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THE MANIFOLD OF CONFORMALLY EQUIVALENT METRICS

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Introduction. Ebin [8] gives a thorough study of the space $\mathscr{M}$ of riemannian metrics on a compact manifold M and of the action of the diffeomorphism group $\mathscr{D}$ of M on $\mathscr{M}$. The purpose of this paper is to study the action of the larger group $\mathscr{C}$ of conformorphisms, or conformal transformations, on $\mathscr{M}$ and on $T^*\mathscr{M}$. On $\mathscr{M}$, the L_2-orthogonal decomposition induced by the action of $\mathscr{C}$ gives a splitting of symmetric tensors into three summands introduced by York [25; 26]. We find submanifolds of $\mathscr{M}$ tangent to the pieces of this decomposition.

The action of $\mathscr{C}$ on $T^*\mathscr{M}$ is symplectic and may be reduced following Marsden-Weinstein [20]. This process parametrizes the space $\mathscr{E}$ of true gravitational degrees of freedom (see [14; 22; 25; 26]). The space is shown to be an (infinite dimensional, weak) symplectic manifold near those points (g, π) with no simultaneous conformal Killing fields. It is argued that near other points, $\mathscr{E}$ has singularities.

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1. The action of the conformorphism group. Let M be a compact oriented C^∞ manifold without boundary. Let $\mathscr{P}$ denote the multiplicative group of positive C^∞ functions on M, which we refer to as *pointwise conformal transformations* and let $\mathscr{D}$ denote the group of C^∞ diffeomorphisms on M, which we refer to as *coordinate transformations*.

Let $\mathscr{C} = \mathscr{D} \times \mathscr{P}$, with the group structure

$$(\varphi_1, p_1) \cdot (\varphi_2, p_2) = (\varphi_1 \circ \varphi_2, p_2 \cdot (p_1 \circ \varphi_2)),$$

be the group of *conformal transformations* or *conformorphisms*. Thus, $\mathscr{C}$ is the semi-direct product of $\mathscr{D}$ and $\mathscr{P}$ with $\mathscr{D}$ acting on $\mathscr{P}$ on the right by $(\varphi, p) \mapsto p \circ \varphi$.

We get a right action of $\mathscr{C}$ on $\mathscr{M}$, the space of C^∞ riemanian metrics on M, denoted $\tilde{A} : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ by $((\varphi, p), g) \mapsto p \cdot \varphi^* g$, where $\varphi^* g$ is the pullback of g by $\varphi \in \mathscr{D}$.

As with $\mathscr{D}$, $\mathscr{C}$ is an infinite dimensional "Lie group" with Lie algebra the semi-direct sum of $\mathscr{X}$, the vector fields on M with C^∞, the C^∞ functions on M (see Ebin [8]).

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For $g \in \mathcal{M}$, let $C_g \subset \mathscr{C}$ be the isotropy group of g; thus

$$C_g = \{(\varphi, p) \in \mathscr{C} | p \cdot \varphi^* g) = g\}.$$

Clearly C_g is isomorphic to the conformal group of g, i.e.,

$$\{\varphi \in \mathscr{D} | \varphi^* g = pg \text{ for some } p \in \mathscr{P}\} \subset \mathscr{D},$$

and thus if dim $M \geqq 3$, C_g inherits the structure of a Lie group (of dimension $\leqq (n+1)(n+2)/2$) with Lie algebra

$$\mathfrak{g}_g = \{(X, N) \in \mathscr{X} \times C^\infty | Ng + L_X g = 0\}$$

(see Kobayashi [**17**]). Also, note that $\mathfrak{g}_g$ is isomorphic to the Lie algebra $\{X \in \mathscr{X} | L_X g = -(2/n)(\delta X) g\}$ of the conformal group of g, i.e., the space of conformal Killing vector fields. Here $(L_X g)_{ij} = X_{i|j} + X_{j|i}$ is the Lie derivative of g with respect to X and $\delta X = -\text{div}\, X = -X^i{}_{|i}$ is minus the divergence of X.

