His numerical convergence studies support our contention that the four and six mode models provide good approximations. However, the *qualitative behavior* of the eigenvalues seems to be the same for the 2 and 4 mode model near the point of concern to us ($\Gamma=2.3\pi^2$; $\rho\simeq108$). Our deductions are based primarily on this qualitative behavior, so for simplicity we describe the 2-mode model in some detail in Section 5.

In addition to the estimates obtained from Galerkin's method, it is also possible to estimate the spectrum numerically by considering the eigenvalue equation $A_\mu x=\lambda x$ directly. Specifically, the eigenvalues of A_μ are those λ for which the quartic equation

$$(1+\alpha\lambda)a^4-\Gamma a^2+\rho a+(\lambda^2+\sqrt{\rho}\,\delta\lambda)=0 \quad (19)$$

has four distinct non-zero roots $a_1,\ldots,a_4$, such that

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ e^{a_1} & a^{a_2} & e^{a_3} & e^{a_4} \\ a_1^2e^{a_1} & a_2^2e^{a_2} & a_3^2e^{a_3} & a_4^2e^{a_4} \end{bmatrix}=0 \quad (20)$$

Expressing the a_i as functions of λ, equation (20) yields a transcendental equation for λ.

5. AN EXAMPLE BASED ON A TWO MODE MODEL OF PANEL FLUTTER

We now make use of results obtained in the previous finite dimensional study[3]. It was shown there that the operator $\mathbf{A}_\mu^2:\mathfrak{R}^4\to\mathfrak{R}^4$ had a double zero eigenvalue at $\mu=(\Gamma,\rho)=(-22.91, 107.8)$, the remaining two eigenvalues being in the left hand plane. $\mathbf{A}_\mu^2$ is a finite dimensional approximation to $A_\mu:X\to X$ and in view of the convergence results of Section 4 we therefore assert that in the ODE on a Banach space (3) the origin $\{0\}$ is a degenerate critical point with a double zero eigenvalue at $\mu=\mu_0\simeq(-22.91, 107.8)$ and that all other eigenvalues have strictly negative real parts. In terms of Theorem 3.1, then, the spectrum of $DF_t^\mu(0)$; $\mu=\mu_0$, splits into two parts: $|\exp(t\sigma_1)|<1$ and $\exp(t\sigma_2)=1$ and the dimension of the latter's eigenspace is two. In view of the global existence, uniquess and smoothness results from §2, we can therefore apply 3.1 and extract a 2-dimensional center manifold $\tilde{M}$.

In a neighborhood U of $\mu_0\in\mathfrak{R}^2$, the control space, all eigenvalues but two remain within the unit circle. From [3] the structure of the bifurcation set is as shown in Fig. 5. We thus assert the existence of a 4-dimensional local *essential model* $\tilde{M}\times U\subset X\times\mathfrak{R}^2$ which completely captures the bifurcational behavior near 0. In particular, referring to the eigenvalue evolutions of Fig. 6, $\tilde{M}\times U$ contains a Hopf bifurcation occurring on B_h and a symmetrical saddle node occurring on B_{s1}. Moreover, finite dimensional computations for the two fixed points $\{\pm x_0\}$ appearing on B_{s1} and existing in region III shows that they are sinks, i.e.

$$|\text{spectrum}(DF_t^\mu(\pm x_0))|<1$$

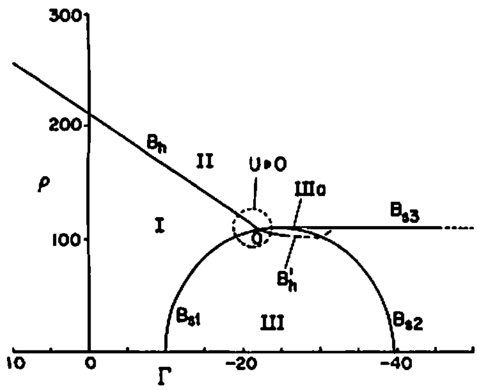

FIG. 5. Partial bifurcation set for the two mode panel ($\alpha = 0.005$, $\delta = 0.1$).

