Journal of Elasticity, Vol. 8, No. 1, January 1978
Sijthoff & Noordhoff International Publishers
Alphen aan den Rijn
Printed in The Netherlands

Classical elastodynamics as a linear symmetric hyperbolic system

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(Received August 1975; revised February 1977)

ABSTRACT

The existence, uniqueness, differentiability and data dependence of solutions of initial-boundary value problems in classical elastodynamics are treated by applying the theory of first-order symmetric hyperbolic systems. Sharp results on the differentiability of solutions are obtained in terms of body force, initial data and boundary conditions.

L'existence, l'unicité, la differentiabilité et la dépendence aux données de la solution de problemes aux conditions initiales aux limites dans le cas de l'elastodynamique classique est traitée en utilisant la theorie des systemes symmétrique hyperbolique de premier ordre. Des résultats finis sont obtenus pour la differentiabilité des solutions, ces resultats dependent des forces de volume, des données initiales et des conditions aux limites.

Introduction

We treat the existence, uniqueness and differentiability of solutions of initial-boundary value problems in classical elastodynamics by applying the theory of first-order symmetric hyperbolic systems.

The possibility that such a program was workable has been alluded to by Brockway [2]. A thorough treatment of uniqueness in linear elastodynamics and some discussion of existence can be found in Knops-Payne [14], which contains many references to relevant works. More recent contributions are Duvaut-Lions [5], Fichera [6], Gurtin [11], Knops-Payne [15], [16], Murray [20] and Wang-Truesdell [23].

Here we consider the Cauchy problem on all of $\mathbb{R}^m$ and the displacement and traction initial-boundary value problems on a bounded region contained in $\mathbb{R}^m$.

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In Section 1 we discuss the existing theorems on linear symmetric hyperbolic systems following Friedrichs [9, 10], Kato [12, 13], Lax-Phillips [17], Massey [19] and Rauch-Massey [22].

In Section 2 we review the equations of classical elastodynamics and appropriate initial and boundary data. We show how the equations can be put in symmetric hyperbolic form in Section 3. Finally, in Section 4 we draw the conclusions about existence, uniqueness and continuous dependence. We are able to get sharp results on the differentiability properties of solutions in terms of body force, initial data and boundary conditions.

The authors thank Professors T. Kato, P. Chernoff and F. Massey for their helpful comments.

1. Linear Symmetric Hyperbolic Systems

These have the form

$$A^0(t,x)\frac{\partial u}{\partial t} - \sum_{i=1}^{m} A^i(t,x)\frac{\partial u}{\partial x^i} + B(t,x)u + f(t,x) \tag{1.1}$$

where $u: \mathbb{R}\times\Omega\to\mathbb{R}^N$, Ω is a region in $\mathbb{R}^m$, A^0, A^i, and B are $N\times N$ matrix valued functions, A^0 and A^i are symmetric and A^0 is positive definite.

We are interested in the initial-boundary value problem, i.e., given a suitable function of u on $\partial\Omega$, the boundary of Ω, and u at $t=0$, find $u(t,x)$ for all $t\in\mathbb{R}$.

The bibliography on such systems is extensive. See, for instance, the references in Courant Hilbert [4] and Fischer-Marsden [7].

Let $C^k(\mathscr{D},\mathscr{R})$ denote the C^k functions with domain $\mathscr{D}$ and range $\mathscr{R}$ and let C^k_b be the C^k functions with derivatives of order $\leq k$ uniformly bounded. Let $H^s(\mathscr{D},\mathscr{R})$, $s\geq 0$, denote the Sobolev space of maps whose (distributional) derivatives of order $\leq s$ are in $L_2(\mathscr{D},\mathscr{R})$. The norms on $H^s(\mathscr{D},\mathscr{R})$ and $L_2(\mathscr{D},\mathscr{R})$ are denoted $\|\cdot\|_{s,\mathscr{D}}$ and $\|\cdot\|_{\mathscr{D}}$, respectively (see, for example, Friedman [8] or Marsden [18] for a discussion of Sobolev spaces and their basic properties).

If h is a function of t and x, let $h(t)$ denote the function of x obtained by freezing t, i.e., $h(t)(x)=h(t,x)$. Let

$$G(t) = (A^0)^{-1}(t)\left[\sum_{i=1}^{m} A^i(t)\frac{\partial}{\partial x^i} + B(t)\right] \quad \text{and} \quad F=(A^0)^{-1}f.$$

Let us first consider the case of $\Omega=\mathbb{R}^m$.

Theorem 1.1. *Consider* (1.1) *on* $\mathbb{R}^m$ *and assume*

(i) A^0, A^i, *and* B *are in* $C^\infty_b([0,T]\times\mathbb{R}^m,\mathbb{R}^{N^2})$.
(ii) A^0 *and* A^i *are symmetric.*
(iii) A^0 *is uniformly positive definite, i.e.,* $A^0(t,x)\geq\delta Id$ *for all* $x\in\mathbb{R}^m$, $t\in[0,T]$, *where* δ *is some fixed positive real number and* Id *is the* $N\times N$ *identity matrix.*
(iv) $f\in H^s([0,T]\times\mathbb{R}^m,\mathbb{R}^N)$.
(v) $u_0\in H^s(\mathbb{R}^m,\mathbb{R}^N)$.

Then there exists a unique solution u of (1.1) *belonging to* $C^r([0,T], H^{s-r}(\mathbb{R}^m,\mathbb{R}^N))$, $0\leq r\leq s$, *such that* $u(0)=u_0$. *The solution varies continuously with the initial data in* $H^s(\mathbb{R}^m,\mathbb{R}^N)$.

Finally, the equations are hyperbolic in the sense that if u_0 *and* f *have compact support then so does* $u(t,x)$ *for each* t.

The case of C^∞ data and the conclusion concerning the support of $u(t,x)$ is a standard result found in Courant-Hilbert [4]. The sharper H^s version here may be found in Kato [12, 13]. (Chernoff [3] also contains a useful exposition of some of these results.)

We consider now the case of bounded regions. Let L and M be given operators that are defined on functions on Ω and $\partial\Omega$, respectively.