Let $\mathscr{D} \cdot g = \mathscr{O}_g = \{\varphi^* g | \varphi \in \mathscr{D}\}$ be the orbit of $g \in \mathcal{M}$ under the action of $\mathscr{D}$ so that $\mathscr{O}_g$ is the set of metrics isometric to g. Let

$$\mathscr{C} \cdot g = \{p\varphi^* g | (\varphi, p) \in \mathscr{C}\}$$

be the orbit of g under the action of $\mathscr{C}$. Thus $\mathscr{C} \cdot g$ is the set of metrics *conformally equivalent to* g. Note that if $M = S^2$, then $\mathscr{C} \cdot g = \mathcal{M}$ by the uniformization theorem (see Wolf [**24**]).

The group $\mathscr{P}$ acts on $\mathcal{M}$ by multiplication,

$$\mathscr{P} \times \mathcal{M} \to \mathcal{M}, \quad (p, g) \mapsto pg.$$

The orbit of $g \in \mathcal{M}$ under $\mathscr{P}$, namely

$$p \cdot g = \{pg | p \in \mathscr{P}\}$$

is the set of metrics *pointwise conformal* to g. If dim $M = 1$, note that $\mathscr{P} \cdot g = \mathcal{M}$.

From Ebin [**8**], $\mathscr{O}_g \in \mathcal{M}$ is a smooth submanifold of $\mathcal{M}$. We shall prove the same for $\mathscr{C} \cdot g$ and $\mathscr{P} \cdot g$, as well as the existence of a slice for the action of $\mathscr{C}$.

In order to sharpen these notions, we work in the corresponding Sobolev spaces. A superscript ^{s,p} denotes the Sobolev class $W^{s,p} = L_s^p$. We always assume that $s > (n/p) + 1$. Thus $\mathcal{M}^{s,p}$, for example, denotes metrics of class $W^{s,p}$ (see Friedman [**15**] and Ebin [**8**] for properties of these spaces). Specifically, we need the fact that $\mathscr{D}^{s,p}$ is a manifold and a topological group in which right multiplication is C^∞ and that the action of $\mathscr{D}^{s+1,p}$ on $\mathcal{M}^{s,p}$ is continuous. Thus, $\mathscr{D}^{s+1,p} \times \mathscr{P}^{s,p}$ is also a topological group, denoted $\mathscr{C}^{s,p}$.

Let us observe that the orbit $\mathscr{P} \cdot g$ of $g \in \mathcal{M}^{s,p}$ under the action of $\mathscr{P}^{s,p}$ on $\mathcal{M}^{s,p}$ is a closed C^∞ submanifold of $\mathcal{M}^{s,p}$ with tangent space at $\bar{g}$ given by

$$S_2{}^C = T_{\bar{g}}(\mathscr{P} \cdot g) = \{h \in S_2{}^{s,p} | h = f\bar{g} \text{ for some } f \in W^{s,p}\}.$$

Indeed, $p \mapsto pg$ is an immersion and a homeomorphism onto its image. For more complicated actions, the following is useful.

1.1 LEMMA. *Let $f : N \to P$ be a C^r map of Banach manifolds, $r \geqq 1$, such that* Ker $Tf \subset TN$ *is a C^r subbundle of TN and for each $x \in N$,* Range $T_x f$ *is closed with a closed complement. Then $f(N)$ is a (locally) immersed submanifold of P and if f is open onto its image, $f(N) \subset \mathscr{P}$ is a submanifold.*

The proof is standard; cf. Lang [**19**] (the version here was pointed out by J. Guckenheimer). Sufficient conditions for f to be open, which we shall use below are these (a) the range of f near $x \in N$ depends only on $f(x)$ [i.e., if $f(x_1) = f(x_2)$ then $f(U_1) = f(U_2)$ for some neighborhoods U_1 of x_1 and U_2 of x_2), and (b) if $f(x_n) \to y$ then x_n has a convergent subsequence in N. This is easy to see using the fact that locally $f(N)$ is immersed. If these conditions hold, $f(N)$ will then be a closed submanifold of P.

We shall also make use of a splitting lemma. Let E, F be vector bundles over M with a fixed riemannian structure (i.e., inner products on the fibers and a volume fixed on M). Let $D : C^\infty(E) \to C^\infty(F)$ be a kth order differential operator, and $D^* : C^\infty(F) \to C^\infty(E)$ its L_2 adjoint (see Palais [**23**]).