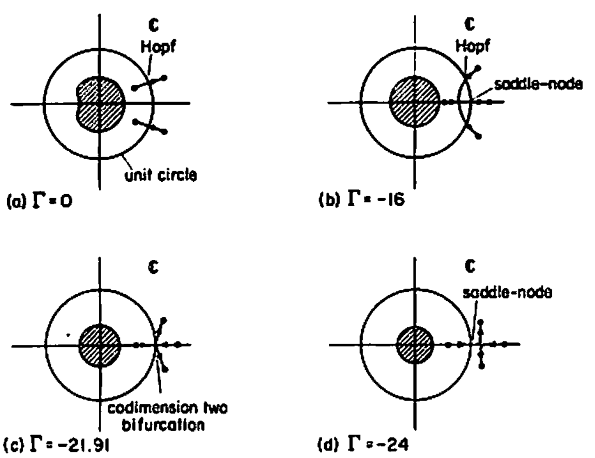

FIG. 6. Eigenvalue evolutions for $DF_\Gamma^\mu(0):X \to X$, Γ fixed, ρ increasing, estimated from 2 mode model. (a) $\Gamma = 0$. (b) $\Gamma = -16$. (c) $\Gamma \simeq -21.91$. (d) $\Gamma = -24$.

below a curve B_h' originating at 0 which we also show on Fig. 5. As μ crosses B_h' transversely $\{\pm x_0\}$ undergo simultaneous Hopf bifurcations before coalescing with $\{0\}$ on B_{s1}. We indicate below how all this behavior is captured by $\tilde{M} \times U$. A fuller description of the bifurcations, including those occurring on B_{s2} and B_{s3}, is provided in [3].

First consider the case where μ cross B_{s2} from region I to region III, not at 0. Here the eigenvalues indicate that a saddle-node bifurcation occurs. In [3] we derive exact expressions for the new fixed points $\{\pm x_0\}$ in the two mode case. This then approximates the behavior of the full evolution equation and the associated semiflow $F_t^\mu: X \to X$ and we can thus assert that a symmetric saddle-node bifurcation occurs on a one-dimensional manifold $\tilde{M}_1$ as shown in Fig. 2b and that the 'new' fixed points $\{\pm x_0\}$ are sinks in region III. Next consider μ crossing $B_h\backslash 0$. Here the eigenvalue evolution shows that a Hopf bifurcation occurs on a two-manifold $\tilde{M}_2$ and use of the stability arguments outlined in Section 3 indicate that the family of closed orbits existing in region II are attracting.

Now let μ cross $B_{s2}\backslash 0$ from region II to region IIIa. Here the closed orbits presumably persist, since they lie at a finite distance from the bifurcating fixed point $\{0\}$. In fact the new points $\{\pm x_0\}$ appearing on B_{s2} are saddles in region IIIa, with two eigenvalues of spectrum $DF_t^\mu(\pm x_0)$ outside the unit circle and all others within it $((\lambda > 1) = 2)$. As this bifurcation occurs, one of the eigenvalues of spectrum $DF_t^\mu(0)$ passes into the unit circle so that throughout regions IIIa and III $(\lambda > 1) = 1$ for $\{0\}$. Finally consider what happens when μ crosses B_h' from region IIIa to III. Here $\{\pm x_0\}$ undergo simultaneous Hopf bifurcations and the stability calculations show that the resultant sinks in region III are surrounded by a family of repelling (unstable) closed orbits.† We do not yet know how the multiple closed orbits of region III interact or whether any other bifurcations occur. However, the stability criterion of Section 2 and Proposition 2.1 imply that if $\rho = 0$ and $\|x\| > 0$ then $dH_h/dt < 0$ and thus oscillatory motions must decay and no closed orbits can exist in X. We are thus led to posit the existence of further bifurcation curves in region III for $\rho > 0$ on which closed orbits are created.