Definition. Given $f\in L_2([0,T]\times\Omega,\mathbb{R}^N)$ and $u_0\in L_2(\Omega,\mathbb{R}^N)$, a function $u\in L_2([0,T]\times\Omega,\mathbb{R}^N)$ is a *strong solution* to

$$\begin{aligned} &(1)\quad Lu=f \quad \text{on} \quad [0,T]\times\Omega,\\ &(2)\quad Mu=0 \quad \text{on} \quad [0,T]\times\partial\Omega,\\ &(3)\quad u(0)=u_0 \quad \text{on} \quad \Omega, \end{aligned} \tag{1.2}$$

if there exists a sequence $\{u_n\}\subset C^\infty_b([0,T]\times\Omega,\mathbb{R}^N)$ with $u_n\to u$, $Lu_n\to f$ in $L_2([0,T]\times\Omega,\mathbb{R}^N)$, $Mu_n=0$ on $[0,T]\times\partial\Omega$ and $u_n(0)\to u_0$ in $L_2(\Omega,\mathbb{R}^N)$.

Theorem 1.2. *Consider* (1.2) *on a bounded open set* Ω *with compact boundary* $\partial\Omega$ *of class* C^∞ *such that* Ω *lies on one side of* $\partial\Omega$. *Let* $N(x)\subset\mathbb{R}^N$ *be a linear subspace varying in a* C^∞ *manner with* $x\in\partial\Omega$* *and let* $M(x)$ *be the orthogonal projection on* $N(x)^\perp$. *Let* $n=(n_1,\ldots,n_m)$ *be the unit outward normal of* $\partial\Omega$, $A_n=\sum_{i=1}^m A^i n_i$ *the boundary matrix, and*

$$L = A^0\frac{\partial}{\partial t} - \sum_{i=1}^{m} A^i\frac{\partial}{\partial x^i} - B.$$

Assume

(i) A^0, A^i, *and* B *are in* $C^\infty_b([0,T]\times\bar\Omega,\mathbb{R}^{N^2})$.
(ii) A^0 *and* A^i *are symmetric.*
(iii) A^0 *is uniformly positive definite on* $[0,T]\times\bar\Omega$.
(iv) A_n *has constant rank in a neighborhood of* $\partial\Omega$.†
(v) $\langle A_n u, u\rangle\leq 0$ *if* $Mu=0$ *and* $N(x)$ *is maximal among subspaces with this property. (Here* $\langle\,,\rangle$ *is the usual inner product in* $\mathbb{R}^N$ *and* $u\in\mathbb{R}^N$.)
(vi) $f\in L_2([0,T]\times\Omega,\mathbb{R}^N)$.
(vii) $u_0\in L_2(\Omega,\mathbb{R}^N)$.

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* Another way of saying this is that $N(x)$ possesses a basis $\{e_i(x)\}_1^p$, $p=\dim N(x)$, where e_i is a C^∞ function of $x\in\partial\Omega$, $1\leq i\leq p$.

† In detail this hypothesis means the following: For x sufficiently close to $\partial\Omega$ there exists a unique normal line through $\partial\Omega$ and x. Let $\bar x$ be the intersection point of this line with $\partial\Omega$. Then the 'unit normal vector' at x, $n(x)=n(\bar x)$. In this way we define $n(x)$, and hence $A_n(x)$, in a neighborhood of $\partial\Omega$.

Then there exists a unique strong solution u of (1.2) *belonging to* $C([0, T], L_2(\Omega, \mathbb{R}^N))$ *in which*

$$\|u(t)\|_{L_2} \le Ce^{\beta t}(\|u_0\|_{L_2} + \|f\|_{[0,t]\times\Omega}) \tag{1.3}$$

for $0 \le t \le T$ *where* C *and* β *are constants.*

Proof. Since A^0 is smooth, symmetric and uniformly positive definite, it has a square root T which also possesses these properties. Changing the unknown to $v = Tu$, we have (using the summation convention)

$$\frac{\partial v}{\partial t} = T^{-1}A^iT^{-1}\frac{\partial v}{\partial x^i} + \left[T^{-1}A^i\frac{\partial T^{-1}}{\partial x^i} + T\frac{\partial T}{\partial t}T^{-1} + T^{-1}BT^{-1}\right]v + T^{-1}f$$

which is of the form

$$\tilde{L}v = \frac{\partial v}{\partial t} - \tilde{A}^i\frac{\partial v}{\partial x^i} - \tilde{B}v = \tilde{f}$$

where $\tilde{A}^i$, $\tilde{B}$, and $\tilde{f}$ satisfy hypotheses (i)–(vi) with $\tilde{A}^0 = Id$. Thus this change of variables (Courant-Hilbert [4]) reduces the problem to $A^0 = Id$. The estimate

$$\|v(t)\|_{L_2} \le Ce^{\beta(t-s)}(\|v(s)\| + \|\tilde{L}v\|_{[s,t]\times\Omega}), \qquad -\infty < s \le t < \infty,$$

can be established for all $v \in C_0^\infty([s, t]\times\Omega, \mathbb{R}^N)$ such that $\tilde{M}v = MTv = 0$ on $[s, t]\times\partial\Omega$.* From the results of Rauch-Massey [22, cf. Proposition 2.1] the proof will follow if the existence of a strong solution can be established for

$$\tilde{L}w = f' \quad \text{on} \quad [0, T]\times\Omega,$$
$$\tilde{M}w = 0 \quad \text{on} \quad [0, T]\times\partial\Omega,$$
$$w(0) = 0 \quad \text{on} \quad \Omega,$$

where $f' \in C_0^\infty([0, T]\times\Omega)$. The technique for carrying this out can be found in Lax-Phillips [17].† □

Remark 1.1. Friedrichs [10] has previously established this result under the hypotheses $f \in H^1([0, T]\times\Omega, \mathbb{R}^N)$ and $u_0 = 0$.

Definition. The functions f, g, and u_0 occurring in equations (1.2) are said to satisfy the *compatibility conditions* if

$$\sum_{i=0}^{p}\binom{p}{i}\left(\left(\frac{\partial}{\partial t}\right)^i M\right)(0)u_{0,p-i} = \left(\left(\frac{\partial}{\partial t}\right)^p g\right)(0) \quad \text{on} \quad \partial\Omega \quad \text{for} \quad 0 \le p \le s-1,$$

where

$$u_{0,p} = -\sum_{i=0}^{p-1}\binom{p-1}{i}G_i(0)u_{0,p-1-i} + \left(\left(\frac{\partial}{\partial t}\right)^{p-1}F\right)(0), \qquad u_{0,0} = u_0, \qquad G_0 = G$$

* This is a simple energy inequality proved as in Courant-Hilbert [4], p. 652.

