BIFURCATIONS OF DYNAMICAL SYSTEMS AND NONLINEAR OSCILLATIONS IN ENGINEERING SYSTEMS

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Dedicated to Eberhard Hopf.

INTRODUCTION

This paper analyzes recent qualitative methods for partial differential equations which are suitable for the analysis of complex bifurcations which may occur in nonlinear engineering systems. We are particularly concerned with flow induced oscillations which occur in, for example, galloping transmission lines or panel flutter and related vibration problems.

We shall present a general framework for the analysis of these problems with the aim of extracting qualitative information, such as the existence and number of periodic orbits or rest points and their stability. This analysis is meant to complement existing techniques

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ı as asymptotic or numerical methods.

The present method uses intrinsically qualitative techniques, h as those of blowing up a singularity and of invariant manifolds. se approaches are particularly powerful for multiparameter problems.

After some dynamical preliminaries, we illustrate the technique blowing up a singularity for bifurcation of fixed points and then s to dynamic bifurcations. The last section discusses applications various engineering systems exhibiting flutter, and in particular : problem of panel flutter.

SOME PRELIMINARIES

In many problems concerning the bifurcation of equilibrium states, is important to keep the full dynamical problem in mind. For ımple, stability is often best understood in the dynamical sense; so it may be useful to know that the bifurcated equilibria lie on invariant manifold of low dimension for the full dynamical problem. course, if the bifurcations include oscillations (periodic orbits), is impossible to ignore the dynamics.

We shall be interested in methods which are applicable to ıltiparameter systems. Indeed, this is often necessary to produce ifurcation diagrams which are insensitive to small perturbations in he equations. (In the literature this is variously studied under he headings "Perturbed Bifurcation Theory" - cf. Keener and Keller 26] or "Catastrophe Theory" - cf. Arnold [2] and Thom [55]. The ost famous example of this is Euler buckling. Viewed as a one arameter system with parameter the beam tension, one gets the raditional picture shown in figure 1.

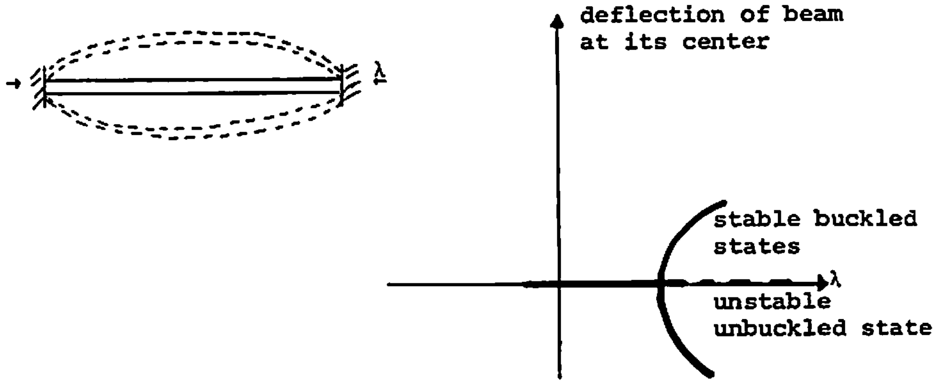
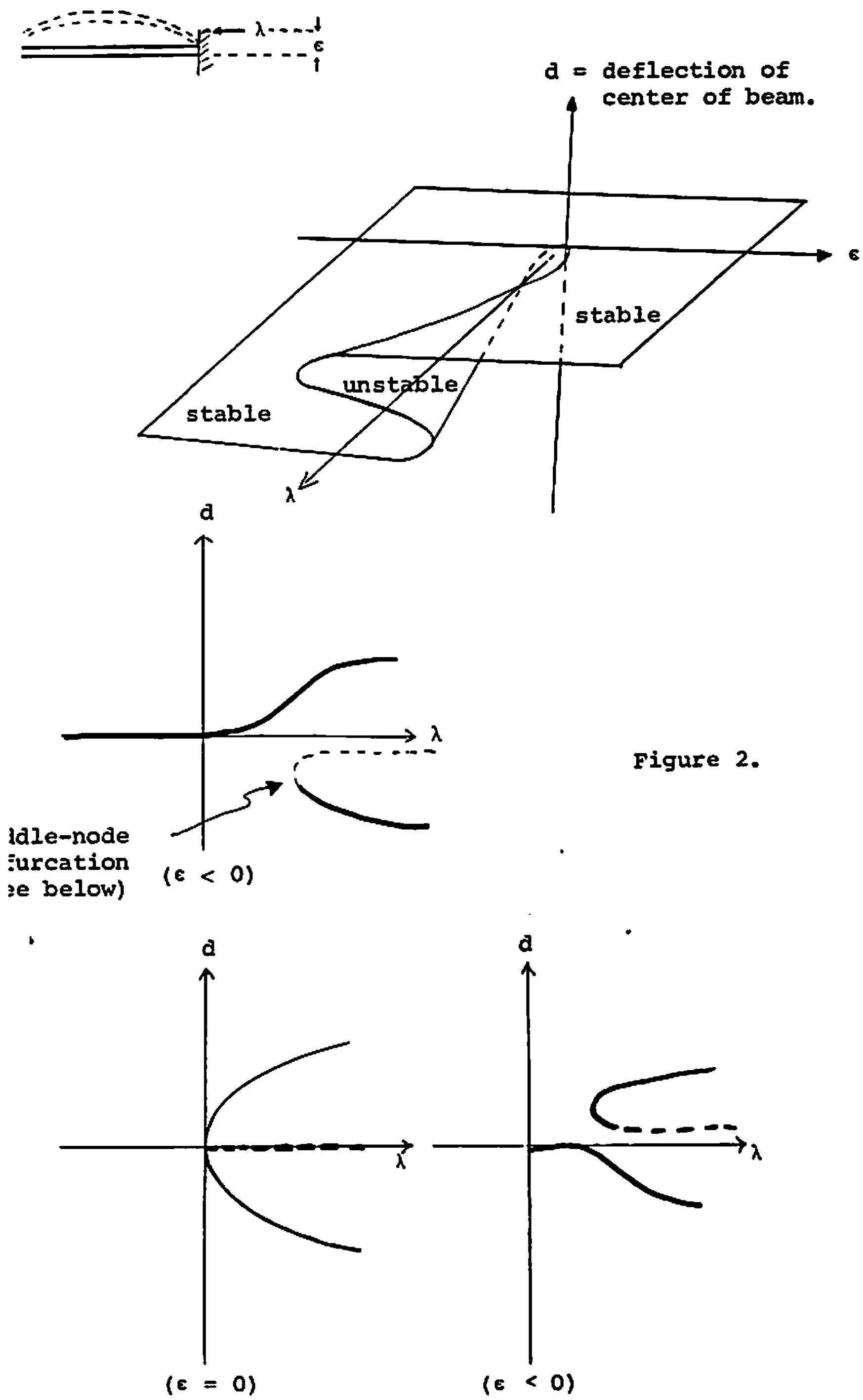


Figure 1

However, this bifurcation diagram is "unstable". It can be tablized by adding a second parameter ϵ , which describes the symmetry of the force λ .† Now we get the more comprehensive ifurcation diagram shown in Figure 2. That in figure 1 is obtained y taking the slice $\epsilon = 0$. This new two-parameter bifurcation liagram is now qualitatively insensitive to further perturbations, ince the cusp singularity is "structurally stable" [55].

†Bifurcations of the fixed points of Duffing's equation $\ddot{x} + \alpha\dot{x} + \gamma x^2\dot{x} - \lambda x + \delta x^3 + \epsilon = 0$ provide a model for this system. See Holmes and Rand [21] for a complete account.


λ
ε
d = deflection of
center of beam.
ε
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d
λ
(ε = 0)
d
λ
(ε < 0)

Figure 2.

The reader can find full discussions of this kind of phenomenon from both the engineering and the mathematical point of view in Zeeman [59], Chow, Hale and Mallet-Paret [8], Thompson and Hunt [56] and Roorda [41-43], and references therein.

§1.1 THE MATHEMATICAL FRAMEWORK

The dynamical framework in which we operate is described as follows. Let $X \subset Y$ be Banach spaces (or manifolds) and let

$$f : X \times \mathbb{R}^p \to Y$$

be a given C^k mapping $(k \geq 2)$. Here $\mathbb{R}^p$ is the parameter space and f may be defined only on an open subset of $X \times \mathbb{R}^p$. The dynamics is given by

$$\frac{dx}{dt} = f(x,\lambda)$$

which defines a semi-flow

$$F_t^\lambda : X \to X$$

by letting $F_t^\lambda(x_0)$ be the solution of $\dot{x} = f(x,\lambda)$ with initial condition $x(0) = x_0$. We assume that this equation defines a local semi-flow on X ; i.e. it has, at least locally in time, unique solutions.

A <u>fixed point</u> is point (x_0,λ) such that $f(x_0,\lambda) = 0$. Therefore, $F_t^\lambda(x_0) = x_0$ i.e. x_0 is an equilibrium point of the ıamics.

A fixed point (x_0,λ) is called stable if there is a neighbor-
ɔd U_0 of x_0 on which $F_t^\lambda(x)$ is defined for all $t \geq 0$ and if : any neighborhood $U \subset U_0$, there is a neighborhood $V \subset U_0$ such ıt $F_t^\lambda(x) \in U$ if $x \in V$ and $t \geq 0$. The fixed point is called ɱptotically stable if, in addition, $F_t^\lambda(x) \to x_0$ as $t \to +\infty$, for in a neighborhood of x_0.

Many nonlinear partial differential equations of evolution type ll into this framework, as we shall see in §4. Also, many semi-near hyperbolic and most parabolic type equations satisfy an ditional smoothness condition; we say F_t^λ is a smooth semi-flow for each t, λ, $F_t^\lambda : X \to X$ (where defined) is a C^k map and s derivatives are strongly continuous in t, λ.