We now have a *partial* picture of the behavior near 0 derived from the two-mode approximation and from use of the stability criterion. The key to completing the analysis lies in the point 0, the 'organising centre' of the bifurcation set at which B_{s2}, B_h and B_h' meet and in the subsystem $\tilde{M} \times U$ which must somehow contain the individual behaviors noted above. Thus $\tilde{M} \times U$ contains all the relevant information and in the neighborhood $V \ni \{0\}$, $\tilde{M}$ is a union of the individual submanifolds $\tilde{M}_1$, $\tilde{M}_2$ etc. In particular, the degenerate singularity occurring at $\{0\} \times \mu_0 \in \tilde{M} \times U$ contains our information in its versal unfolding[22]. In view of corollary 3.2 we can regard F_t^μ restricted to $\tilde{M}$ as generated by a C^l vector field (for any $l < \infty$). Now Takens has analysed the singularities of such two parameter vector fields on two-manifolds[5]. (In fact Takens' analysis is for the C^∞ planar case but he informs us that his theorems go through with minor changes for our present situation, since although we work in a two-manifold we are only interested in a *local* analysis). We therefore make use of his results to pick the unique generic two parameter family of vector fields which fits our case, taking into account the symmetry; i.e. we demand that this family shares the behavior already detected and that it provides a coherent completion of our partial picture. This leads us to

†A calculation of Brian Hassard shows that these conclusions are also valid for the four mode (8-dimensional) model.

Conjecture 5.1. Flutter and divergence near $\{0\} \times \mu_0 \in X \times \Re^2$ *can be modeled by a two parameter vector field* $\tilde{X}_\mu$ *on a two manifold* $\tilde{M}$, *where* $\tilde{X}_\mu$ *is differentiably equivalent to Takens'* $m=2$; $-$*normal form*[5].

Thus the $m=2$; $-$normal form provides our essential model, see Fig. 7. Note that this allows

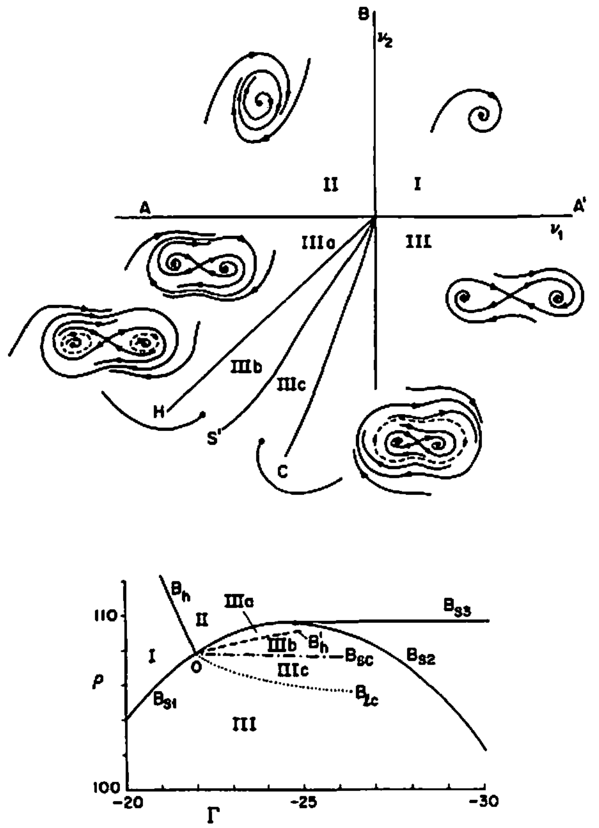

FIG. 7. A local model for bifurcations of the panel near 0, $(\rho, \Gamma) \simeq (107.8, -21.91)$; $\alpha = 0.005$, $\delta = 0.1$. (a) Takens' $m=2$; $-$'normal form[5], with associated vector fields. (b) The modified panel bifurcation set.

us to complete the partial bifurcation set of Fig. 5. We show a schematic evolution of attractors in Fig. 8, where the hysteresis effect associated with the coexistence of multiple attracting regimes and the resultant 'strong' bifurcations[22] should be noted.

One of the deeper and more significant features of bifurcation diagrams obtained this way is their structural stability, i.e., a slight perturbation of the equations governing the system, taking any imposed symmetries into account, will perturb the bifurcation diagram but will not alter its qualitative features.