† Unfortunately, the details of this technique are lengthy and not amenable to a concise exposition. We refer the interested reader to the original source.

and

$$G_i = \sum_{j=1}^{m}\left(\left(\frac{\partial}{\partial t}\right)^i((A^0)^{-1}A^j)\frac{\partial}{\partial x^j} + \left(\frac{\partial}{\partial t}\right)^i((A^0)^{-1}B)\right), \qquad i \ge 1.$$

(G is defined above Theorem 1.1.)

The compatibility conditions are clearly necessary for the solution u of $(1.2)_{1,3}$ with $Mu = g$, to be in $C^r([0, T], H^{s-r}(\Omega, \mathbb{R}^N))$, $0 \le r \le s$.

Theorem 1.3. *Let the hypotheses be the same as in Theorem 1.2 through* (iii). *Assume*

(iv)′ A_n *is nonsingular on* $[0, T]\times\partial\Omega$.

(v)′ $f \in H^s([0, T]\times\Omega, \mathbb{R}^N)$.

(vi)′ $u_0 \in H^s(\Omega, \mathbb{R}^N)$.

(vii)′ f *and* u_0 *satisfy the compatibility conditions with* $g = 0$.

Then there exists a unique strong solution u *of* (1.2) *belonging to* $C^r([0, T], H^{s-r}(\Omega, \mathbb{R}^N))$ *for* $0 \le r \le s$, *which varies continuously with the initial data in* $H^s(\Omega, \mathbb{R}^N)$. *In addition, there exists a constant* C_s, *independent of* f *and* u_0, *such that*

$$|||u(t)|||_{s,\Omega} \le C_s(\|u_0\|_{s,\Omega} + \|f\|_{s,[0,t]\times\Omega} + |||f(0)|||_{s-1,\Omega}) \tag{1.4}$$

for $0 \le t \le T$, *where*

$$|||h(t)|||^2_{s,\Omega} = \sum_{j=0}^{s}\left\|\left(\left(\frac{\partial}{\partial t}\right)^j h\right)(t)\right\|^2_{s-j,\Omega}.$$

The proof of this theorem can be found in Rauch-Massey [22]. Unfortunately the condition that A_n is nonsingular is not met in the applications we have in mind. One can circumvent this condition in various situations by appealing to special properties of the system of equations under consideration.

Briefly, one employs Theorem 1.2 in place of Rauch-Massey's Proposition 2.1. Then the proof goes through if one can establish the following two lemmas:

Lemma 1.1. (1.4) holds for all $u \in H^{s+1}([0, T]\times\Omega, \mathbb{R}^N)$ satisfying (1.2).

Lemma 1.2. There exist sequences $(u_{0n}) \subset H^r(\Omega, \mathbb{R}^N)$, $(f_n) \subset H^r([0, T]\times\Omega, \mathbb{R}^N)$, $r \ge s+2$ with $u_{0n} \to u_0$ in $H^s(\Omega, \mathbb{R}^N)$, $f_n \to f$ in $H^s([0, T]\times\Omega, \mathbb{R}^N)$ such that u_{0n}, f_n satisfy the compatibility conditions with $g = 0$ for $0 \le p \le s+1$ for each n.

These are Rauch-Massey's Lemmas 3.2 and 3.3, respectively, which they establish with the aid of (iv)′. The way these conditions can be proved for a particular symmetric hyperbolic system without (iv)′ is illustrated for the wave equation in the example below.

Remark 1.2. It suffices to prove Lemma 1.2 in the weaker form where (u_{0n}) and (f_n) are required only to satisfy the compatibility condition for $0 \le p \le s$. Having done this, one approximates u_{0n} and f_n by sequences satisfying the compatibility condition for $0 \le p \le s+1$.

The case $Mu=g$ can be reduced to the case $Mu=0$ by a standard procedure, as follows.

THEOREM 1.4. *Let the hypotheses be the same as in Theorem* 1.3 *through* (vi)'. *Assume* $g\in H^{s+1/2}([0,T]\times\partial\Omega,\mathbb{R}^N)$* *and* f, g, *and* u_0 *satisfy the compatibility conditions. Then there exists a unique solution* u *of* $(1.2)_{1,3}$, *with* $Mu=g$, *belonging to* $C^r([0,T],H^{s-r}(\Omega,\mathbb{R}^N))$, $0\le r\le s$.

Proof. There exists a function $\tilde g\in H^{s+1}([0,T]\times\Omega,\mathbb{R}^n)\subset C^r([0,T],H^{s-r}(\Omega,\mathbb{R}^N))$ such that $\tilde g|_{\partial\Omega}=g$ (see, e.g., Palais [21]). The functions $f'=f-L\tilde g$ and $u_0'=u_0-\tilde g(0)$ satisfy the compatibility conditions with $g=0$. The solution w of (1.2) with f' and u_0' replacing f and u_0, respectively, is in $C^r([0,T],H^{s-r}(\Omega,\mathbb{R}^N))$, $0\le r\le s$, by Theorem 1.3. Thus $u=\tilde g+w$ also has these properties. □

Remark 1.3. If the requirements of Theorem 1.3 are met for all $s\ge 0$, then the solution $u\in C^\infty([0,T],C_b^\infty(\Omega,\mathbb{R}^N))$.

Remark 1.4. Assume that G and M are t-independent and $f=0$. Then the compatibility conditions become simply $MG^ru_0=0$, $0\le r\le s-1$. Then Theorem 1.3 says that the closure of G in H^s, $\bar G:\mathscr{D}(\bar G^s)\cap H^s(\Omega,\mathbb{R}^N)\to L_2(\Omega,\mathbb{R}^N)$ is the generator of a strongly continuous one-parameter group. Consequently, by the general theory of semigroups (see, e.g., Yosida [25]), if u_0 is in the domain of $\bar G$, so is $u(t)$ for each t. Often one can show that the domain of $\bar G$ consists of smoother functions; the following example illustrates the point.