For general conditions under which a semi-flow is smooth, see :sden and McCracken [32]. One especially simple case occurs when

$$f(x,\lambda) = A_\lambda x + B(x,\lambda) ,$$

ıere $A_\lambda : X \to Y$ is a linear generator depending continuously on and $B : Y \times \mathbb{R}^p \to Y$ is a C^k map. This result is readily :oved by the variation of constants formula

$$x(t) = e^{tA_\lambda} x_0 + \int_0^t e^{(t-s)A_\lambda} f(x(s),\lambda)ds$$

See Segal [48] for details).

Standard estimates and the proof for ordinary differential equations now prove the following (see Marsden and McCracken [32] for details):

LIAPUNOV'S THEOREM. *Suppose* F_t^λ *is a smooth flow,* (x_0,λ) *is a fixed point and the spectrum of the linear semi-group*

$$U_t^\lambda = D_x F_t^\lambda(x_0) : X \to X$$

(The Fréchet derivative with respect to $x \in X$ *) is* $e^{t\sigma}$ *where* σ *lies in the left half plane a distance* $> \delta > 0$ *from the imaginary axis. Then* x_0 *is asymptotically stable and for* x *sufficiently close to* x_0 *we have an estimate*

$$\|F_t^\lambda(x) - x_0\| \le Ce^{-t\delta} \quad .$$

If we are interested in the location of fixed points, we solve the equation

$$f(x,\lambda) = 0 ,$$

and the stability of a fixed point x_0 will be determined by the spectrum σ of the linearization at x_0 :

$$A_\lambda = D_x f(x_0,\lambda) \ .$$

(We assume the operator is non-pathological --- eg has discrete spectrum --- so $\sigma(e^{tA_\lambda}) = e^{t\sigma(A_\lambda)}$.) In critical cases where the spectrum lies on the imaginary axis, stability has to be determined

›ther means. It is at criticality where, for example, a curve of ›d points $x_0(\lambda)$ changes from stable to unstable, that a ırcation can occur, as we shall see in §2 .

The second major point we wish to make is that within the text of smooth semi-flows, the usual invariant manifold theorems m ordinary differential equations carry over.

In bifurcation theory it is often useful to apply the invariant ifold theorems to the suspended flow

$$F_t : X \times \mathbb{R}^p \to X \times \mathbb{R}^p$$
$$(x,\lambda) \to (F_t^\lambda (x),\lambda)$$

› invariant manifold theorem states that if the spectrum of the ıearization A_λ at a fixed point (x_0,λ) splits into $\sigma_S \cup \sigma_C$, ›re σ_S lies in the left half plane and σ_C is on the imaginary is, then the flow F_t leaves invariant manifolds M_S and M_C ngent to the eigenspaces corresponding to σ_S and σ_C respective- ; M_S is the stable and M_C is the center manifold. (One can low an unstable manifold too if that part of the spectrum is nite). By Liapunov's theorem, orbits on M_S converge to (x_0,λ) ponentially. For suspended systems, note that we always have $\in \sigma_C$.

The idea of the proof is this: we apply invariant manifold ıeorems for smooth maps with a fixed point to each F_t separately. ince F_t and F_s commute $(F_t \circ F_s = F_{t+s} = F_s \circ F_t)$, it follows nat these invariant manifolds can be chosen in common for all the

F_t .

For bifurcation problems the center manifold theorem is the most relevant, so we summarize the situation. (See Marsden and McCracken [32] for details).

CENTER MANIFOLD THEOREM FOR FLOWS. Let Z be a Banach space admitting a C^∞ norm away from 0 and let F_t be a C^0 semi-flow defined on a neighborhood of 0 for $0 \le t \le \tau$. Assume $F_t(0) = 0$ and for each $t > 0$, $F_t : Z \to Z$ is a C^{k+1} map whose derivatives are strongly continuous in t. Assume that the spectrum of the linear semigroup $DF_t(0) : Z \to Z$ is of the form $e^{t(\sigma_s \cup \sigma_c)}$ where $e^{t\sigma_c}$ lies on the unit circle (i.e. σ_c lies on the imaginary axis) and $e^{t\sigma_s}$ lies inside the unit circle a nonzero distance from it, for $t > 0$; i.e. σ_s is in the left half plane. Let Y be the generalized eigenspace corresponding to the part of the spectrum on the unit circle. Assume $\dim Y = d < \infty$.

Then there exists a neighborhood V of 0 in Z and a C^k submanifold $M_c \subset V$ of dimension d passing through 0 and tangent to Y at 0 such that

(a) If $x \in M_c$, $t > 0$ and $F_t(x) \in V$, then $F_t(x) \in M_c$.

(b) If $t > 0$ and $F_t^n(x)$ remains defined and in V for all $n = 0, 1, 2, \ldots$, then $F_t^n(x) \to M_c$ as $n \to \infty$.

See Figure 3 for a sketch of the situation.

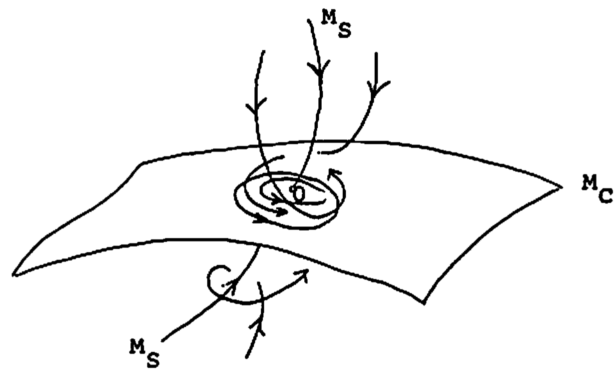

Figure 3

For example, suppose we have a curve of fixed points $x_0(\lambda)$, $\in \mathbb{R}$ which become unstable as λ crosses λ_0 and two stable xed points branch off, as in figure 1. Then all three points will e on the center manifold for the suspended system. Taking λ = nstant slices yields an invariant manifold M_C^λ for the parametrized stem; see figure 4.

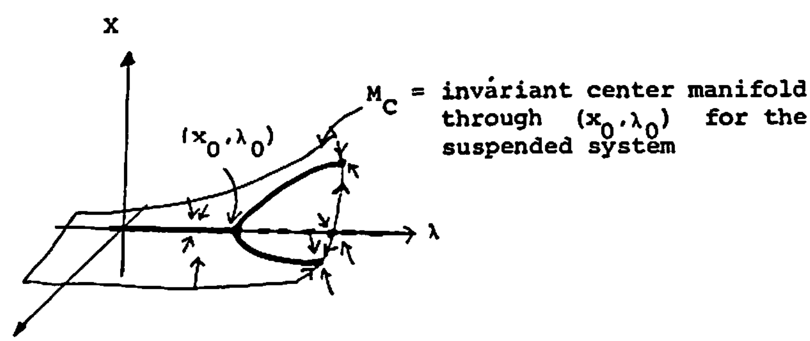

Figure 4 (also, see p. 170)

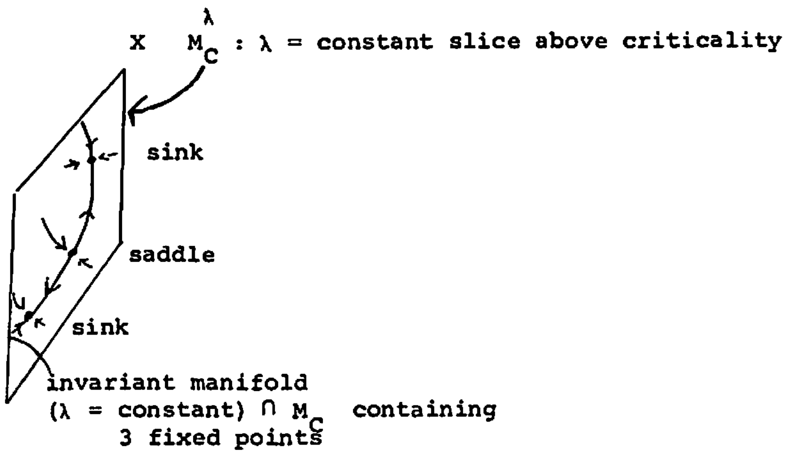

Figure 4 (cont'd)

Although the center manifold is only known implicity, it can greatly simplify the problem qualitatively by reducing an initially inifinite dimensional problem to a finite dimensional one. Likewise, questions of stability become questions on the center manifold itself. For example, it becomes clear, at least under a non-degneracy condition, that in the context of figure 4, supercritical branches are stable and subcritical branches are unstable. (For center manifolds of higher dimension, however, this is not true in general - see McLeod and Sattinger [30] for instance and the discussion in §3).

§2. BIFURCATION OF FIXED POINTS

Most of the literature on bifurcation theory deals with bifurcation of fixed points. For example, see Cesari [7], Sather [45, 46], Crandall and Rabinowitz [9, 10], Nirenberg [35], Sattinger

'], articles in Keller and Antman [25], and the references therein. the applications side, many of the papers grew out of work of :ter (see Thompson and Hunt [56] for references).

Here we shall give a relatively simple geometrical framework for ıling with bifurcations at multiple eigenvalues for multiparameter stems. The approach follows Buchner, Marsden and Schecter [6] and nbines some ideas in Nirenberg [35] with the method of blowing up singularity. A general stability analysis is complex, as indicated McLeod and Sattinger [30]. In specific problems this can sometimes reduced to that for single parameter systems or an eigenvalue alysis can be done numerically, as we shall indicate in §4. erefore, we shall not discuss stability at this point any further.

As above, fixed points are determined by the zeros of a C^k map

$$f : X \times \mathbb{R}^p \to Y$$

Let $x_0(\lambda)$ be a given p-parameter manifold of solutions of $x,\lambda) = 0$ i.e. $f(x_0(\lambda),\lambda) = 0$ for λ in an open set in $\mathbb{R}^p$. t $x_0(\lambda_0) = x_0$. Following standard terminology, we say that $_0,\lambda_0)$ is a <u>bifurcation point</u> if every neighborhood of (x_0,λ_0) ntains a solution (x,λ) of $f(x,\lambda) = 0$ with $x \neq x_0(\lambda)$. The set all solutions near (x_0,λ_0), including $(x_0(\lambda),\lambda)$, constitute e <u>bifurcation set</u>. From a more general point of view, it seems sirable to define a bifurcation point as one near which the set of lutions changes topological type as λ varies.