1.2 SPLITTING LEMMA. (cf. Berger-Ebin [**3**]). *Assume D has injective symbol or that D^* has injective symbol. Then*

$$W^{s,p}(F) = \text{Range } D \oplus \text{Ker } D^*.$$

On the right, D is regarded mapping $W^{s+k,p}(E) \to W^{s,p}(F)$ and $D^* : W^{s,p}(F) \to W^{s-k,p}(E)$. Here $\infty \geqq s \geqq k$, $1 < p < \infty$. If D has injective symbol, Ker D *is finite dimensional and consists of C^∞ elements.*

Remarks. 1) This is proved in Berger-Ebin [**3**] in case $p = 2$ and D has injective symbol. It is not difficult to give a direct proof of 1.2 using the elliptic estimates and Rellich's theorem (see, e.g. [**1**; **16**; **18**] in which "elliptic" means "injective symbol"). The main point to be proved is that the range of D is closed. The case of D^* with injective symbol can be deduced from that for D, for if D^* has injective symbol, Range (D) = Range (DD^*) and as DD^* is elliptic, this is closed.

2) If an operator D maps into a product space $F = F_1 \oplus \ldots \oplus F_l$ with different orders in each factor $D_i : C^\infty(E) \to C^\infty(F_i)$ and if one computes the symbol of each D_i separately and their direct sum is injective, then the basic elliptic estimates, and hence the splitting lemma still holds. Similarly, if $E = E_1 \oplus \ldots \oplus E_m$, the symbol may be computed as an $m \times l$ matrix of sub-symbols. Finite dimensionality of Ker D requires order $(D_i) \geqq 1$; if some D_{ij} has order zero, one must assume dim $E <$ dim F. (These remarks are based on the work of Agmon-Douglis and Nirenberg [**1**]; see also Hörmander [**16**]).

3) Applying the splitting theorem to the Laplacian on a Riemannian manifold results at once in the Hodge decomposition for forms.

For fixed $g \in \mathscr{M}^{s+r,p}$, the orbit map

$$\Psi_g : \mathscr{C}^{s+1,p} \to \mathscr{M}^{s,p}, \quad (\varphi, p) \mapsto p\varphi^* g$$

is of class C^r, $r \geqq 0$. For $r \geqq 1$, its tangent (derivative) at the identity (id, 1) is

$$T\Psi_g(\mathrm{id}, 1) = \tau_g : \mathscr{X}^{s+1,p} \times W^{s,p} \to S_2^{s,p}$$
$$\tau_g(X, N) = L_X g + Ng.$$

Here, $W^{s,p}$ stands for the real valued functions on M of class $W^{s,p}$, and $S_2^{s,p}$ denotes the symmetric two tensors of class $W^{s,p}$. (See Ebin [8] for the relevant facts needed to prove differentiability of the orbit map.)

The L_2-adjoint of τ_g is

$$\tau_g^* : S_2 \to \mathscr{X} \times C^\infty; \quad h \mapsto (2\delta h, \mathrm{tr}\, h)$$

where $(\delta h)^i = -h^{ji}{}_{|j}$ is the contravariant divergence of h, and $\mathrm{tr}\, h = h^i{}_i$ is the trace of h. Note that $\mathrm{Ker}\, \tau_g^* = S_2^{TT}$, the transverse ($\delta h = 0$), traceless ($\mathrm{tr}\, h = 0$) tensors. If dim $\geqq 3$, it is infinite dimensional [4]).

The symbol of τ_g is (see the remarks following (1.2)),

$$\sigma_\xi(v, s) = sg + v^b \otimes \xi + \xi \otimes v^b$$

where $\xi \in T_x^*M$, $v \in T_xM$, $s \in \mathbf{R}$ and $v^b \in T_x^*M$ is the corresponding one form via g.

It is easy to see that for $\xi \neq 0$, and dim $M \geqq 2$, σ_ξ is injective, for if $sg + v^b \otimes \xi + \xi \otimes v^b = 0$, taking the trace yields $s = -(2/n)v \cdot \xi$ and contracting with $v^b \otimes \xi$ gives $(1 - 2/n)(v \cdot \xi)^2 + \|v\|^2\|\xi\|^2 = 0$, so $v = 0$ and $s = 0$.