EXAMPLE. Consider the wave equation $\ddot\phi=\Delta\phi$ on $\Omega\subset\mathbb{R}^m$ with boundary conditions $\phi=0$ on $\partial\Omega$. Assume Ω and $\partial\Omega$ satisfy the hypotheses of Theorem 1.2. Let $\phi_i=\partial\phi/\partial x^i$, $u=(\phi,\phi_1,\dots,\phi_m,\dot\phi)$, $A^0=Id$ and

$$A^1=\begin{pmatrix}0&\cdot&\cdot&\cdot&0\\0&\cdot&\cdot&\cdot&1\\0&\cdot&\cdot&\cdot&0\\\cdot&&&&\cdot\\\cdot&&&&\cdot\\\cdot&&&&\cdot\\0&1&0&\cdots&0\end{pmatrix},\quad\text{etc.}$$

* There are numerous definitions of fractional Sobolev spaces, all of which are equivalent. Perhaps the simplest definition is provided by the Fourier transform. For example, $f\in H^{s+1/2}(\mathbb{R}^n,\mathbb{R})$ if $f\in H^s(\mathbb{R}^n,\mathbb{R})$, s an integer, and D^sf (the sth generalized derivative of f) is in $H^{1/2}(\mathbb{R}^n,\mathbb{R})$. A function g is said to be in $H^{1/2}(\mathbb{R}^n,\mathbb{R})$ if the function $h(k)=(1+|k|^2)^{1/4}\hat g(k)$ is in $L_2(\mathbb{R}^n,\mathbb{R})$, where $\hat{}$ indicates the Fourier transform, $k=(k_1,\dots,k_n)$ being the transform variable.

For an open region $\mathscr{D}\subset\mathbb{R}^n$, $H^{s+1/2}(\mathscr{D},\mathbb{R})$ can be defined as follows. A function $f\in H^{s+1/2}(\mathscr{D},\mathbb{R})$ if $f\in H^s(\mathscr{D},\mathbb{R})$ and D^sf has an extension to a function in $H^{1/2}(\mathbb{R}^n,\mathbb{R})$.

The case in which $\mathscr{D}$ is a compact manifold, such as $\partial\Omega$, can be reduced to an open region in $\mathbb{R}^n$, as [...]e, by choosing an open covering of $\mathscr{D}$ by coordinate charts.

Here,

$$A_n=\begin{pmatrix}0&\cdot&\cdot&\cdot&0\\0&\cdot&\cdot&\cdot&n_1\\0&\cdot&\cdot&\cdot&n_2\\\cdot&&&&\cdot\\\cdot&&&&\cdot\\\cdot&&&&\cdot\\0&n_1&n_2&\cdots&0\end{pmatrix}$$

which has rank 2 and $N=\{u\mid\dot\phi=0$ on $\partial\Omega\}$. Thus $(A_nu,u)=0$ for $u\in N$ and this form is definite on no larger subspace.

Lemma 1.1 can be established by induction on s as follows: When $s=0$ we have (1.3). Thus assume the estimate (1.4) has been obtained with $|||u(t)|||_{s-1,\Omega}$ replacing the left-hand side of (1.4). By a standard calculation we can also obtain (1.4) with $|||(\partial u/\partial t)(t)|||_{s-1,\Omega}$ in place of the left-hand side. Thus it remains to establish a similar estimate for $|||(\partial u/\partial x^i)(t)|||_{s-1,\Omega}$. Since

$$u=(\phi,\phi_1,\dots,\phi_m,\dot\phi),$$

$$\frac{\partial u}{\partial t}=(\dot\phi,\dot\phi_1,\dots,\dot\phi_m,\ddot\phi),$$

and

$$\frac{\partial u}{\partial x^i}=(\phi_i,\phi_{1i},\dots,\phi_{mi},\dot\phi_i),$$

where $\phi_{ij}=\partial^2\phi/\partial x^i\partial x^j$, it suffices to establish the estimate for $\|\phi_{ij}(t)\|_{s-1,\Omega}$, $1\le i$, $j\le m$. Since $(\Delta\phi)(t)=\ddot\phi(t)\in H^{s-1}(\Omega,\mathbb{R})$ and the boundary conditions hold, we get $\phi(t)\in H^{s+1}(\Omega,\mathbb{R})$ and $\|\phi(t)\|_{s+1,\Omega}\le c\|\ddot\phi(t)\|_{s-1,\Omega}$. (This is a standard elliptic estimate for the Laplace operator; see Agmon [1]). Thus $\|\phi_{ij}(t)\|_{s-1,\Omega}\le c|||(\partial u/\partial t)(t)|||_{s-1,\Omega}$, $1\le i,j\le m$, which gives us (1.4).

In the present circumstances, the compatibility conditions are simply MG^ru_0 on $\partial\Omega$, $0\le r\le s-1$. Thus to establish Lemma 1.2, we require a sequence $\{u_{0l}\}\subset H^{s+2}(\Omega,\mathbb{R})$ such that $MG^ru_{0l}=0$ on $\partial\Omega$, $l\ge 1$, $0\le r\le s+1$, and $u_{0l}\to u_0$ in $H^s(\Omega,\mathbb{R})$. In more explicit notation, we have that $\Delta^k\phi_0=0$, $1\le k\le s/2$ and $\Delta^l\dot\phi_0=0$, $0\le l\le(s-1)/2$, on $\partial\Omega$, where k and l are integers, and we must establish the existence of sequences $\{\phi_{0l}\}$, $\{\dot\phi_{0l}\}\subset H^{s+2}(\Omega,\mathbb{R})$ such that the compatibility conditions are satisfied for each $l\le 1$, $1\le k\le(s+1)/2$, $0\le l\le s/2$, and which converge to ϕ_0, $\dot\phi_0$, respectively, in $H^s(\Omega,\mathbb{R})$. Because of Remark 1.2, it suffices to carry out the construction for $1\le k\le(s+1)/2$, $0\le l\le(s-1)/2$, if s is odd, or for $1\le k\le s/2$, $0\le l\le s/2$, if s is even. Let us pursue the case in which s is odd.