If (x_0,λ_0) is a bifurcation point, then $D_x f(x_0,\lambda_0) : X \to Y$, the Fréchet derivative of f with respect to x at (x_0, λ_0), is not a surjection. This is a trivial consequence of the implicit function theorem. Nevertheless, this criterion is effective in singling out condidates for bifurcation points.

Let $X_1 = \ker D_x f(x_0, \lambda_0)$ and assume

(i) X_1 splits; i.e. $X = X_1 \oplus X_2$ for a closed subspace $X_2 \subset X$ and (ii) X_1 is finite dimensional.

We refer to Buchner, Marsden and Schecter [6] for the case in which X_1 is allowed to be infinite dimensional.

Likewise, assume

(iii) Range $D_x f(x_0, y_0) = Y_1$ is closed and has a closed complement Y_2 ; $Y = Y_1 \oplus Y_2$, $\dim Y_2 < \infty$.

Let P be the projection of Y to Y_1 and let $x_2 = u(x_1, \lambda)$ be the unique solution of

$$f(x_1 + x_2, \lambda) = 0$$

for $x_1 \in X_1$, $x_2 \in X_2$ near (x_0, λ_0) . This is guaranteed by the implicit function theorem. Thus, the equation $f(x, \lambda) = 0$ is equivalent to the <u>bifurcation equation</u>:

$$(I - P) f(x_1 + u(x_1, \lambda), \lambda) = 0$$

The reduction to the bifurcation equation, called the Liapunov-Schmidt procedure, is analogous to the reduction to the center manifold. In fact, as described above, the bifurcation of fixed points takes place within a center manifold for the dynamical systems.

Usually, but not always, $D_x f$ is a Fredholm map and so the bifurcation equation is a finite dimensional problem. (An exception occurs in general relativity; see Fischer, Marsden and Moncrief [13].) The methods of Buchner, Marsden and Schecter [6] do not require this assumption.

Now we give the main result on bifurcation at simple eigenvalues. e hypotheses are stated in a form convenient for verification and weakened below with no change in the proof). The result is essenlly the same as in Crandall and Rabinowitz [9] , to which the readis referred for examples. The proof, however, is more geometrily satisfying. If is due to Nirenberg [35] , based on a suggestion Duistermaat .

EOREM (BIFURCATION AT SIMPLE EIGENVALUES). *Assume* p = 1, *dim* X_1 = *dim* $Y_2 = 1$,

$$\frac{\partial f}{\partial \lambda}(x_0,\lambda_0) = 0, \quad \frac{\partial^2 f}{\partial \lambda^2}(x_0,\lambda_0) \in Y_1$$

and $\frac{\partial^2 f}{\partial\lambda\partial x}(x_0,\lambda_0)\cdot x_1 \notin X_1$ *where* X_1 = *span* (x_1) , $\|x_1\| = 1$

en the bifurcation set near (x_0,λ_0) *consists of two intersecting, ansversal,* C^{k-2} *curves.*

OF. Let ℓ be a linear functional orthogonal to Y_1 and let

$$\varphi : X_1 \times \mathbb{R} \to \mathbb{R},$$

$$\varphi(x_1,\lambda) = \ell(f(x_1 + u(x_1,\lambda), \lambda)$$

that, $\varphi^{-1}(0)$ is the bifurcation set near (x_0,λ_0) . Easy calculans show that

$$\varphi(x_0,\lambda_0) = 0$$

$$d\varphi(x_0,\lambda_0) = 0$$

$$d^2\varphi(x_0,\lambda_0) = \begin{pmatrix} * & \ell\left(\frac{\partial^2 f}{\partial\lambda\partial x}(x_0,\lambda_0)\cdot x_1\right) \\ \ell\left(\frac{\partial^2 f}{\partial\lambda\partial x}(x_0,\lambda_0)\cdot x_1\right) & 0 \end{pmatrix}$$

Thus (x_0,λ_0) is a non-degenerate critical point for φ of index 1. Thus by a C^{k-2} change of coordinates, $\varphi(y,\mu) = \frac{1}{2}(y^2 - \mu^2)$ by the Morse lemma. Hence $\varphi^{-1}(0)$ is two C^{k-2} curves. □

With this point of view, we can now turn to the general case. We shall need a preliminary definition.

DEFINITION. Let Z_1, Y_2 be Banach spaces and $B : Z_1 \times Z_1 \to Y_2$ be a continuous symmetric bilinear form. Let $Q(v) = \frac{1}{2}B(v,v)$ be the associated quadratic form. Let $C = Q^{-1}(0)$; i.e. C is the cone of zeros of Q. We say Q is in general position at $v \in C$, $v \neq 0$ if the linear map $w \mapsto B(v,w)$ of Z_1 to Y_2 is surjective. If we say Q is in general position on C we mean it is so at each non-zero point of C.

THEOREM (THE GENERAL CASE). *Let* $f : X \times \mathbb{R}^p \to Y$ *be as above. Assume* (i), (ii) *and* (iii) *hold, and that*

$$\frac{\partial f}{\partial \lambda}(x_0,\lambda_0) = 0 \, .$$

Let $B = (I - P)D^2f(x_0,\lambda_0)$ *restricted to* $X_1 \times \mathbb{R}^p = Z_1$ *and* Q *be the associated quadratic form. Assume that* Q *is in general position on* C.

Then the bifurcation set near (x_0,λ_0) *is homeomorphic to* $C = Q^{-1}(0)$ *via a homeomorphism that takes* (x_0,λ_0) *to* 0 *and is a* C^k *diffeomorphism away from* (x_0,λ_0).

If $v \in Q^{-1}(0)$, *there is a* C^{k-2} *curve* $(x(s),\lambda(s))$ *of solutions to* $f(x,\lambda) = 0$ *tangent to* v *at* (x_0,λ_0) *and the union of these curves constitutes the bifurcation set.*

REMARKS. 1. If one only knows Q is in general position at a particular $v \in C$, then v is still tangent to a C^{k-2} curve in the bifurcation set.

2. The proof may enable one to determine the structure of the bifurcation set even if the hypotheses fail. One may have to rescale the variables by different amounts in different directions and follow the method outlined in the blowing up lemma below.

3. If $p-1 + \dim X_1 = \dim Y_2 = m$, the bifurcation set consists on 2s curves of class C^{k-2} through (x_0, λ_0) where $1 \le s \le 2^{m-1}$ (This follows by using Bezout's theorem from algebraic geometry todetermine the number of rays in $Q^{-1}(0)$).

The theorem is proved by appeal to a rather general result. The :erest in this approach is that the techniques are completely :aightforward and applicable to a wide variety of situations. We : the following lemma:

)WING-UP LEMMA. Let H be Euclidean n-space, V Euclidean m-space 1 $g : H \to Y$ a C^k map, $k \ge 3$. Assume

$$Q \text{ is in general position on } C = Q^{-1}(0)$$

Then there is a neighborhood U of O in H such that $^{1}(0) \cap U$ is homeomorphic to $Q^{-1}(0)$ via a homeomorphism that takes to O and is a C^k diffeomorphism away from O . Moreover, if $\in Q^{-1}(0)$, there is a C^{k-2} curve $\alpha(s) \in g^{-1}(0)$, $-\delta \le s \le \delta$ with $)) = 0$, $\alpha'(0) = v$.

Here is how the lemma yields the theorem. Let $: H = X_1 \times \mathbb{R}^p \to Y_2 = V$ be defined by

$$g(x_1, \lambda) = (I - P)f(x_1 + u(x_1, \lambda), \lambda)$$

early g is of class C^k . Also, noting that $Du(x_0, \lambda_0) = 0$ (from e definition of u and $\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) = 0$), one calculates that

(a) $Dg(x_{10}, \lambda_0) = 0$

and (b) $D^2 g(x_{10}, \lambda_0) = (I - P)\, D^2 f(x_0, \lambda_0)$, restricted to $X_1 \times \mathbb{R}^p$.

Therefore, by our assumptions in the main theorem, the blowing-up lemma applies to g . Since the zeros of $f(x, \lambda)$ are the graph of u over the zeros of g , the conclusions of the theorem follow.

Here is the idea of the proof of the blowing-up lemma. Let S be the unit sphere in H . Set

$$\tilde{g} : S \times \mathbb{R} \to V ,$$

$$\tilde{g}(x,r) = \frac{1}{r^2} g(rx)$$

By Taylor's theorem,

$$g(x) = Q(x) + R(x)$$

so where R is C^{k-2} , so

$$\tilde{g}(x,r) = Q(x) + \frac{1}{r^2} R(rx)$$

and since R vanishes like r^3 , $\tilde{g}$ is C^{k-2} . Away from $r = 0$, the zeros of $\tilde{g}$ and g are in 1-1 correspondence. If we identify $S \times \{0\}$ and S , we have thus blown up the singularity of g at 0 to the unit sphere S . Near S , the structure of the set of zeros of $\tilde{g}$ can be analyzed easily since 0 is a regular value of g on S (by hypothesis) and $\tilde{g}^{-1}(0)$ intersects S transversally. By pushing this structure down to X_1 by the map $(x,r) \mapsto rx$, we get the result.

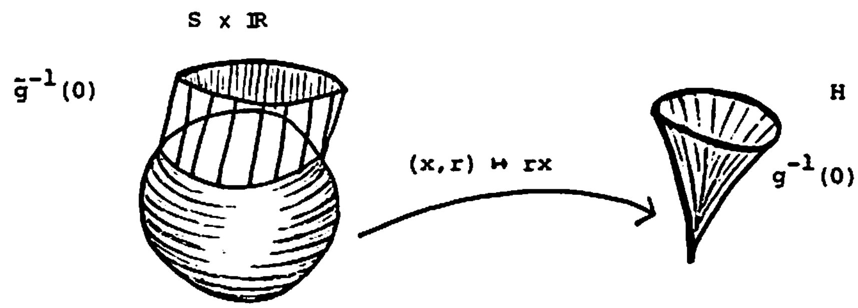

Figure 5.