1.3 *Remark*. If dim $\geqq 3$, dim $E <$ dim F (E and F are the domain and range bundles for τ_g) and so Ker τ_g is finite dimensional. Thus one gets a proof that if dim $M \geqq 3$, the conformal group of g is a Lie group using the general result of Palais (Kobayashi [7, p. 13]).

From 1.2, $S_2 = \mathrm{Range}\, \tau_g \oplus \mathrm{Ker}\, \tau_g^*$. That lemma as stated needs the assumption that g is C^∞, but actually $g \in \mathscr{M}^{s+r,p}$, $r \geqq 1$ is sufficient for the decomposition in $W^{s,p}$, by examination of the proofs.

If we apply the trivial pointwise orthogonal splitting

$$S_2 = S_2^T \oplus S_2^C, \quad h = (h - (1/n)(\mathrm{tr}\, h)g) + (1/n)(\mathrm{tr}\, h)g$$

where $S_2^T = \{h \in S_2 | \mathrm{tr}\, h = 0\}$, $S_2^C = \{fg | f \in C^\infty\}$, to the range of τ_g, we get the finer splitting

$$S_2 = S_2^{TT} \oplus S_2^C \oplus (S_2^T \cap \mathrm{Range}\, \tau_g)$$

written as $h = h^{TT} + (1/n)(\mathrm{tr}\ \ h)g + ((2/n)(\delta X)g + L_X g)$ where $S_2^{TT} = \mathrm{Ker}\, \tau_g^*$, the transverse traceless tensors. This is the splitting of York [**25**; **26**]. Summarizing:

1.4 THEOREM. *Let* dim $M \geqq 2$ *and* $g \in \mathscr{M}^{s+r,p}$. *Then*

$$S_2^{s,p} = \mathrm{Range}\, \tau_g \oplus \mathrm{Ker}\, \tau_g^* = $$
$$= (S_2^{TT})^{s,p} \oplus (S_2^C)^{s,p} \oplus (S_2^T \cap \mathrm{Range}\, \tau_g)^{s,p}$$

where $\tau_g : \mathscr{X}^{s+1,p} \times W^{s,p} \to S_2^{s,p}$ *and* $\tau_g^* : S_2^{s,p} \to (\mathscr{X}^{s-1,p}, W^{s,p})$.

Using (1.4), we can prove that the conformal orbit is a manifold:

1.5 THEOREM. *Let* $\dim M \geqq 3$, $g \in \mathcal{M}^{s+r,p}$, *and* $\mathcal{C} \cdot g$ *be its orbit under* $\mathcal{C}^{s+1,p} = \mathcal{D}^{s+1,p} \times \mathcal{P}^{s,p}$. *Then* $\mathcal{C} \cdot g \subset \mathcal{M}^{s,p}$ *is a* C^r *closed submanifold with tangent space at* g *given by*

$$T_g(\mathcal{C} \cdot g) = \{L_X g + Ng | X \in \mathcal{X}^{s+1,p}, N \in W^{s,p}\}.$$

Proof. Let $\Psi : \mathcal{C}^{s+1} \to \mathcal{M}^{s,p}$ be the orbit map through g, $\Psi(\varphi, p) = p\varphi^* g$. Then Ψ is a C^r map and has tangent at the identity given by

$$T_{(\mathrm{id},1)}\Psi(X, N) = \tau_g(X, N),$$

and at (φ, p),

$$T_{(\varphi,p)}\Psi(X \circ \varphi, p(N \circ \varphi)) = p\varphi^*\{\tau_g(X, N)\}$$

which is just τ_g extended to be right invariant. Thus $\ker T_{\varphi,p}\Psi$ is $\operatorname{Ker} \tau_g$ extended to be right invariant over $\mathcal{C}^{s,p}$. This is a finite dimensional C^r subbundle of $T\mathcal{C}^{s,p}$ by the arguments of Ebin-Marsden [**9**, Appendix A]. By 1.4, the range of $T\Psi$ is closed with closed complement.