The genericity and structural stability arguments behind Takens' classification make our conjecture a strong one. To verify it conclusively one would have to compute the vector field $\tilde{X}_\mu$ or at least the k-jet of that field, $k=3$, generating the flow $\tilde{F}^\mu_t = (F^\mu_t$ restricted to $\tilde{M})$. In the finite dimensional case the original 'complete' vector field X^n_μ on $\Re^{2n}$ is known and the computation reduces to that of a vector field $\tilde{\tilde{X}}^n_\mu$ on the eigenspace $\Re^2$ associated with the part of the spectrum of $DX^n_\mu(0)$ with zero real part (cf. [27].) It is then possible to show that $\tilde{\tilde{X}}^n_\mu$ is equivalent to $\tilde{X}^n_\mu$, the vector field on $\tilde{M}^n$ and an analysis of $\tilde{\tilde{X}}^n_\mu$ using Takens' Hamilton Bifurcation theory[5] would thus provide a conclusive result. Note that Takens' theory only applies to two-dimensional vector fields and that restriction to the center manifold is thus essential. In particular, detection of the 'Figure 8' global saddle-connection on B_{sc} and the bifurcation of closed orbits on B_{lc} relies on the computation of certain integrals on curves and domains in $\Re^2$.

In the infinite dimensional case a corresponding computation seems much more difficult, but

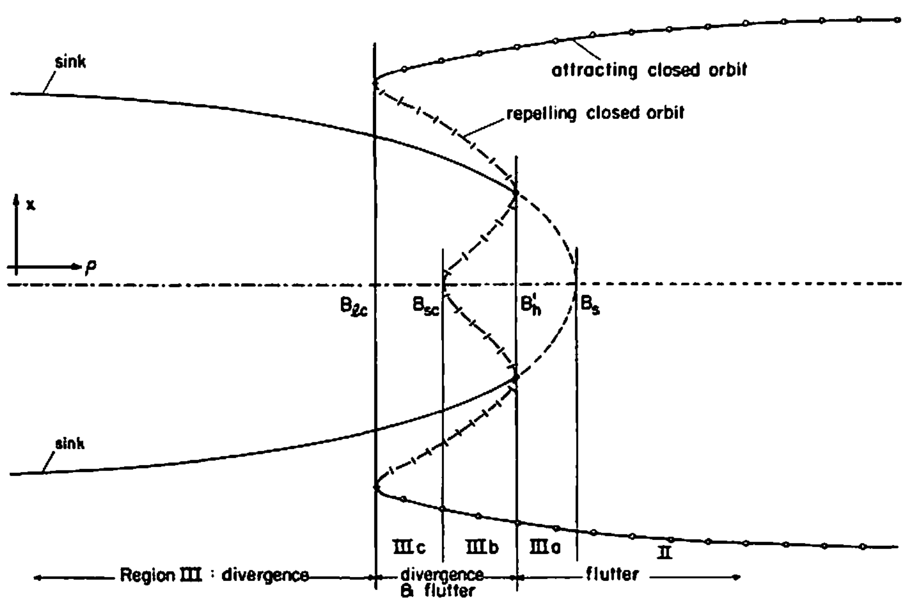

FIG. 8. Evolution of attractors for the panel, $\Gamma \simeq -24$, ρ increasing.

computation and analysis of $\tilde{X}^n_\mu$ and the convergence results discussed in Section 4 should suffice. However, we leave these computations for later study.

Remark. As shown in [18], the Takens $m=2$: − normal form can be written as an ODE as follows

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\nu_1 x_1 - \nu_2 x_2 - x_1^2 x_2 - x_1^3: \qquad (21)$$

as the nonlinear oscillator

$$\ddot{x} + \nu_2 \dot{x}_1 + \nu_1 x + x^2 \dot{x} + x^3 = 0 \qquad (22)$$

Here (x_1, x_2) can be regarded as a local coordinate system on $\tilde{M}$. Thus the essential model can be viewed as a 'nonlinear normal mode'. The parameters ν_1 and ν_2 of (21–22) and Fig. 7(a) are (nonlinear) functions of Γ and ρ. We should stress that (x_1, x_2) are *not* obtained by taking the natural basis of X as in the Galerkin work of Section 4, but that they are related to $\tilde{M}$ and hence to the eigenspace associated with the double zero eigenvalue occurring at 0 for $\mu = \mu_0$. However, the above remark does indicate that simple nonlinear oscillators are indeed justifiable as *local* models for flutter and divergence.