. DYNAMIC BIFURCATION THEORY

Bifurcation theory for dynamical systems is much less developed .an that for fixed points. Indeed the variety of bifurcation possible ıd their structure is much more complex. We shall briefly outline :re some examples of dynamic bifurcations and then state a general .an for attacking a complex bifurcation problem.

We begin by describing the simplest bifurcations for one para- :ter systems. In a sense these bifurcations are the generic local ıes. (See Sotomayor [50] and Takens [53] for details) . If one ıposes a symmetry, however, what is generic may change, as we shall ‹plain.

ıDDLE NODE. This is a bifurcation of fixed points; a saddle and a ınk come together and annihilate one another, as shown in Figure 6 . simple real eigenvalue of the sink crosses the imaginary axis at ıe moment of bifurcation; one for the saddle crosses in the oppo- ite direction. The suspended center manifold is 2-dimensional . ıe symmetric situation of a saddle-source is also possible.

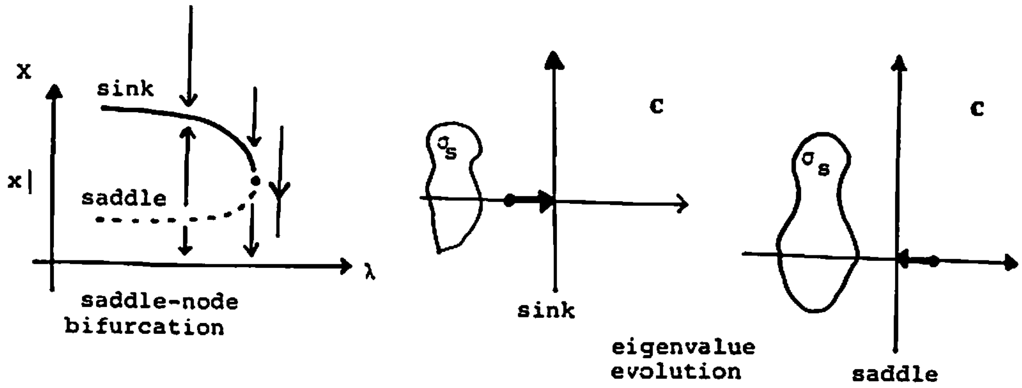

Figure 6.

If an axis of symmetry is present, as will be the case in the example of panel flutter treated in §4 , then a symmetric bifurcation can occur, as in Figure 7 . As in our discussion of Euler buckling, a small asymmetric perturbation or imperfection 'unfolds' this into a simple non bifurcation path and a saddle node. In Figures 6 and 7 we also indicate the vector field flow directions schematically.

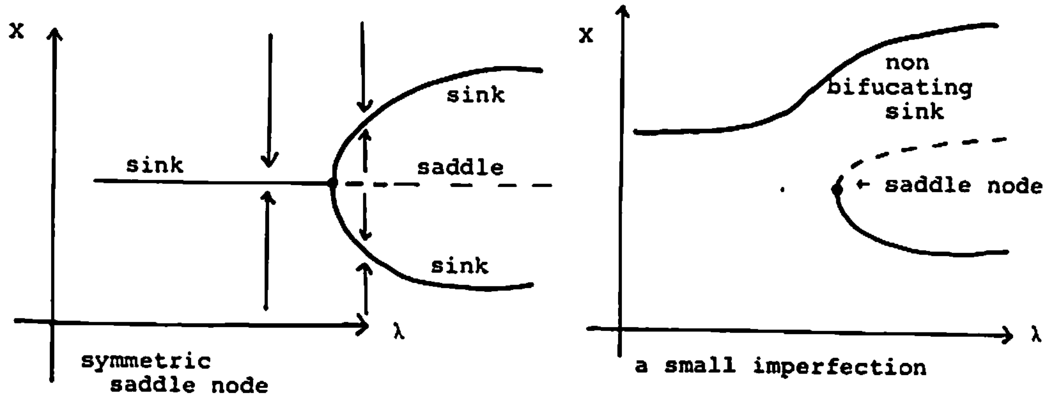

Figure 7.

HOPF BIFURCATION. This is a bifurcation to a periodic orbit; here a sink becomes a saddle by two complex conjugate non-real eigenvalues ıssing the imaginary axis. As with the symmetric saddle node, the :urcation can be sub-(unstable closed orbits) or super-(stable closed ›its) critical. (See Marsden and McCracken [32] for calculations determine which is which) . Figure 8 depicts the supercritical :racting case. Here the suspended center manifold is 3-dimensional.

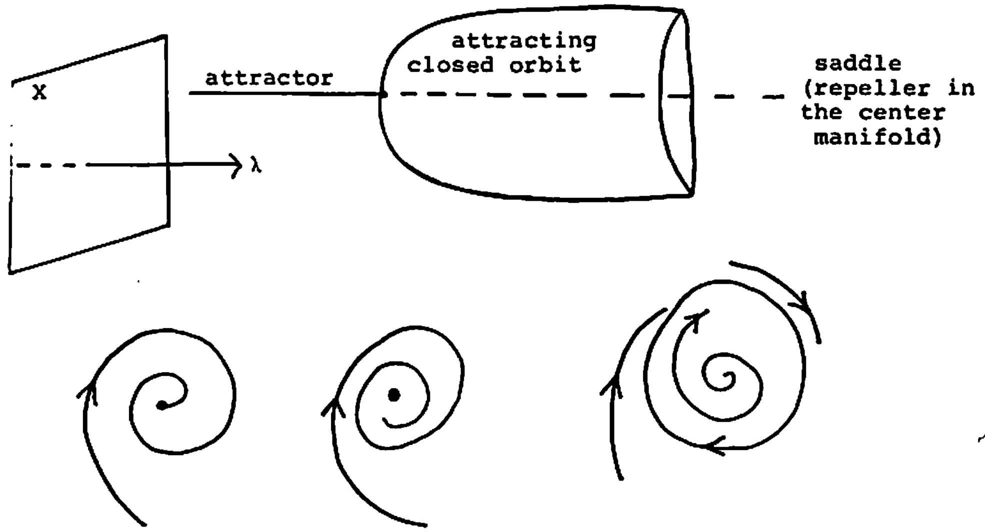

Figure 8 The Hopf Bifurcation.

These two bifurcations are local in the sense that they can be .alyzed by linearization about a fixed point. There are however, ›me global bifurcations which are more difficult to detect. A ıddle connection is shown in Figure 9; cf Takens [54] , Arnold [2] .

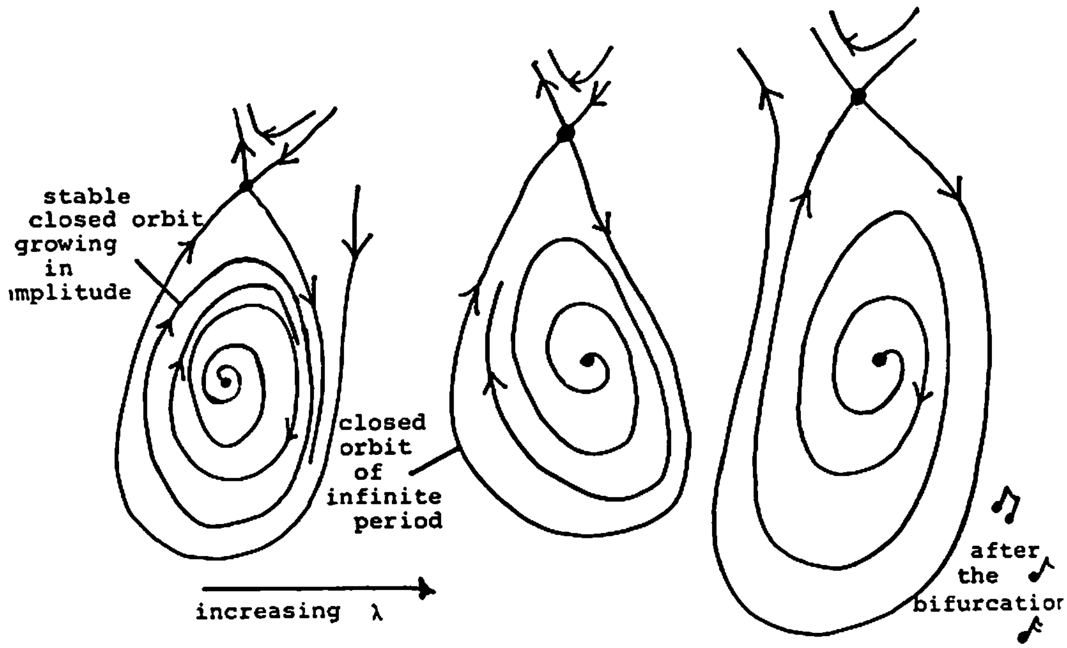

Figure 9. A Saddle Connection

Here the stable and unstable seperatrices of the saddle point pass through a state of tangency (when they are identical) and thus cause the annihilation of the attracting closed orbit.

These global bifurcations can occur as part of local bifurcations of systems with additional parameters. This approach has been developed by Takens [54] who has classified generic or 'stable' bifurcations of two parameter families of vector fields on the plane. This is an outgrowth of extensive work of the Russian school led by Andronov [1] . An example of one of Taken's bifurcations with a symmetry imposed is shown in Figure 10. (The labels are for later use.)