To finish the proof we need to verify that Ψ is open onto its image and has closed range. This will be done using the remarks following (1.1). By right invariance of Ψ, the local ranges depend only on the image. Next one shows that if $p_n \varphi_n^* g \to \gamma$, then $\{(\varphi_n, p_n)\}$ has a covergent subsequence in $\mathcal{C}^{s,p}$. Since the proof proceeds in the same way as Ebin [**8**], Prop. 6.13, we shall omit the details (replace his $T\zeta_n$ by $\sqrt{p_n}\, T\varphi_n$ where appropriate).

One can also prove this result, as in Ebin [**8**] by putting a C^r manifold structure on $\mathcal{C}^{s,p}/C_g^{s+1,p}$ and showing that the map from this to $\mathcal{C}^{s+1,p} \cdot g$ is an injective immersion which is a homeomorphism. The fact that C_g need not be compact causes no difficulty in the proof. The proof above is the same argument in different language.

In analogy with the slice theorem for the action of $\mathcal{D}$ on $\mathcal{M}$ (Ebin [**8**]), we now prove that there exists a slice for the action of $\mathcal{C}$ on $\mathcal{M}$; assume $\dim M \geqq 3$.

1.6 THEOREM. *If* $g \in \mathcal{M}^{s+r,p}$, $r \geqq 1$, *then the action of* $\mathcal{C}^{s+1,p}$ *on* $\mathcal{M}^{s,p}$ *admits a slice* $\tilde{S}$ *at* g; *i.e.* $\tilde{S}$ *is a submanifold containing* g *such that*

(1) *if* $(\eta, p) \in C_g^{s+1,p}$, $\tilde{A}((\eta, p), \tilde{S}) = \tilde{S}$;

(2) *if* $(\eta, p) \in \mathcal{C}^{s+1,p}$ *and* $\tilde{A}((\eta, p), \tilde{S}) \cap \tilde{S} \neq \emptyset$, *then* $(\eta, p) \in C_g^{s,p}$;

(3) *there is a local cross-section* $\chi : \mathcal{C}^{s+1,p}/C_g^{s+1,p} \to \mathcal{C}^{s+1,p}$ *defined on a neighborhood* U *of the identity coset such that* $([\eta, p], g_1) \mapsto \tilde{A}(\chi[\eta, p], g_1)$ *is a homeomorphism of* $U \times \tilde{S}$ *with a neighborhood of* g. *(One can also allow* $s = \infty$.*)*

Proof. Introduce the space of tensor densities

$$\mathcal{W}^{s,p} = \{g \otimes \mu_g^{-2/n} | g \in \mathcal{M}^{s,p}\}.$$

This space is "conformally invariant"; i.e., $pg \otimes (\mu_{pg})^{-2/n} = g \otimes \mu_g^{-2/n}$ so that $\mathscr{P}^{s,p}$ leaves each element of $\mathscr{W}^{s,p}$ fixed. Thus the orbit space $\mathscr{M}^{s,p}/\mathscr{P}^{s,p}$ (also see (2.1)) is canonically isomorphic to $\mathscr{W}^{s,p}$.

$\mathscr{D}^{s+1,p}$ acts on $\mathscr{W}^{s,p}$ by pullback and the orbits of $\mathscr{D}^{s+1,p}$, lifted back to $\mathscr{M}^{s,p}$ via $\pi : \mathscr{M}^{s,p} \to \mathscr{M}^{s,p}/\mathscr{P}^{s,p} \approx \mathscr{W}^{s,p}$, are just the orbits of $\mathscr{C}^{s+1,p}$ on $\mathscr{M}^{s,p}$. Proceeding along the standard lines of the Ebin-Palais Slice Theorem it is straightforward to show that the action of $\mathscr{D}^{s+1,p}$ on $\mathscr{W}^{s,p}$ has a slice, which can be identified with a slice $\tilde{S}$ for the action of $\mathscr{C}^{s+1,p}$ on $\mathscr{M}^{s,p}$. As in Theorem 1.5, the fact that the isotropy groups for the action of $\mathscr{D}^{s+1,p}$ on $\mathscr{W}^{s,p}$ need not be compact causes no difficulty in the proof. As in [**8**], one can also allow $s = \infty$. This completes the proof.