First, note that $u_0\in H^s(\Omega,\mathbb{R})$ implies $\phi_0\in H^{s+1}(\Omega,\mathbb{R})$. Let $h^{(k)}=\Delta^k\phi_0$, where k is any nonnegative integer. For $1\le k\le s/2$, $h^{(k)}\in H_0^{s+1-2k}(\Omega,\mathbb{R})=\{h\mid h\in H^{s+1-2k}(\Omega,\mathbb{R})$ and $h=0$ on $\partial\Omega\}$ whereas for $k=(s+1)/2$, $h^{(k)}\in L_2(\Omega,\mathbb{R})$. Take a sequence $\{h_l^{(k)}\}\subset H_0^2(\Omega,\mathbb{R})$ such that $h_l^{(k)}\to h^{(k)}$ in $L_2(\Omega,\mathbb{R})$. Then [...]

$h_i^{(k-1)}$ by the elliptic boundary value problem $\Delta h_i^{(k-1)} = h_i^{(k)}$ in Ω, $h_i^{(k-1)} = 0$ on $\partial\Omega$. It follows that $h_i^{(k-1)} \in H_0^2(\Omega, \mathbb{R})$. The difference $h_i^{(k-1)} - h^{(k-1)}$ satisfies $\Delta(h_i^{(k-1)} - h^{(k-1)}) = h_i^{(k)} - h^{(k)}$ in Ω, $h_i^{(k-1)} - h^{(k-1)} = 0$ on $\partial\Omega$. Thus we have the estimate $\|h_i^{(k-1)} - h^{(k-1)}\|_{2,\Omega} \leq c\|h_i^{(k)} - h^{(k)}\|_{\Omega}$, where c is a constant, which implies $h_i^{(k-1)} \to h^{(k-1)}$ in $H_0^2(\Omega, \mathbb{R})$. We repeat this process until $k = 1$. In this case we have $h^{(1)} = \Delta\phi_0 \in H_0^{s-1}(\Omega, \mathbb{R})$ and therefore get a sequence $\{h_i^{(1)}\} \subset H_0^{s+1}(\Omega, \mathbb{R})$ which converges to $h^{(1)}$ in $H_0^{s-1}(\Omega, \mathbb{R})$. The solution of $\Delta h_i^{(0)} = h_i^{(1)}$ in Ω, $h_i^{(0)} = \nu_i$ on $\partial\Omega$, where $\{\nu_i\} \subset H^{s+3}(\Omega, \mathbb{R})$, $\nu_i \to \phi_0$ in $H^{s+1}(\Omega, \mathbb{R})$ is in $H^{s+3}(\Omega, \mathbb{R})$ and the difference $h_i^{(0)} - \phi_0$ satisfies $\Delta(h_i^{(0)} - \phi_0) = h_i^{(1)} - h^{(1)}$ in Ω, $h_i^{(0)} - \phi_0 = \nu_i - \phi_0$ on $\partial\Omega$. Therefore $\|h_i^{(0)} - \nu_i\|_{s+1,\Omega} \leq c(\|h_i^{(1)} - h^{(1)}\|_{s-1,\Omega} + \|\nu_i - \phi_0\|_{s+1,\Omega})$, where c is a constant, from which it follows that $h_i^{(0)} \to \phi_0$ in $H^{s+1}(\Omega, \mathbb{R})$. Since $\{h_i^{(0)}\} \subset H^{s+3}(\Omega, \mathbb{R})$ and $\Delta^k h_i^{(0)} \in H_0^{s+3-2k}(\Omega, \mathbb{R})$, $i \geq 1$, $1 \leq k \leq (s+1)/2$, $\{h_i^{(0)}\}$ satisfies the required conditions, i.e., $\phi_{0i} = h_i^{(0)}$, $i \geq 1$.

The construction of $\{\dot\phi_{0i}\}$ is entirely analogous, i.e., there exists a sequence $\{\dot h_i^{(0)}\} \subset H^{s+2}(\Omega, \mathbb{R})$, $\Delta^l \dot h_i^{(0)} \in H_0^{s+2-2l}(\Omega, \mathbb{R})$, $i \geq 1$, $0 \leq l \leq (s-1)/2$, such that $\dot h_i^{(0)} \to \dot\phi_0$ in $H_0^s(\Omega, \mathbb{R})$.

The case s is even proceeds analogously.

Thus, from Theorem 1.3, one obtains the fact that if $(\phi_0, \dot\phi_0) \in H^{s+1}(\Omega, \mathbb{R}) \times H^s(\Omega, \mathbb{R})$ and satisfies the compatibility conditions, then there exists a unique solution of $\ddot\phi = \Delta\phi$ such that $\phi \in C^r([0, t], H^{s+1-r}(\Omega \times \mathbb{R}))$, $0 \leq r \leq s+1$.

2. Classical Elastodynamics

We begin by summarizing our notations and the equations we will be studying.

We refer vectors and tensors to a fixed system of cartesian coordinates and let indices run from 1 to m. The summation convention is employed.

A *body* Ω is an open connected set in $\mathbb{R}^m$ (we allow $\Omega = \mathbb{R}^m$). Its motion is given by the *displacements* $u_i(t, x)$, where $t \in [0, T]$ and $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$.

The equations of the classical theory are

$$\begin{aligned} \rho \frac{\partial^2 u_i}{\partial t^2} &= \frac{\partial T_{ij}}{\partial x^j} + \rho b_i, \\ T_{ij} &= T_{ji}, \\ T_{ij} &= c_{ijkl} e_{kl}, \\ 2e_{ij} &= \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \end{aligned} \tag{2.1}$$

where T_{ij} is the *Cauchy stress tensor*, e_{ij} is the *infinitesimal strain tensor*, c_{ijkl} are the *elasticities*, b_i is the *external body force* and ρ is the *mass density*. We assume ρ, $c_{ijkl} : \Omega \to \mathbb{R}$, $\rho > 0$ and $b_i : [0, T] \times \Omega \to \mathbb{R}$ are assigned functions. The elasticities satisfy the symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk}.$$

Combining the equations in (2.1) results in the displacement form of the equations of motion:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x^j}\left(c_{ijkl} \frac{\partial u_k}{\partial x^l}\right) + \rho b_i \tag{2.2}$$

An initial value problem consists of equation (2.2) on $[0, t] \times \mathbb{R}$ and initial conditions

$$u_i(0) = \alpha_i, \qquad \frac{\partial u_i}{\partial t}(0) = \beta_i \tag{2.3}$$

on Ω, where α_i and β_i are assigned functions. If Ω has a boundary, we append to (2.2) and (2.3) boundary conditions. In the sequel we consider two cases:

$$u_i = \gamma_i \quad \text{on} \quad [0, t] \times \partial\Omega, \tag{2.4}$$

the *displacement problem* and

$$T_i = n_j c_{ijkl} \frac{\partial u_k}{\partial x^l} = \delta_i \quad \text{on} \quad [0, t] \times \partial\Omega, \tag{2.5}$$

the *traction problem*, where n_j is the unit outward normal and γ_i, δ_i are assigned functions.