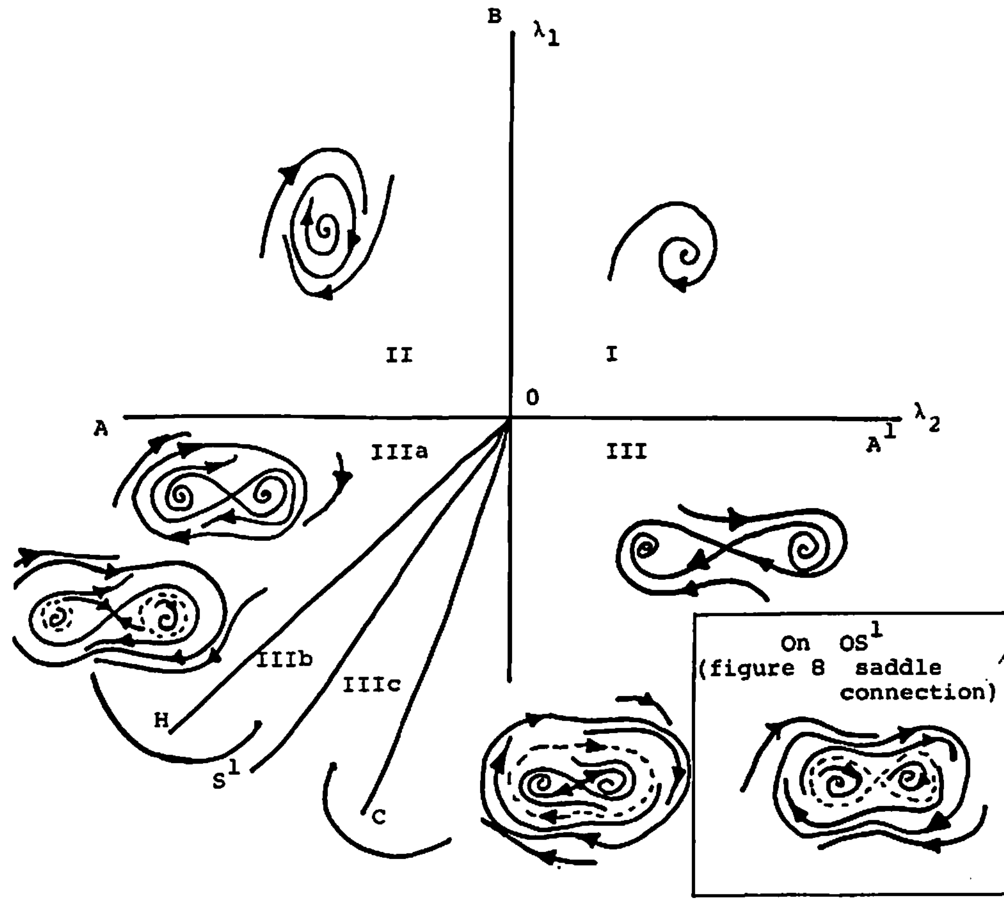

Figure 10. Takens' (2,-) normal form showing the 'local phase portrait in each region on parameter space (Takens' [54])

Some of the phenomena captured by the bifurcations outlined bove have been known to engineers for many years. In particular e might mention the jump phenomenon of Duffings equation (see imoshenko [57], Holmes and Rand [21]) and the more complex bifurcaional behavior of the forced van der Pol oscillator (Hayashi [15], :olmes and Rand [22] ; [22] contains a proof that the planar varia:ional equation of the latter oscillator undergoes a saddle connection ›ifurcation as in Figure 9 .

Now we can outline an approach to bifurcation problems (cf. :olmes and Rand [20]) . First of all, the analysis is for two para- ıeter systems which posess, near a fixed point (x_0, λ_0), a three di- ıensional suspended center manifold (i.e. for each λ , a two dimen- ıional invariant manifold for the dynamics). Typically, a fixed point ıill have a real double zero eigenvalue at a certain parameter value ınd we are interested in bifurcations near this <u>organizing center</u>.)ne first fills in as much of the bifurcation diagram as possible, ısing linearization to detect Hopf and saddle node bifurcation. Second, ɔne <u>assumes</u> (taking any symmetry into account) that the bifurcation diagram itself is stable to small perturbations. This is justified since ɔne is presumably working with a model which only approximates some physical situation [20] . Finally, the correct bifurcation diagram is obtained by looking through Takens' list for a diagram(s)† consistent with the information obtained.

We shall illustrate how this procedure works in a concrete problem in §4 .

§4. FLUTTER IN ENGINEERING SYSTEMS

Before giving a particular example analyzed by the methods of §3 , we discuss some ideas and examples of flutter in general.

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†See Takens [54] . Here, in §§5,6, Takens lists generic bifurcations of 2 parameter vectorfields on the plane (or on two-manifolds) having singularities with double zero eigenvalues. He allows the vectorfields to have rotational symmetry but assumes that there is no "higher" degeneracy in the nonlinear terms. See Takens [51] for an example where the latter does occur. It is not strictly correct to speak of a "list" of two-parameter bifurcation, since the various analyses has not been conveniently gathered in one article.

All too often, engineers are content with only a linear ana-
sis. For example, flutter is often viewed as the presence of two
mplex conjugate eigenvalues with positive real part. (cf. Zeigler
)]). The non-linear system may be fluttering (i.e. have a closed
bit) or not, as shown in Figure 11 . Mathematically, the develop-
nt of spontaneous flutter is best detected through the Hopf bifurca-
on, remembering that the periodic orbits could be unstable and the
furcation subcritical.

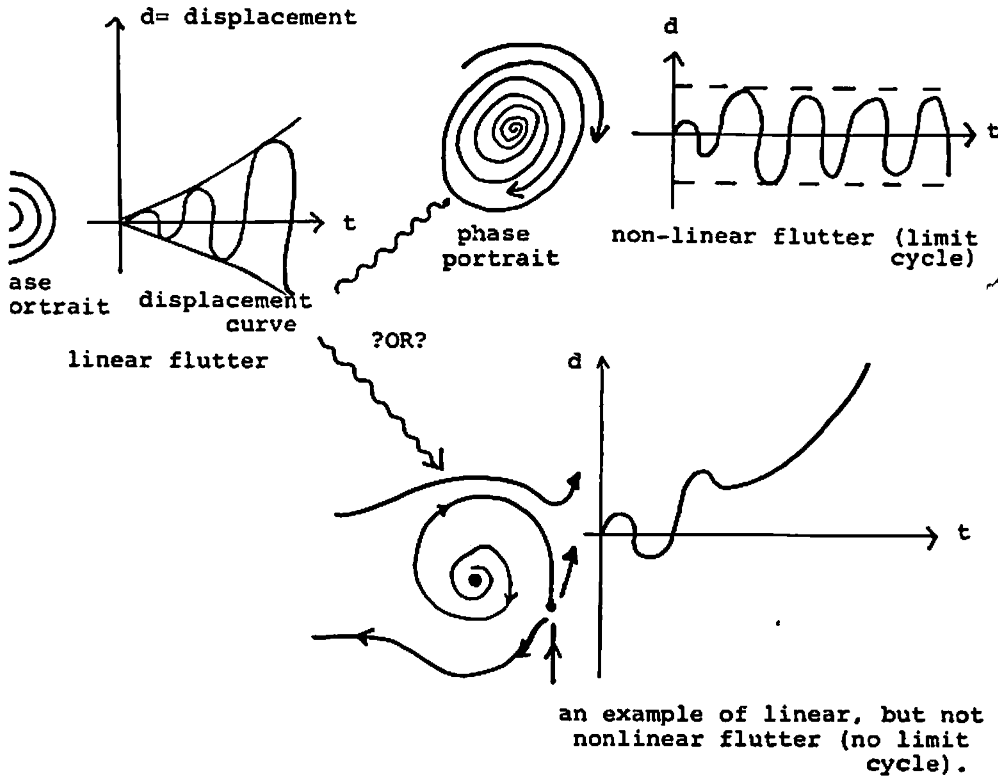
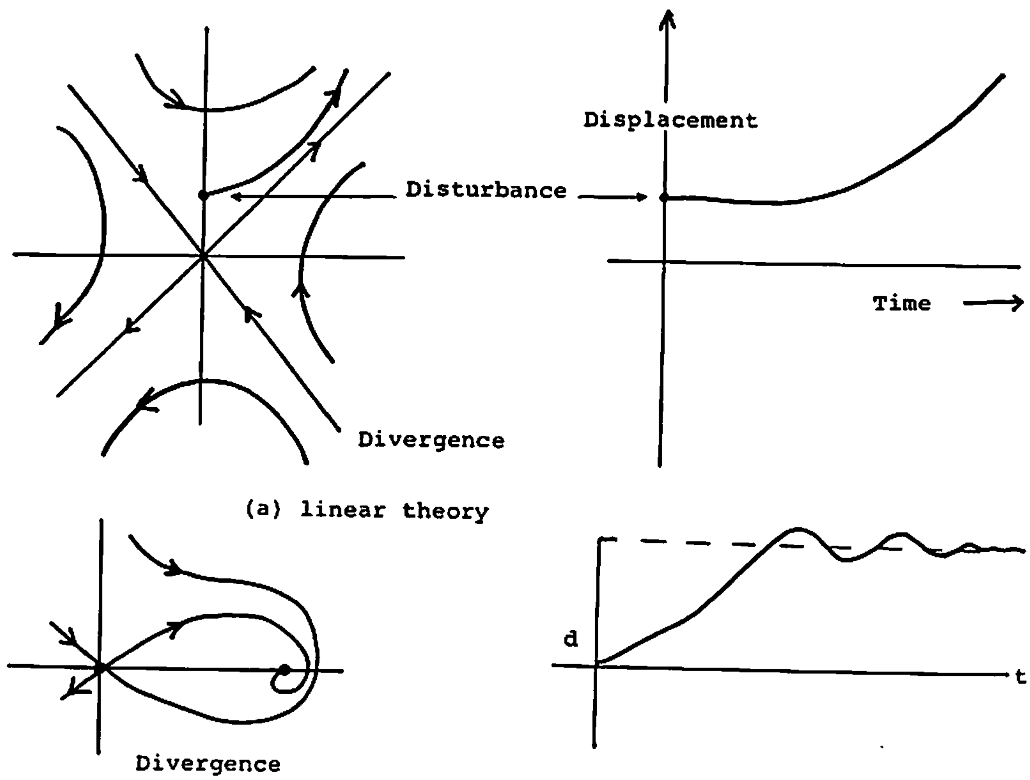


Figure 11.

Similar remarks may be made about divergence (a saddle point
r source) as shown in Figure 12 .



(a) linear theory

(b) a nonlinear possibility

Figure 12.

There are, in broad terms, three kinds of flutter of interest to the engineer. Here we briefly discuss these types. Our bibliography is not intended to be exhaustive, but merely to provide a starting point for the interested reader.

(a) AIRFOIL OR WHOLE WING FLUTTER ON AIRCRAFT

Here linear stability methods <u>do</u> seem appropriate since virtually any oscillations are catastrophic. Control surface flutter probably comes under this heading also. See Bisplinghoff and Ashley [3] and Fung [14] for examples and discussion.