The usual consequences of the existence of a slice, such as locally decreasing conformal groups, and generically trivial conformal groups, $C_g = (\mathrm{id}, 1)$, if $\dim M \geq 3$, follow directly.

If μ is a volume form on M, let $\mathscr{N}_\mu = \{g \in \mathscr{M} | \mu(g) = \mu\}$. Then $\mathscr{N}_\mu$ (or $\mathscr{N}_\mu^{s,p}$) is a submanifold of $\mathscr{M}$ (or $\mathscr{M}^{s,p}$) with tangent space $S_2{}^T$ (see Ebin [**8**]).

Remark. If S_g is a slice at g for the action of $\mathscr{D}$, then we can choose

$$\tilde{S}_g \doteq S_g \cap \mathscr{N}_{\mu(g)}$$

with tangent space $\ker \tau_g^* = S_2{}^{TT} = S_2{}^0 \cap S_2{}^T$, which is orthogonal to the conformal orbit $\mathscr{C} \cdot g$ of g. Indeed, by the construction in Ebin [**8**] S_g is the L_2-exponentiation of $S_2{}^0$, $\mathscr{N}_{\mu(g)}$ is the L_2-exponentation of S^T, and $\tilde{S}_g$ is the L_2-exponentation of $S_2{}^{TT} = S_2{}^T \cap S_2{}^0$.

Thus the decomposition $S_2 = \ker \tau_g^* \oplus \text{Range } \tau_g$ can be written as

$$T_g\mathscr{M} = T_g\tilde{S}_g \oplus T_g(\mathscr{C} \cdot g)$$

the second summand describing the deformations along the conformal orbit $\mathscr{C} \cdot g$ and the first summand describing the deformations along the slice $\tilde{S}_g$ for this action. This is the analog of the canonical splitting (Ebin [**8**])

$$T_g\mathscr{M} = T_gS_g \oplus T_g\mathscr{O}_g = S_2{}^0 \oplus \alpha_g(\mathscr{X}).$$

Remark. One can approach the decomposition (1.4) using the conformal Lie derivative in place of the Lie derivative; i.e. $LX = L_Xg + (2/n)(\delta X)g$. In this approach one uses the action of $\mathscr{D}$ on the space $\mathscr{W}$ of (pointwise) conformally invariant tensor densities $g \otimes \mu_g^{-2/n}$ ($\mu_g = \mu(g)$ is the volume element of g) rather than the action of $\mathscr{C}$ on $\mathscr{M}$. See Fischer-Marsden [**13**, § 2] below, and Cantor [**6**].

Finally we remark that one can form the orbit space $\mathscr{M}/\mathscr{C}$, the space of equivalence classes of metrics conformal to one another. This space is sometimes called "conformal superspace" (York [**25**; **26**]). Since the conformal groups C_g change from metric to metric, $\mathscr{M}/\mathscr{C}$ is not itself a manifold. However, like the stratification of $\mathscr{M}/\mathscr{D}$ (Fischer [**10**]), it also can be stratified into sets of manifolds. See York [**25**] for a linearized version of this stratification.

2. The geometry of York's decomposition. In this section we construct manifolds which are tangent to the summands in York's decomposition (Theorem 1.4). First we consider the geometry behind the pointwise orthogonal splitting $S_2 = S_2^C \oplus S_2^T$ by showing that the splitting is globally integrable.

In the action $\mathscr{P}^{s,p} \times \mathscr{M}^{s,p} \to \mathscr{M}^{s,p}$, the isotropy at each g is the identity so the action is free. Moreover, the map

$$\mathscr{P}^{s,p} \times \mathscr{M}^{s,p} \to \mathscr{P}^{s,p} \times \mathscr{M}^{s,p}, \quad (p, g) \mapsto (p, pg)$$

is proper, for if p lies in a compact set $K \subset \mathscr{P}^{s,p}$, and pg lies in a compact set $H \subset \mathscr{M}^{s,p}$, g lies in $(1/K)H$, where $K > 0$ is a lower bound for $\{p \in K\}$. Thus the action of $\mathscr{P}^{s,p}$ on $\mathscr{M}^{s,p}$ is free and proper, so that the quotient space $\mathscr{M}^{s,p}/\mathscr{P}^{s,p}$ is a manifold.