3. Symmetric Hyperbolic Form of the Equations

We now exhibit the way in which (2.1) can be written in symmetric hyperbolic form. We give the formulation for $m = 3$; the setup for arbitrary m is similar.

Let $u = (\boldsymbol{u}, \boldsymbol{T}, \boldsymbol{v})$ where $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{T} = (T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{23})$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, the velocity. Let $\boldsymbol{e} = (e_{11}, e_{22}, e_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23})$ where $\gamma_{ij} = 2e_{ij}$. The constitutive equation $T_{ij} = c_{ijkl} e_{kl}$ can be put in the matrix form $\boldsymbol{T} = E\boldsymbol{e}$. We will need the condition that E is symmetric and uniformly positive definite. Translated in terms of the c_{ijkl}'s these requirements become $c_{ijkl} = c_{klij}$ and $c_{ijkl}(x)\psi_{ij}\psi_{kl} \geq \delta\psi_{ij}\psi_{ij}$ for all symmetric ψ_{ij}, a fixed $\delta > 0$ and all $x \in \Omega$. Define

$$A^0 = \begin{pmatrix} Id & \cdot & \cdot \\ \cdot & E^{-1} & \cdot \\ \cdot & \cdot & \rho Id \end{pmatrix}, \qquad A^i \frac{\partial}{\partial x^i} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & D \\ \cdot & D^t & \cdot \end{pmatrix},$$

where Id is the 3×3 identity matrix,

$$D = \begin{pmatrix} \partial/\partial x^1 & \cdot & \cdot \\ \cdot & \partial/\partial x^2 & \cdot \\ \cdot & \cdot & \partial/\partial x^3 \\ \partial/\partial x^2 & \partial/\partial x^1 & \cdot \\ \partial/\partial x^3 & \cdot & \partial/\partial x^1 \\ \cdot & \partial/\partial x^3 & \partial/\partial x^2 \end{pmatrix},$$

and dots denote zeros. Let

$$B=\begin{pmatrix} \cdot & \cdot & Id \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad \text{and} \quad f=\begin{pmatrix} \cdot \\ \cdot \\ \rho b \end{pmatrix},$$

where $b=(b_1, b_2, b_3)$.

With the preceding definitions, (1.1) is a symmetric hyperbolic system equivalent to (2.1). This formulation was suggested by Wilcox [24].

4. Existence, Uniqueness and Differentiability Theorems in Classical Elastodynamics

We now combine the developments of Sections 1 to 3 to obtain existence, uniqueness and differentiability theorems for classical elastodynamics. Our assumptions are as follows:

(i) ρ and c_{ijkl} are in $C_b^\infty(\bar{\Omega}, \mathbb{R})$.
(ii) $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$.
(iii) $c_{ijkl}(x)\psi_{ij}\psi_{kl} \geq \delta\psi_{ij}\psi_{ij}$ for all symmetric ψ_{ij}, a fixed $\delta>0$ and all $x\in\bar{\Omega}$.
(iv) $b_i \in H^s([0,T]\times\Omega, \mathbb{R})$.
(v) $\alpha_i \in H^{s+1}(\Omega, \mathbb{R})$, $\beta_i \in H^s(\Omega, \mathbb{R})$. $\qquad (A_1)$

First consider the case of $\Omega=\mathbb{R}^m$. Let $\alpha=(\alpha_1, \ldots, \alpha_m)$, $\beta=(\beta_1, \ldots, \beta_m)$, etc.

THEOREM 4.1. *Consider (2.2) on $\Omega=\mathbb{R}^m$ with initial conditions given by (2.3). Let assumptions A_1 hold.*

Then there exists a unique solution u_i of (2.2) such that

(i) $u_i \in C^r([0,T], H^{s+1-r}(\Omega, \mathbb{R}))$, $0\leq r\leq s+1$.
(ii) $u_i(0)=\alpha_i$, $(\partial u_i/\partial t)(0)=\beta_i$ *on* Ω.
(iii) $(\alpha, \beta)\mapsto(u, \partial u/\partial t)$ *is a C^0 mapping from $H^{s+1}(\Omega, \mathbb{R}^m)\times H^s(\Omega, \mathbb{R}^m)$ into itself.* $\qquad (C)$

Proof. First note that $E^{-1}\in C_b^\infty(\Omega, \mathbb{R}^{(m(m+1)/2)^2})$ and $\rho b\in H^s([0,T]\times\Omega, \mathbb{R}^m)$, hence so are A^0 and f of Section 3. Putting (2.2) into the symmetric hyperbolic form of Section 3 and applying Theorem 1.1 yields that u_i, T_{ij}, and v_i are in $C^r([0,T], H^{s-r}(\Omega, \mathbb{R}))$, $0\leq r\leq s$. The additional differentiability for u_i can be obtained in several ways. For example, $T_{ij}(t)\in H^r(\Omega, \mathbb{R})$ implies $e_{ij}(t)\in H^r(\Omega, \mathbb{R})$ and $2(\partial e_{ij}/\partial x^j)(t)=(\Delta u_i+\partial^2 u_j/\partial x^i\partial x^j)(t)\in H^{r-1}(\Omega, \mathbb{R})$, which is an elliptic operator on $(u_1, \ldots, u_m)$. Therefore $u_i(t)\in H^{r+1}(\Omega, \mathbb{R})$. (This can be proved exactly as $\Delta h\in H^r(\Omega, \mathbb{R})\Rightarrow h\in H^{r+2}(\Omega, \mathbb{R})$ is proved, i.e., by means of the Fourier transform; cf. Yosida [25].) □

Remark 4.1. There is no nonuniqueness due to rigid body motions because all quantities involved are in function spaces whose members $\to 0$ as $|x|\to\infty$.