CROSS FLOW OSCILLATIONS

The familiar flutter of sun-blinds in a light wind comes under .s heading. The "galloping" of power transmission lines and of tall .ldings and suspension bridges provide examples which are of more :ect concern to engineers: the famous Tacoma Narrows bridge disaster s caused by cross flow oscillations. In such cases (small) limit :le oscillations are acceptable (indeed, they are inevitable), and a nonlinear analysis is appropriate.

Cross-flow flutter is due to the oscillating force caused by on-Karman" vortex shedding behind the body, Figure 13 .

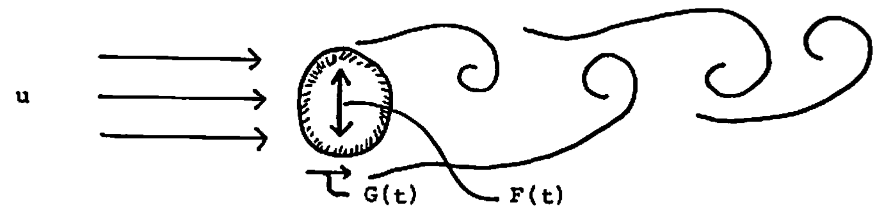


Figure 13. Cross flow oscillations

e alternating stream of vortices leads to an almost periodic force t) transverse to the flow in addition to the in-line force G(t) ; t) varies less strongly than F(t) . The flexible body responds to t) and, when the shedding frequency (a function of fluid velocity, and the body's dimensions) and the body's natural or resonance ·equency are close, then "lock in" or entrainment can occur and large nplitude oscillations are observed. Experiments strongly suggest a .mit cycle mechanism and engineers have traditionally modelled the .tuation by a van der Pol oscillator or perhaps a pair of coupled scillators. See the symposium edited by Naudascher [34] for a num- er of good survey articles; the review by Parkinson is especially elevant. In a typical treatment, Novak [36] discusses a specific cample in which the behaviour is modelled by a free van der Pol type scillator with nonlinear damping terms of the form

$$a_1\dot{x} + a_2\dot{x}^2 + a_3\dot{x}^3 + \ldots$$

ch equations possess a fixed point at the origin $x = \dot{x} = 0$ and n also possess multiple stable and unstable limit cycles. These cles are created in bifurcations as the parameters $a_1, a_2, \ldots,$ ich contain windspeed terms, vary. Bifurcations involving the xed point and global bifurcations in which pairs of limit cycles are eated both occur (cf. Novak [36] figures 3,8,9). Parkinson also scusses the phenomenon of entrainment which can be modelled by the rced van der Pol oscillator.

In a more recent study, Landl [28] discusses such an example ich displays both "hard" and "soft" excitation, or, in Arnold's rm [2] , strong and weak bifurcations:

$$\ddot{x} + \delta\dot{x} + x = a\Omega^2 C_L$$

$$\ddot{C}_L + (\alpha - \beta C_L^2 + \gamma C_L^4)\dot{C}_L + \Omega^2 C_L = b\dot{x} .$$

ere $\cdot \equiv \frac{d}{dt}$ and $\alpha, \beta, \gamma, \delta, a, b$ are generally positive constants or a given problem (they depend upon structural dimensions, fluid roperties, etc.). Ω is the vortex shedding frequency. As Ω varies ne system can develop limit cycles leading to a periodic variation n C_L, the lift coefficient. The term $a\Omega^2 C_L$ then acts as a periodic riving force for the first equation, which represents one mode of ibration of the structure. This model, and that of Novak, appear to isplay generalised Hopf bifurcations (see Takens [51]) .

In related treatments allowance has been made for the effects f (broad band) turbulence in the fluid stream by including stochas- ic excitations. Vacaitis et. al [58] proposed such a model for the scillations of a two degree of freedom structure and carried out some umerical and analogue computer studies. Recently Holmes and Lin [17] applied qualitative dynamical techniques to a deterministic version of this model prior to stochastic stability studies of the full model (Lin and Holmes [29]). The Vacaitis model assumes that the von Karman vortex excitation can be replaced by a term

$$F(t) \equiv F\cos(\Omega t + \Psi(t))$$

where Ω is the (approximate) vortex shedding frequency and $\Psi(t)$ is a random phase term. In common with all the treatments cited above the actual mechanism of vortex generation is ignored and "dummy" drag and lift coefficients are introduced. These provide discrete analogues of the actual fluid forces on the body. Iwan and Blevins [24] and St. Hilaire [49] have gone a little further in attempting to relate such force coefficients to the fluid motion but the problem appears so difficult that a rigorous treatment is still impossible. The major problem is, of course, our present inability to solve the Navier-Stokes equations for viscous flow around a body. Potential flow solutions are of no help here, but recent advances in numerical techniques may be useful. Ideally a rigorous analysis of the fluid motion should be coupled with a continuum mechanical analysis of the structure. For the latter, see the elegant Hamiltonian formulation of Marietta [31] for example.

The common feature of all these treatments (with the exception of Marietta's) is the implicit reduction of an infinite dimensional problem to one of finite dimensions, generally to a simple nonlinear oscillator. The use of center manifold theory and the concepts of genericity and structural stability suggests a way in which this reduction might be rigorously justified. To illustrate this we turn to the third broad class of flutter, which we discuss in more detail.

:) AXIAL FLOW INDUCED OSCILLATIONS

In this class of problems, oscillations are set up directly ırough the interaction between a fluid and a surface across which : is moving. Examples are oscillations in pipes and (supersonic) anel flutter. Experimental measurements (vibration records from nu- lear reactor fuel pins, for example) indicate that axial flow induced scillations present a problem just as severe as the more obvious one f cross flow oscillations. See the monograph by Dowell [11] for an ccount of panel flutter and for a wealth of further references. scillations of beams in axial flow and of pipes conveying fluid have een studied by Païdoussis [37,38] and Brooke-Benjamin [4,5] ; see aïdoussis [38] for a good survey. Figure 14 shows the three situ- tions.

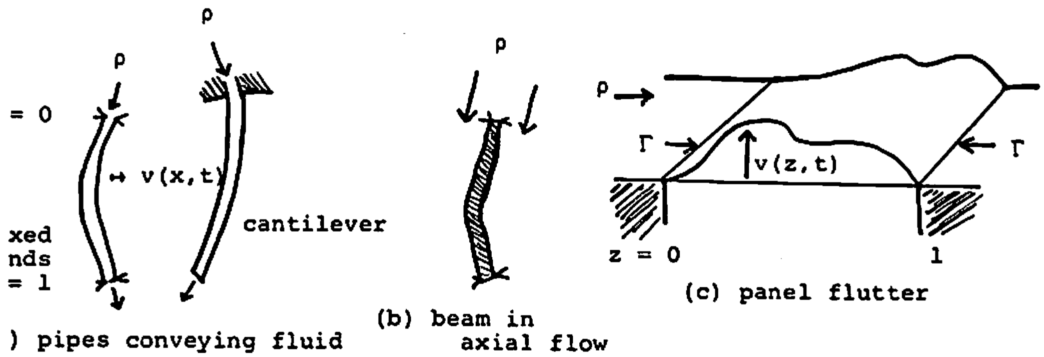


Figure 14. Axial flow-induced oscillations.

n addition to the effects of the fluid flow velocity ρ, the struc- ural element might also be subject to mechanical tensile or compress- ve forces Γ which can lead to buckling instabilities even in the ıbsence of fluid forces.

The equations of motion of such systems, written in one dimen- ional form and with all coefficients suitably nondimensionalised, an be shown to be of the type

$$\alpha\dot{v}'''' + v'''' - [\kappa\int_0^1 (v'(\xi))^2 d\xi + \sigma\int_0^1 (v'(\xi)\dot{v}'(\xi))d\xi]v'' + \ddot{v} +$$

$$(*) \quad + [\text{linear fluid and mechanical loading terms in } v'', \dot{v}', v', \dot{v}] = 0$$

ere $\alpha, \sigma > 0$ are structural viscoelastic damping coefficients and > 0 is a (nonlinear) measure of membrane stiffness; $v = v(z,t)$ and $= \partial/\partial t$; $' = \partial/\partial z$ (cf. Holmes [16]). Brooke-Benjamin [4] , aïdoussis [37,38] and Dowell [11,12] , for example, provide derivaions of specific equations of this type. The fluid forces are again pproximated, but in a more respectable manner.

In the case of panel flutter, if a static pressure differenial exists across the panel, the right hand side carries an addiional parameter P . Similarly, if mechanical imperfections exist s hat compressive loads are not symmetric, then the "cubic" symmetry f (*) is destroyed (cf. §1, figures 1 and 2, above) .

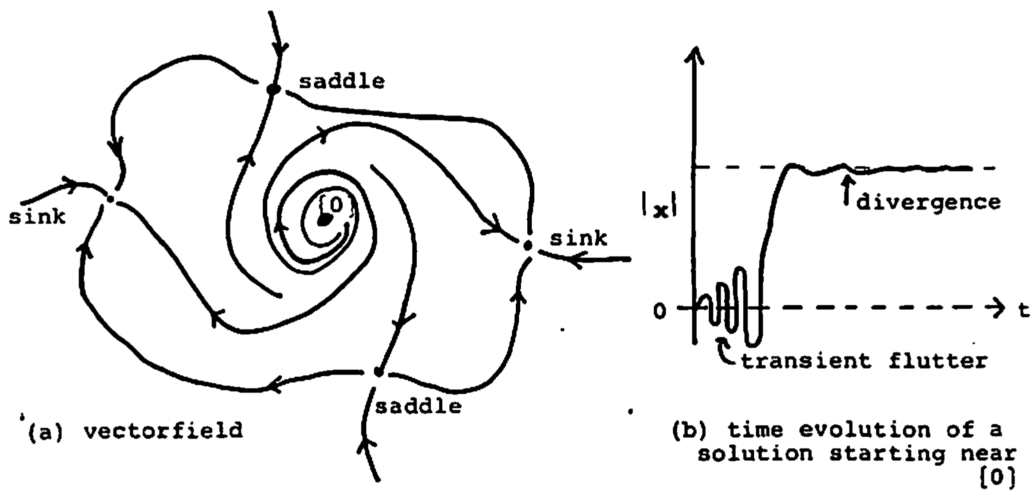
Problems such as those of figure 14 have been widely studied oth theoretically and experimentally, although, with the notable xception of Dowell and a number of other workers in the panel flutter rea, engineers have concentrated on linear stability analyses. Such nalyses can give misleading results, as we shall see. In many of hese problems, engineers have also used low dimensional models, even hough the full problem has infinitely many degrees of freedom. Such procedure can actually be justified if careful use is made of the enter manifold theorem.