We summarize as follows:

2.1 Proposition. *The orbit $\mathscr{P} \cdot g$ of $g \in \mathscr{M}^{s,p}$ under $\mathscr{P}^{s,p}$ is a closed C^∞ submanifold of $\mathscr{M}^{s,p}$ with tangent space $S_2^C = \{h \in S_2^{s,p} | h = fg \text{ for some } f \in W^{s,p}\}$. Also $\mathscr{M}^{s,p}/\mathscr{P}^{s,p}$ is a C^∞ manifold such that the projection $\pi : \mathscr{M}^{s,p} \to \mathscr{M}^{s,p}/\mathscr{P}^{s,p}$ is a submersion.*

Now let $\mathscr{V}$ denote the space of positively oriented volume forms on M, and recall (1.6) that

$$\mathscr{M}^{s,p}/\mathscr{P}^{s,p} \approx \mathscr{W}^{s,p} = \{g \otimes \mu_g^{-2/n} | g \in \mathscr{M}^{s,p}\}.$$

2.2 Theorem. (a) *$\mathscr{M}^{s,p}$ is diffeomorphic to $\mathscr{M}^{s,p}/\mathscr{P}^{s,p} \times \mathscr{V}^{s,p}$ [or $\mathscr{W}^{s,p} \times \mathscr{V}^{s,p}$] via $g \mapsto (\mathscr{P}^{s,p} \cdot g, \mu(g))$ [or $g \mapsto (g \otimes \mu_g^{-2/n}, \mu(g))$].*

(b) *$T_g\mathscr{M}^{s,p} = T_g(\mathscr{P}^{s,p} \cdot g) \oplus T_g\mathscr{N}_{\mu_g}^{s,p}$ and if $\mathscr{P}^{s,p}g'$ meets $\mathscr{N}_{\mu_{g'}}^{s,p}$, it does so in a single point g.*

Proof. First (b). The assertion $T_g\mathscr{M} = T_g(\mathscr{P} \cdot g) \oplus T_g\mathscr{N}_{\mu_g}$ is just $S_2 = S_2^C \oplus S_2^T$, the pointwise orthogonal decomposition above. If $g_1, g_2 \in \mathscr{P} \cdot g' \cap \mathscr{N}_{\mu_{g'}}$, then $g_2 = pg_1$ and $\mu(g_1) = \mu(g_2)$. But $\mu(g_2) = \mu(pg_1) = p^{n/2}\mu(g_1)$ so $p = 1$, and $g_2 = g_1$.

The result (a) also follows since $g \mapsto (\mathscr{P} \cdot g, \mu(g))$ is one to one; as it is obviously onto with invertible derivative it is a diffeomorphism.

The result may be paraphrased by saying that there is a "coordinate grid" for $\mathscr{M}$, the coordinates of g being $\mathscr{P} \cdot g$ and $\mu(g)$. The manifolds where these coordinates are constant are orthogonal whenever they intersect, and their tangent spaces split $T_g\mathscr{M}$.

The decomposition $S_2 = \operatorname{Ker} \tau_g^* \oplus \operatorname{Range} \tau_g$ is a splitting of the tangent space of $\mathscr{M}$ into a piece along the conformal orbit $\mathscr{C} \cdot g$ and along the slice $\tilde{S}_g$ for this action (see Fig. 1(a)).

Of course, the splitting $T_g\mathscr{M} = T_g\tilde{S}_g \oplus T_g(\mathscr{C} \cdot g)$ is not globally integrable since the isotropy groups of the action of $\mathscr{C}$ on $\mathscr{M}$ are not locally constant.

To intersect this splitting with the previous one, we first consider a decomposition of $T_g(\mathscr{C}\cdot g)$; see Figure 1(b).