We now consider the case in which Ω has a boundary. We assume:

Ω is a bounded open set contained in $\mathbb{R}^m$ with compact boundary $\partial\Omega$ of class C^∞ such that Ω lies on one side of $\partial\Omega$. $\qquad (A_2)$

Let

$$L=\left[\frac{\partial}{\partial x^j} c_{ijkl} \frac{\partial}{\partial x^l}\right] \quad \text{and} \quad L_n=\left[n_j c_{ijkl}\frac{\partial}{\partial x^l}\right].$$

THEOREM 4.2. (*displacement problem*) *Consider (2.2) on Ω with initial conditions given by (2.3) and boundary conditions given by (2.4). Let assumptions A_1 and A_2 above hold and assume*

(i) $\gamma\in H^{s+3/2}([0,T]\times\partial\Omega, \mathbb{R}^m)$.
(ii) *The compatibility conditions hold; namely*

$$L^r\alpha+\sum_{k=1}^{r} L^{r-k}\rho(\partial/\partial t)^{2k-2}b(0)=\rho(\partial/\partial t)^{2r}\gamma(0), \qquad 0\leq r\leq s/2,$$

$$L^l\beta+\sum_{k=1}^{l} L^{l-k}\rho(\partial/\partial t)^{2k-1}b(0)=\rho(\partial/\partial t)^{2l+1}\gamma(0), \qquad 0\leq l\leq (s-1)/2,$$

on $\partial\Omega$.

Then the conclusions C in Theorem 4.1 hold and $u_i=\gamma_i$ on $\partial\Omega$.

THEOREM 4.3. (*traction problem*) *Consider (2.2) on Ω with initial conditions given by (2.3) and boundary conditions given by (2.5). Let assumptions A_1 and A_2 hold and assume:*

(i) $\delta\in H^{s+1/2}([0,T]\times\partial\Omega, \mathbb{R}^m)$.
(ii) *The compatibility conditions hold; namely*

$$L_n\left(L^r\alpha+\sum_{k=1}^{r} L^{r-k}\rho(\partial/\partial t)^{2k-2}b(0)\right)=\rho(\partial/\partial t)^{2r}\delta(0), \qquad 0\leq r\leq (s-1)/2,$$

$$L_n\left(L^l\beta+\sum_{k=1}^{l} L^{l-k}\rho(\partial/\partial t)^{2k-1}b(0)\right)=\rho(\partial/\partial t)^{2l+1}\delta(0), \qquad 0\leq l\leq (s-2)/2,$$

on $\partial\Omega$.

Then the conclusions C in Theorem 4.1 hold and $n_j T_{ij}=\delta_i$ on $\partial\Omega$.

Remark 4.2. If the requirements of Theorems 4.1–4.3 are met for all $s\geq 0$, then $u_i\in C^\infty([0,T], C_b^\infty(\Omega, \mathbb{R}))$ (cf. Fichera [6]).

Proof of Theorems 4.2 and 4.3. In view of Theorem 1.4, it suffices to prove Theorems 4.2 and 4.3 for the case of homogeneous boundary data. Note that the boundary matrix A_n has constant rank $2m$ and that $N(x)=\{u \mid v_i=0\}$ for the displacement problem whereas $N(x)=\{u \mid n_j T_{ij}=0\}$ for the traction problem. In each case N is a linear subspace of $\mathbb{R}^N$ for each $x\in\partial\Omega$, varying smoothly with x, and $(A_n u, u)=2v_i T_{ij} n_j=0$ is definite on no larger subspace.

To establish Lemma 1.1 we proceed as in the example of the wave equation. Since in the present circumstances

$$u = (u, T, v),$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}, \frac{\partial T}{\partial t}, \frac{\partial v}{\partial t}\right),$$

and

$$\frac{\partial u}{\partial x^i} = \left(\frac{\partial u}{\partial x^i}, \frac{\partial T}{\partial x^i}, \frac{\partial v}{\partial x^i}\right),$$

we require an estimate for all of $\partial u/\partial x^i$. Again we employ results of elliptic operator theory.* The boundary value problem $Lu(t) = \rho(\partial v/\partial t - b)(t) \in H^{s-1}(\Omega, \mathbb{R}^m)$, $u(t) = \alpha$ on $\partial\Omega$ *or* $L_n u(t) = 0$ on $\partial\Omega$, implies $u(t) \in H^{s+1}(\Omega, \mathbb{R}^m)$ and

$$\|u(t)\|_{s+1,\Omega} \leq c(\|(\partial v/\partial t)(t)\|_{s-1,\Omega} + \|b(t)\|_{s-1,\Omega} + \|\alpha\|_{s+1,\Omega}).$$

Estimates for $\|(\partial u/\partial x^i)(t)\|_{s,\Omega}$ and $\|(\partial T/\partial x^i)(t)\|_{s-1,\Omega}$ in terms of the right hand side of (1.4) follow immediately.

To obtain an estimate for $\partial v/\partial x^i$ we proceed similarly. From the boundary value problem $Lv(t) = \operatorname{div}(\partial T/\partial t)(t) \in H^{s-2}(\Omega, \mathbb{R}^m)$, $v(t) = 0$ on $\partial\Omega$ *or* $L_n v(t) = 0$ on $\partial\Omega$, implies $v(t) \in H^s(\Omega, \mathbb{R}^m)$ and $\|v(t)\|_{s,\Omega} \leq c\,\|\operatorname{div}(\partial T/\partial t)(t)\|_{s-2,\Omega}$ from which it follows that $\|(\partial v/\partial x^i)(t)\|_{s-1,\Omega} \leq c\,|||(\partial u/\partial t)(t)|||_{s-1,\Omega}$. Thus Lemma 1.1 is proved.

To establish Lemma 1.2 for the displacement problem we need to exhibit sequences $\{\alpha_l\} \subset H^{s+3}(\Omega, \mathbb{R}^m)$, $\{\beta_l\} \subset H^{s+2}(\Omega, \mathbb{R}^m)$, $\{b_l\} \subset H^{s+2}([0, T] \times \Omega, \mathbb{R}^m)$ such that the compatibility conditions (Theorem 4.2, (ii)) are satisfied with $\gamma(t) = \gamma(0)$ for each $l \geq 1$, $0 \leq r \leq (s+1)/2$, $0 \leq l \leq s/2$, and $\alpha_l \to \alpha$ in $H^{s+1}(\Omega, \mathbb{R}^m)$, $\beta_l \to \beta$ in $H^s(\Omega, \mathbb{R}^m)$, $b_l \to b$ in $H^s([0, T] \times \Omega, \mathbb{R}^m)$. By Remark 1.2 it suffices to carry this out for $0 \leq r \leq (s+1)/2$, $0 \leq l \leq s/2$, for s is odd and for $0 \leq r \leq s/2$, $0 \leq l \leq s/2$ for s even. We proceed in a similar fashion to the example of the wave equation.