Often the location of fixed points and the evolution of spectra bout them has to be computed by making a Galerkin or other approxiation and then using numerical techniques. There are obvious conergence problems (see Holmes and Marsden [18,19]), but once this is one, the organizing centers and dimension of the center manifolds

:an be determined relatively simply.

}4.1 PIPES CONVEYING FLUID AND SUPPORTED AT BOTH ENDS

Pipe flutter is an excellent illustration of the difference between the linear prediction of flutter and what actually happens in :he PDE model. The phase portrait on the center manifold in the non-linear case is shown in figure 15, at parameter values for which the linear theory predicts "coupled mode" flutter. (cf. Païdoussis-Issid [38] and Plaut-Huseyin [40]) .



(a) vectorfield

(b) time evolution of a solution starting near {0}

Figure 15.

In fact, we see that the pipe merely settles to one of the stable buckled rest points with no non-linear flutter. (See Holmes [16] for details.) The presence of imperfections should not substantially change

is situation (see Zeeman [59] for a global analysis of a similar
ɔ-mode buckling problem).

The absence of flutter in the nonlinear case can be seen by fferentiating a suitable Liapunov function along solution curves of ≥ PDE. In the pipe flutter case the PDE is

$$\alpha\dot{v}'''' + v'''' - \{\Gamma - \rho^2 + \gamma(1-z) + K|v'|^2 + \sigma(v',\dot{v}')\}v'' + + 2\sqrt{\beta}\rho\dot{v}' + \gamma v' + \delta\dot{v} + \ddot{v} = 0 \quad ,$$

see Païdoussis-Issid [38], Holmes [16]). Here $|\cdot|$ and $(\cdot\,,\cdot)$ note the usual L_2 norm and inner product and solutions $x = \{v,\dot{v}\}$ e in a Hilbert space $X = H_0^2([0,1]) \times L_2([0,1])$ (see Holmes and Marsden 8,19] and §4.2 below for more details of the specific analytic amework for such a problem). For our Liapunov function we choose e energy: in this case given by

$$H(x(t)) = \tfrac{1}{2}|\dot{v}|^2 + \tfrac{1}{2}|v''|^2 + \frac{\Gamma-\rho^2}{2}|v'|^2 + \frac{K}{4}|v'|^4 + \frac{\gamma}{2}([1-z]v',v') \quad ,$$

f. Païdoussis-Issid [38], appendix I). Differentiating $H(x(t))$ ong solution curves yields

$$\frac{dH}{dt} = -\delta|\dot{v}|^2 - \alpha|\dot{v}''|^2 - \sigma(v',\dot{v}')^2 - 2\sqrt{\beta}\rho(\dot{v}',\dot{v})$$

nce $(\dot{v}',\dot{v}) \equiv 0$ and $\delta, \alpha, \sigma > 0$, dH/dt is negative for all $v > 0$ d thus all solutions must approach rest points $x_i \in X$. In partic- ar, for $\Gamma > \Gamma_0$, the first Euler buckling load, all solutions proach $x_0 = \{0\} \in X$ and the pipe remains straight. Thus a term : the type $\rho\dot{v}'$ cannot lead to nonlinear flutter. In the case of beam in axial flow terms of this type <u>and</u> of the type $\rho^2 v'$ both :cur and nonlinear flutter evidently can take place (see Païdoussis

7] for a linear analysis) . Experimental observations actually in- cate that fluttering motions more complex than limit cycle can occur.

We should note, however, that cantilevered pipes can flutter: ooke-Benjamin [5] has some excellent photographs of a two-link del. Here flutter is caused by the so-called follower force at the ee end which introduces an additional term into the energy equation see Brooke-Benjamin [4]).

1.2 PANEL FLUTTER

Now we turn to an analysis of panel flutter. We consider the ne-dimensional" panel shown in Figure 14(c)† and we shall be inter- sted in bifurcations near the trivial zero solution. The equation f motion of such a thin panel, fixed at both ends and undergoing cylindrical" bending (so spanwise bending) can be written as

$$\alpha\dot{v}'''' + v'''' - \{\Gamma + K\int_0^1 (v'(\xi))^2 d\xi + \sigma\int_0^1 (v'(\xi)\dot{v}'(\xi))d\xi\}v'' +$$

$$+ \rho v'' + \sqrt{\rho}\,\delta\dot{v} + \ddot{v} = 0 \,, \qquad (1)$$

see Dowell [12], Holmes [16] . Here $\cdot \equiv \partial/\partial t$, $' \equiv \partial/\partial z$ and we have included viscoelastic structural damping terms α , σ as well as aerodynamic damping $\sqrt{\rho}\delta$. K represents nonlinear (membrane) stiffness, ρ the dynamic pressure and Γ an in-plane tensile load. All quantities are nondimensionalised and associated with (1) we have boundary conditions at $z = 0,1$ which might typically be simply

---

† A two dimensional or von-Karmen panel is presumably a good deal more complicated. For the bifurcation of fixed points, see Chow, Hale and Mallet-Paret [8].

pported $(v = v'' = 0)$ or clamped $(v = v' = 0)$. In the following make the physically reasonable assumption that $\alpha, \sigma, \delta, K$ are xed > 0 and let the control parameter $\mu = \{\rho, \Gamma | \rho \geq 0\}$ vary. In ntrast to previous studies (Dowell [11,12]) in which (1) and milar equations were analyzed for specific parameter values and initial conditions by numerical integration of a finite dimensional alerkin approximation, here we study the qualitative behavior of (1) der the action of μ .

To proceed with the methods of §3 , we first redefine (1) as n ODE on a Banach space, choosing as our basic space $= H_0^2([0,1]) \times L^2([0,1])$, where H_0^2 denotes H^2 functions† in 0,1] which vanish at 0,1 . Set $\|\{v,\dot{v}\}\|_X = (|\dot{v}|^2 + |v''|^2)^{\frac{1}{2}}$, here $|\cdot|$ denotes the usual L^2 norm and define the linear operat

$$A_\mu = \begin{pmatrix} 0 & I \\ C_\mu & D_\mu \end{pmatrix} \quad ; \quad \begin{aligned} C_\mu v &= -v'''' + \Gamma v'' - \rho v' \\ D_\mu \dot{v} &= -\alpha \dot{v}'''' - \sqrt{\rho}\ \delta \dot{v} \ . \end{aligned} \tag{2}$$

he basic domain of A_μ, $D(A_\mu)$ consists of $\{v,\dot{v}\} \in X$ such that $\in H_0^2$ and $v + \alpha\dot{v} \in H^4$; particular boundary conditions necessitate urther restrictions. After defining the nonlinear operator $3(v,\dot{v}) = (0, [K|v'|^2 + \sigma(v',\dot{v}')]v'')$, where $(\ , \ )$ denotes the L^2 inner product, (1) can be rewritten as

$$\frac{dx}{dt} = A_\mu x + B(x) \equiv G_\mu(x) \ ;$$

$$x = \{v,\dot{v}\};\ x(t) \in D(A_\mu) \ . \tag{3}$$

---

† H^2 consists of functions which, together with their first and

We next define an energy function $H : X \to \mathbb{R}$ by

$$H\{v,\dot{v}\} = \tfrac{1}{2}|\dot{v}|^2 + \tfrac{1}{2}|v''|^2 + \frac{\Gamma}{2}|v'|^2 + \frac{K}{4}|v'|^4 \tag{4}$$

and compute

$$\frac{dH}{dt} = -\rho(v',\dot{v}) - \sqrt{\rho}\,\delta|\dot{v}|^2 - \alpha|\dot{v}''|^2 - \sigma(v',\dot{v})^2 \quad .$$

Using the methods of Segal [48] one shows that (3) and hence (1) defines a unique smooth local semi-flow F_t^{μ} on X . Using the energy function (4) and some arguments of Parks [39], one shows that $H(x(t))$ is bounded and hence that F_t^{μ} is in fact <u>globally</u> defined for all $t \geq 0$.

By making 2-mode and 4-mode approximations, one finds that for $\sigma = 0.0005$, $\delta = 0.1$, the operator A_{μ} has a double zero eigenvalue at $\mu = (\rho, \Gamma) \approx (110, -22.6)$, (the point 0 in figure 16) the remaining eigenvalues being in the left half plane. (See Holmes [16] and Holmes-Marsden [18, 19].) Thus around the zero solution we obtain a four dimensional† suspended center manifold. Referring to the eigenvalue evolution at the zero solution in Figure 17, which is obtained numerically, we are able to fill in the portions of the bifurcation diagram shown in Figure 16.

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† Note that the control parameter μ is now two dimensional.

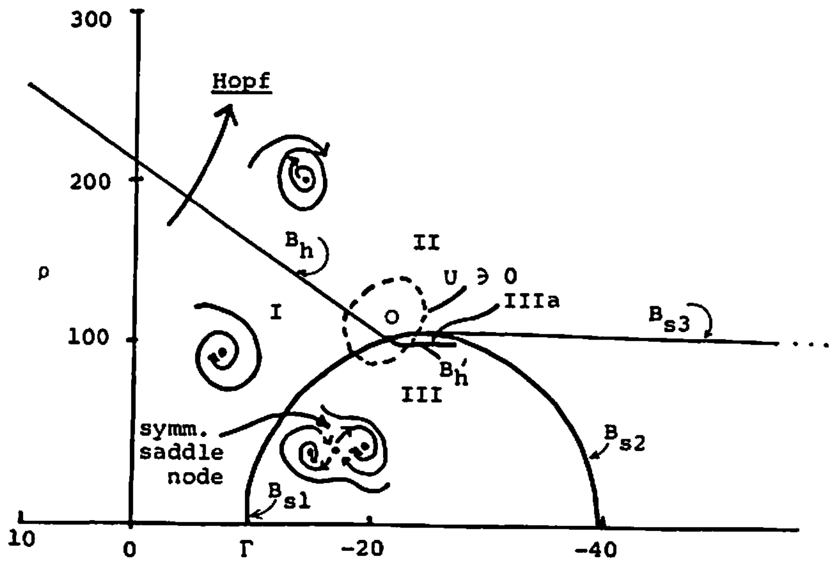

Figure 16. Partial bifurcation set for the two mode panel (α=0.005, δ= 0.1).