If the symmetry inherent in equation (1) is broken by, for example, the addition of a static pressure differential P_0 on the panel (cf. [1]), then it is not clear how the symmetry of the normal form of equations (21)–(22) will break. The bifurcation to divergence is easy to understand, since it will be governed by the cusp catastrophe and will break as indicated in Figs 2b and 2c; cf. [25]. However, Takens has suggested that the resultant *dynamic* bifurcations may involve degenerate singularities of codimension ≥ 3 and that perhaps a four parameter unfolding may be necessary. In a numerical study of a similar equation related to chemical reactions [28], Sel'kov has some interesting numerical results which indicate that asymmetrical saddle connections still take place, although the 'Figure 8' symmetry is broken.

We close with some comments on further globalization. The essential model obtained above is only valid locally near the point 0. Similar models can be obtained in the neighborhoods of other organising centers in the bifurcation set; see [3]. It is not immediately clear how to piece these organising centers together or if they require the four or higher mode model for their accurate detection. Our procedure near 0 was to piece together codimension one bifurcations using Takens' classification. If one adds a third parameter, such as α, it appears that the various organising centers coalesce for a certain value of α. Thus piecing together the organising centers appears to be a problem of *at least* a codimension *three* bifurcation on a two-manifold. (The most obvious guess would be an organisation of Takens' normal forms in pairs about five fixed points.)†

If the center manifold has dimension ≥ 3 e.g. if the 'grand' organising center alluded to above had a triple zero eigenvalue, then the picture may be much more complicated. Indeed, one might expect strange or chaotic motions. See [4, 25]. Moreover, the fact that codimension ≥ 3 bifurcations on two, let alone three, manifolds have not been classified makes these problems of further globalization very difficult.

6. CONCLUSIONS

In this paper we have outlined a new approach to the analysis of continuum mechanical or distributed parameter problems governed by nonlinear PDEs. We have taken the specific example of panel flutter to illustrate our thesis, but the techniques are clearly applicable to a wide range of similar problems and studies of hydrodynamic instability[4] and nonlinear buckling[26] have already appeared. The method stresses the qualitative aspects of behavior and relies on the extraction of an essential model which captures the local bifurcational behavior. In addition to permitting the analysis of complicated dynamic behavior, this drastic reduction in dimension suggests that models previously derived heuristically by engineers may be rigorously justifiable and also provides insight into their limitations. The great difficulty of globalization, beyond codimension 2, or fitting the local models together, is important here.

In this present work the main existence and uniqueness Theorems 2.5, 2.6 and 2.7 are quite general and in many applications the major part of the work would be in checking propositions similar to 2.1–4 and in estimating the spectra of suitably linearised operators. Note that global existence is not necessary for use of center manifold theory: in [4] an analysis of the Navier–Stokes equations is carried out, and only local existence has been proven for the three dimensional case.

In this short account it has not been possible to discuss our analysis in detail and in particular we have abbreviated the bifurcation theoretic aspects. The present study is also incomplete in some respects: the major conjecture 5.1 remains

†T. Poston has conjectured that the codimension here is at least 8 (personal communication).

to be established with rigour and additional convergence estimates may be needed. However, this paper should provide an introduction to a number of new techniques and an indication of how they can be used in a coherent scheme of analysis. We stress that this approach should be seen as complementary to existing techniques for solving nonlinear PDEs, such as asymptotic methods and the numerical integration of the finite dimensional evolution equation employed by Dowell[1, 2]. Knowledge of the qualitative structure of solutions in X or $\mathfrak{R}^n$ and in particular of the attracting sets is essential if numerical solutions are to be interpreted to maximum advantage.

Acknowledgements—This work was done with the financial support of the Science Research Council of the UK (PH) and the National Science Foundation of the U.S.A. and the Carnegie Foundation (JM). During part of the work PH held a research fellowship at the Institute of Sound and Vibration Research, Southampton University, and JM was Visiting Professor at Heriot-Watt University, Edinburgh. Both authors would like to express their thanks to John Ball, Jack Carr, David Chillingworth, Brian Hassard, Carlos Mota-Soares, Tim Poston and Floris Takens for valuable discussions, comments and calculations.