Assume s is odd and let $h^{(k)} = L^k \alpha$. For $1 \leq r \leq s/2$, $h^{(r)} \in H_b^{s+1-2r}(\Omega, \mathbb{R}^m) = \{h \mid h \in H^{s+1-2r}(\Omega, \mathbb{R}^m)$ and $h = -\sum_{k=1}^{r} L^{r-k}\rho(\partial/\partial t)^{2k-2} b(0)$ on $\partial\Omega\}$, whereas for $r = (s+1)/2$, $h^{(r)} \in H^{s+1-2r}(\Omega, \mathbb{R}^m)$. Select a sequence $\{b_l\} \subset H^{s+2}([0, T] \times \Omega, \mathbb{R}^m)$ such that $b_l \to b$ in $H^s([0, T] \times \Omega, \mathbb{R}^m)$. Let $r = (s+1)/2$ and take a sequence $\{h_l^{(r)}\} \subset H^2(\Omega, \mathbb{R}^m)$, such that $h_l^{(r)} = -\sum_{k=1}^{r} L^{r-k}\rho(\partial/\partial t)^{2k-2} b_l(0)$ on $\partial\Omega$, and $h_l^{(r)} \to h^{(r)}$ in $L_2(\Omega, \mathbb{R}^m)$. Then define $h_l^{(r-1)}$ by the boundary value problem: $Lh_l^{(r-1)} = h_l^{(r)}$ in Ω, $h_l^{(r-1)} = -\sum_{k=1}^{r-1} L^{r-1-k}\rho(\partial/\partial t)^{2k-2} b_l(0)$ on $\partial\Omega$. Thus we have $h_l^{(r-1)} \in H_{bl}^4(\Omega, \mathbb{R}^m)$, where $H_{bl}^s = H_b^s$ with b_l replacing b, and $h_l^{(r-1)} \to h^{(r-1)}$ in $H^2(\Omega, \mathbb{R}^m)$. We repeat this procedure until we have $\{h_l^{(1)}\} \subset H_{bl}^{s+1}(\Omega, \mathbb{R})$ which converges to $h^{(1)}$ in $H^{s-1}(\Omega, \mathbb{R}^m)$. The solution of $Lh_l^{(0)} = h_l^{(1)}$ in Ω, $h_l^{(0)} = \nu_l$ on $\partial\Omega$, where $\{\nu_l\} \subset H^{s+3}(\Omega, \mathbb{R}^m)$, $\nu_l \to \alpha$ in $H^{s+1}(\Omega, \mathbb{R}^m)$, is in $H^{s+3}(\Omega, \mathbb{R}^m)$ and converges to α in $H^{s+1}(\Omega, \mathbb{R}^m)$. Thus $\alpha_l = h_l^{(0)}$ and b_l are the desired sequences.

The construction of β_l proceeds analogously, as does the case s is even, and thus we omit the details. This completes lemma 1.2 for the displacement problem and thus proves Theorem 4.2.

* The hypotheses on the coefficients $c_{\alpha\beta\gamma\delta}$ ensure that L is a strongly elliptic operator (cf. Fichera [6]).

To accomplish the same end for the traction problem, we proceed essentially as above. For this case we use the set $\{h \mid h \in H^{s+1-2r}(\Omega, \mathbb{R}^m)$ and $L_n h = -L_n(\sum_{k=1}^{r} L^{r-k}\rho(\partial/\partial t)^{2k-2} b(0)$ on $\partial\Omega)\}$ and take $\delta(t) = 0$ in the compatibility conditions. The traction boundary conditions put global constraints upon the sequences $h_l^{(r)}$. For example, each $h_l^{(r)}$ may need to be altered by a constant so that the corresponding elliptic boundary value problem can be solved. This is no problem because the differentiability class of the $h_l^{(r)}$ is unaffected and the limit $h^{(r)}$ must also satisfy the same constraint. □

Remark 4.3. Various Sobolev inequalities are often useful, e.g., if $s > m/2 + k$, where m is the dimension of the domain, then $H^s \subset C_b^k$. Thus for the important cases $m = 2$ and $m = 3$, $u_l(t) \in H^{s+2}(\Omega, \mathbb{R}^m)$ implies $u_l(t) \in C_b^2(\Omega, \mathbb{R}^m)$.

Remark 4.4. A theorem, analogous to the above, can clearly be proved for the *contact problem* in which

$$\left.\begin{array}{ll} u_i = \gamma_i, & i \in M \subset \{1, \ldots, m\} \\ T_i = \delta_i, & i \in \{1, \ldots, m\} - M \end{array}\right\} \quad \text{on} \quad [0, T] \times \partial\Omega.$$

The components γ_i, δ_i may be referred to an arbitrary curvilinear coordinate system, as long as the resulting subspace $N(x)$ varies smoothly with x.

Remark 4.5. The *mixed problem*, in which

$$u_i = \gamma_i \quad \text{on} \quad [0, T] \times \partial\Omega_1,$$

$$T_i = \delta_i \quad \text{on} \quad [0, T] \times \partial\Omega_2,$$

where $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ and $\overline{\partial\Omega_1 \cup \partial\Omega_2} = \partial\Omega$, is at present outside the realm of applicability of the existing theorems for symmetric hyperbolic systems (cf. Theorem 1.3). In this case the boundary subspace fails to be smoothly varying with x. However, for the case of one space dimension ($m = 1$), two-point initial-boundary value problems, in which displacement is specified on one end and traction on the other, would be accomodated by the present theorem.

The existence and uniqueness of a weak solution to the mixed problem is treated in Duvaut-Lions [5].

Remark 4.6. The present theory also does not cover the exterior problem, i.e., when Ω is the exterior of some region. However, there should be no obstruction to there being a theorem to cover this case.

Remark 4.7. We note that the hypotheses necessary to achieve the differentiability results presented herein are rather stringent. Thus we conclude that for a typical elastodynamic problem arising in practice the existence of a C^∞ solution is the exception rather than the rule.

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