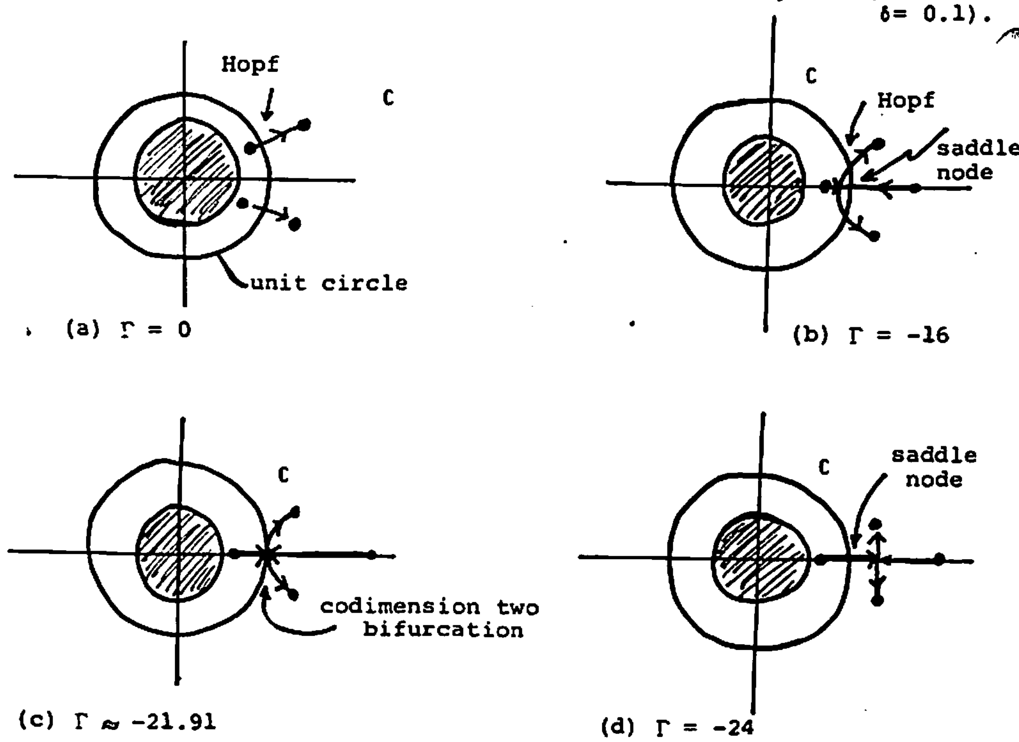

Figure 17. Eigenvalue evolutions for $DF_t^{\mu}(0) : X \to X$,

In particular a supercritical Hopf bifurcation occurs crossing B_h and a symmetrical saddle node on B_{s1}, as shown. These are the flutter and buckling or divergence instabilities detected in previous studies such as Dowell's. Moreover, finite dimensional computations for the two fixed points $\{\pm x_0\}$ appearing on B_{s1} and existing in region III show that they are sinks $(|\text{spectrum}\ (DF_t^{\mu}(\pm x_0))| < 1)$ below a curve B_h' originating at 0 which we also show on figure 16. As μ crosses B_h' transversally, $\{\pm x_0\}$ undergo simultaneous Hopf bifurcations before coalescing with $\{0\}$ on B_{s1}. A fuller description of the bifurcations, including those occurring on B_{s2} and B_{s3}, is provided by Holmes [16]. First consider the case where μ crosses B_{s2} from region I to region III, not at 0. Here the eigenvalues indicate that a saddle-node bifurcation occurs. In Holmes [16] exact expressions are derived for the new fixed points $\{\pm x_0\}$ in the two mode case. This then approximates the behaviour of the full evolution equation and the associated semiflow $F_t^{\mu} : X \to X$ and we can thus assert that a symmetric saddle-node bifurcation occurs on a one dimensional manifold as shown in figure 1 and that the "new" fixed points are sinks in region III. Next consider μ crossing $B_h \setminus 0$. Here the eigenvalue evolution shows that a Hopf bifurcation occurs on a two-manifold and use of the stability calculations from Marsden and McCracken [32]† indicate that the family of closed orbits existing in region II are attracting.

Now let μ cross $B_{s2} \setminus 0$ from region II to region IIIa. Here the closed orbits presumably persist, since they lie at a finite distance from the bifurcating fixed point $\{0\}$. In fact the new points $\{\pm x_0\}$ appearing on B_{s2} are saddles in region IIIa, with two eigenvalues of spectrum $DG_{\mu}(\pm x_0)$ outside the unit circle and all others within it $(\#(\lambda > 1) = 2)$. As this bifurcation occurs one of

† This has been confirmed for eight and twelve mode models by B.Hassard using his and Wan's stability formula (to appear in J. Math. An. Appl.)

e eigenvalues of spectrum $DF_t^{\mu}(0)$ passes into the unit circle so at throughout regions IIIa and III $(\lambda > 1) = 1$ for $\{0\}$. Finally con-der what happens when μ crosses B_h' from region IIIa to III. re $\{\pm x_0\}$ undergo simultaneous Hopf bifurcations and the stability lculations show that the resultant sinks in region III are surround-by a family of repelling closed orbits. We do not yet know how e multiple closed orbits of region III interact or whether any other furcations occur but we now have a partial picture of behaviour near derived from the two-mode approximation and from use of the sta-lity criterion. The key to completing this analysis lies in the int 0, the "organizing centre" of the bifurcation set at which $_2$, B_h, and B_h' meet.

According to our general scheme, we now postulate that our furcation diagram near 0 is stable to small perturbations in our pproximate) equations. We look in Takens' classification and find at exactly one of them is consistent with the information found in gure 16, namely the one shown in Figure 10. Thus we are led to the mplete bifurcation diagram shown in Figure 18 with the oscillations various regions as shown in Figure 10.

ıe modified panel ıfurcation set.

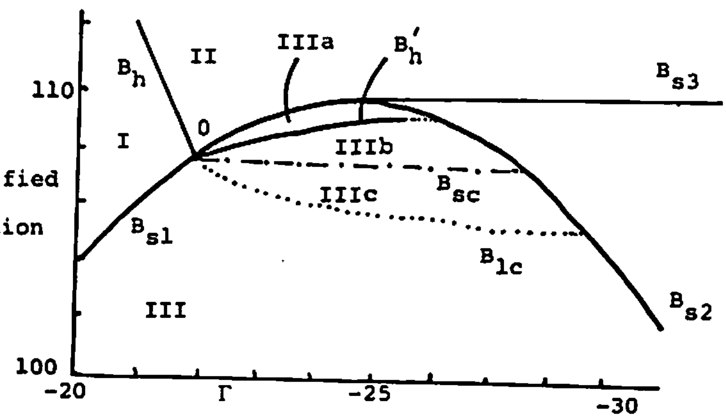

Figure 18. A local model for bifurcations of the panel near 0, $(\rho,\Gamma)\cong(110, -22.6)$; $\alpha=0.005$, $\delta=0.1$. (Numerical values derived from two mode model). For vectorfields in Regions I-IIIc, see Figure 10.

In principle one could check this out rigorously by proving that ur vector field on the center manifold has the appropriate normal form. uch a calculation is probably rather long, but possible. See Holmes 16] and Marsden [18,19] for additional comments. Also, it is not lear how the presence of a small imperfection or static pressure ifferential would affect the symmetric vectorfields of Figure 10 .

Although the eigenvalue computations used in this analysis were erived from two and four models (in which A_μ of eqn (2) is replaced y a 4 x 4 or 8 x 8 matrix and X is replaced by a vector space somorphic to $\mathbb{R}^4$ or $\mathbb{R}^8$) , the convergence estimates of [18,19] ndicate that in the infinite dimensional case the behaviour remains ualitatively identical. In particular, for $\mu \in U$, a neighbourhood f 0 , all eigenvalues but two remain in the negative half-plane. hus the dimension of the center manifold does not increase and our our dimensional "essential model" , a two parameter vectorfield on a wo manifold, provides a local model for the onset of flutter and livergence. We are therefore justified in locally replacing the in- inite dimensional semi-flow $F_t^\mu : X \to X$ by a finite dimensional system. Moreover, the actual vectorfields and bifurcation set shown n figure 10 can be realised by the nonlinear oscillator

$$\ddot{y} + \lambda_2\dot{y} + \lambda_1 y + \gamma y^2\dot{y} + \eta y^3 = 0 \qquad ; \gamma, \eta > 0$$

or

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -\lambda_1 y_1 - \lambda_2 y_2 - \gamma y_1^2 y_2 - \eta y_1^3 \tag{5}$$

(see Holmes and Rand [23] for a complete analysis of this system.)

In engineering terms (5) might be thought of as a "nonlinear normal mode" [44] of the system of equation (1), with λ_1, λ_2 representing equivalent linear stiffness and damping. (Note however that he relationship between the coordinates y_1, y_2 and any conveniently hosen basis in the function space X is likely to be nonlinear: in articular, a single "natural" normal mode model of the panel flutter roblem cannot exhibit flutter, although it can diverge (see Holmes 16]) ; flutter occurs through coupling between the natural (linear) ormal modes.)

We have thus seen how a simple nonlinear oscillator of van der ol-Duffing type might provide an essential model for panel flutter. he methods outlined in this article may be useful in many other omplex problems involving nonlinear oscillations.

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