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APPENDIX TO SECTION 2

In this appendix we give a few supplementary remarks which may be useful in studying equations of the type (11). We begin by abstracting the essence of the energy argument in 2.1.

Proposition 2.10. Suppose the conditions of 2.5 and 2.7 hold and there is a C^1 function $H: X \to \mathfrak{R}$ such that

(i) *there is a monotone increasing function $\phi:[a,\infty)\to[0,\infty)$, where $[a,\infty)\supset$ Range of H, satisfying $\|x\|\leq\phi(H(x))$;*

(ii) *there is a constant $K\geq 0$ such that if $u(t)$ satisfies (11),*

$$\frac{d}{dt}H(u(t))\leq KH(u(t)).$$

Then $F_t(u_0)$ is defined for all $t\geq 0$ and $u_0\in X$.

If, in addition, H is bounded on bounded sets and

(iii) $$\frac{d}{dt}H(u(t))\leq 0 \quad \text{if} \quad \|u(t)\|\geq B,$$

then any solution of (11) remains uniformly bounded in X for all time; i.e., given $u_0\in D(A)$, there is a constant $C=C(u_0)$ such that $\|u(t)\|\leq C$ for all $t\geq 0$. Thus, the hypotheses of 2.7 hold.

Proof. By (ii),

$$H(u(t))\leq H(u_0)\exp(Kt)$$

so by (i),

$$\|u(t)\|\leq\phi(H(u_0)\exp(Kt))$$

Thus, global existence follows by 2.7. Let

$$H_B = \sup\{H(u) \mid \|u\| \leq B\}$$

so, by (iii),

$$H(u(t)) \leq \max\{H(u_0), H_B\}$$

Hence, by (i), we can take

$$C = \phi(\max\{H(u_0), H_B\})$$ ■

We next consider Liapunov's theorem 2.9. In critical cases, when the spectrum of $\exp(DG(x_0))$ is on the unit circle, stability of x_0 may be more difficult to determine. However, it is important for it usually determines the direction of bifurcation (subcritical or supercritical) near the given system. A well-known criterion for this in terms of energy functions is as follows:

Proposition 2.11. Let $H: X \to \Re$ be C^1, let x_0 be a fixed point of the system (11), and suppose:

(i) *for x in a neighborhood U of x_0, $H(x) \geq 0$ and there is a strictly monotone continuous functions $\psi: [0, \infty) \to [0, \infty)$ such that*

$$\|x - x_0\| \leq \psi(H(x));$$

(ii) *there is a continuous monotone function $f: [0, \infty) \to [0, \infty)$, locally Lipschitz on $(0, \infty)$ such that*

(a) $\frac{dH}{dt}(F_t(x)) \leq -f(H(F_t(x))$ *for* $x \in U$

and

(b) *solutions of $\dot{r} = -f(r)$ tend to zero as $t \to +\infty$.*

Then x_0 is asymptotically stable.

Proof. Let $r(t)$ be the solution of $\dot{r} = -f(r)$ with $r(0) = H(x)$, $x \in U$, $x \neq x_0$. Then, by (ii), $H(F_t(x)) \leq r(t)$. Hence $H(F_t(x)) \to 0$ as $t \to +\infty$. Thus, by (i), $F_t(x)$ remains near x_0 and converges to it as $t \to +\infty$. ■

Example 2.12. (A special case of a result of Ball and Carr[12]). Consider the critical Duffing equation $\ddot{u} + \dot{u} + u^3 = 0$. We show that the origin is asymptotically stable and that solutions decay like $C/\sqrt{t}$ as $t \to +\infty$. We consider the function

$$H(u, \dot{u}) = (u + \dot{u})^2 + \dot{u}^2 + u^4$$

and find

$$\frac{d}{dt} H(u, \dot{u}) = -2u^4 - 2\dot{u}^2$$

$$\leq \begin{cases} -2 & \text{if} \max(|u|, |\dot{u}|) \geq 1 \\ -\frac{1}{6}[H(u, \dot{u})]^2 & \text{if} \max(|u|, |\dot{u}|) \leq 1 \end{cases}$$

Thus, with $f(r) = -(1/6)r^2$ we find $H(u, \dot{u}) \leq C/t$ as $t \to \infty$, so $|u(t)| + |\dot{u}(t)| \leq C'/\sqrt{t}$ as $t \to \infty$.

In some bifurcation problems and, in particular, the one we study here, the equations have critical points whose location is known only numerically and the parameter values at criticality are known only numerically as well. In such situations, the technique above cannot be used since it is sensitive to small perturbations. For this reason other methods for examining critical cases are needed. For the Hopf bifurcation we can use stability formulas developed in that theory; see Hopf[13], Marsden and McCracken[4], Hassard and Wan[14], and Section 3.

Now we turn to a theorem on the convergence of the solutions for equations approximating (11). These will be Galerkin or other approximations in examples like panel flutter.

Proposition 2.13. Let the hypotheses of 2.5. hold for $G = A + B$ and for $G_n = A_n + B_n$, $n = 1, 2, \ldots$ on a fixed Banach space X.† Assume

(i) $\|\exp(tA_n)\| \leq M \exp(t\beta)$, *for all $t \geq 0$ and $n = 1, 2, \ldots$;*

(ii) $(\lambda - A_n)^{-1} \to (\lambda - A)^{-1}$ *strongly, for $\lambda > \beta$;*

(iii) $B_n \to B$ *locally uniformly on X; and*

(iv) *there is an open set U about any given point, and a constant $K \geq 0$ such that*

$$\|B_n(x) - B_n(y)\| \leq K\|x - y\|, \quad n = 1, 2, \ldots; x, y \in U$$

Then $F_t^n(u_0) \to F_t(u_0)$ locally uniformly, where F_t^n is the local semiflow of G_n. Convergence holds for all $t \geq 0$ for which the semiflows are defined.

Proof. Let

$$u(t) = \exp(tA)u_0 + \int_0^t \exp((t-s)A)B(u(s))ds$$

and

$$u_n(t) = \exp(tA_n)u_0 + \int_0^t \exp((t-s)A_n)B_n(u_n(s))ds$$

so that

$$\|u(t) - u_n(t)\| \leq \|\exp(tA)u_0 - \exp(tA_n)u_0\|$$
$$+ \int_0^t \|\exp((t-s)A)B(u(s)) - \exp((t-s)A_n)B(u(s))\|ds$$
$$\times \int_0^t M \exp(t\beta)\|B(u(s)) - B_n(u_n(s))\|ds.$$

By the Trotter–Kato theorem (Kato [8, p. 502]), $\exp(tA_n) \to \exp(tA)$ (strongly), uniformly on bounded t-intervals. Thus the first two terms $\to 0$ as $n \to \infty$. The last term is bounded by

$$M \exp(t\beta) \int_0^t \|B(u(s)) - B_n(u(s))\|ds$$
$$+ M \exp(t\beta) \int_0^t \|B_n(u(s)) - B_n(u_n(s))\|ds.$$

The first term $\to 0$ by (iii) and the second is bounded by

$$KM \exp(t\beta) \int_0^t \|u(s) - u_n(s)\|ds$$

by (iv), for t sufficiently small. Thus, by Gronwall's inequality, $u_n(t) \to u(t)$ in X as $n \to \infty$ uniformly in t for t small. A routine argument now establishes convergence for all t for which the semiflows are defined. ■

Remark. If (i) holds and if $D(A_n) \supset D(A)$, $n = 1, 2, \ldots$ and if $A_n u \to Au$ for $u \in D(A)$, then (ii) will hold (from the resolvent formula); see Kato[8, p. 429].

This result will actually show that the semiflows defined by the Galerkin approximations to the panel problem (see [3]) converge to the semiflow of the full nonlinear partial differential equation (3) of the panel.

†If G_n is associated with a subspace $X_n \subset X$ and $P_n: X \to X_n$ is a projection, $G_n \circ P_n$ will be associated with X itself.