2.3 LEMMA. *Let* $g \in \mathscr{M}^{s+r,p}$. *Then* $\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)}{}^{s,p}$ *is a* C^r *closed submanifold of* $\mathscr{M}^{s+r,p}$ *with tangent space* $\{h = (2/n)(\delta X)g + L_X g \mid X \in \mathscr{X}^{s+1,p}\}$ *at* g. *Thus we have the splitting* $T_g(\mathscr{C}\cdot g) = T_g(\mathscr{P}\cdot g) \oplus T_g(\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)})$.

Proof. Let $\tilde{\mu} : \mathscr{C}\cdot g \to \mathscr{V}^{s,p}$, $g_1 \mapsto \mu(g_1)$. For $h = fg_1 + L_X g_1 \in T_{g_1}(\mathscr{C}\cdot g)$, $D\tilde{\mu}(g_1)\cdot h = \frac{1}{2}\operatorname{tr} h\, \mu(g) = ((n/2)f - \delta X)\mu(g)$. This derivative is surjective, since for $\rho \in W^{s,p}$, a solution of $(n/2)f - \delta X = \rho$ is $X = 0, f = (2/n)\rho$. Thus $\tilde{\mu}^{-1}(\mu(g)) = \mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)}{}^{s,p}$ is a submanifold with the stated tangent space.

Remarks. (1) Since $\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)}{}^{s,p}$ is the intersection of $\mathscr{C}\cdot g$ with a $\mu =$ constant coordinate in $\mathscr{M}$, it is diffeomorphic to a submanifold of $\mathscr{W}^{s,p}$. This submanifold is, naturally, the orbit of $g \otimes \mu(g)^{-2/n}$ under $\mathscr{D}^{s+1,p}$.

(2) The summand $T_g(\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)})$ represents the infinitesimal deformations that preserve both the conformal class of g and its volume element. Note that intersecting the splitting $S_2 = T_g(\mathscr{P}\cdot g) \oplus T_g\mathscr{N}_{\mu(g)}$ with $T_g(\mathscr{C}\cdot g)$ gives the splitting for $T(\mathscr{C}\cdot g)$ in (2.4). There is also a corresponding coordinate grid for this splitting:

$$\mathscr{C}\cdot g \approx \mathscr{C}\cdot g/\mathscr{P} \times \{\mu \in \mathscr{V} \mid \mu = \mu(g), g \in \mathscr{C}\cdot g\},$$

which represents a finite version of this splitting.

Using the splitting in (2.3) and $T_g\mathscr{M} = T_g\tilde{S}_g \oplus T\,\mathscr{C}\cdot g$, York's decomposition

$$S^2 = S_2{}^{TT} \oplus S_2{}^C \oplus (\text{Range } \tau_g \cap S_2{}^T)$$

can be written in terms of tangent spaces as

$$(2.5)\quad T_g\mathscr{M} = T_g\tilde{S}_g \oplus T_g(\mathscr{P}\cdot g) \oplus T_g(\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)}),$$

$\tilde{S}_g = S_g \cap \mathscr{N}_{\mu(g)}$ where the first summand is orthogonal to the orbit through g by $\mathscr{C}$ (or orthogonal to the orbit through g by $\mathscr{D}$ and preserves the volume element), the second summand preserves the pointwise conformal class of g, and third summand preserves both the conformal equivalence class and the volume element, and these three deformations are mutually orthogonal (see Figure 1(c)).

Further Remarks. (1) For applications to general relativity, the interesting component is the one which changes the conformal equivalency class, namely $T_g\tilde{S}_g$. The other two summands preserve this class and therefore for relativitists are of lesser interest (see O'Murchadha and York [**22**], York [**25**; **26**] and Fischer-Marsden [**14**] for further details).

(2) The factors $T_g\tilde{S}_g \oplus T_g(\mathscr{C}\cdot g \cap \mathscr{N}_{\mu(g)})$ in 2.5 are naturally isomorphic to the splitting of $T_{g\otimes\mu_g{}^{-2/n}}\mathscr{W}$ along a slice and the orbit for the action of $\mathscr{D}$ on $\mathscr{W}$ and $T_g(\mathscr{P}\cdot g)$ is along the orthogonal $\mathscr{W}$-constant manifolds in the coordinate grid $\mathscr{M} \approx \mathscr{W} \times \mathscr{V}$.

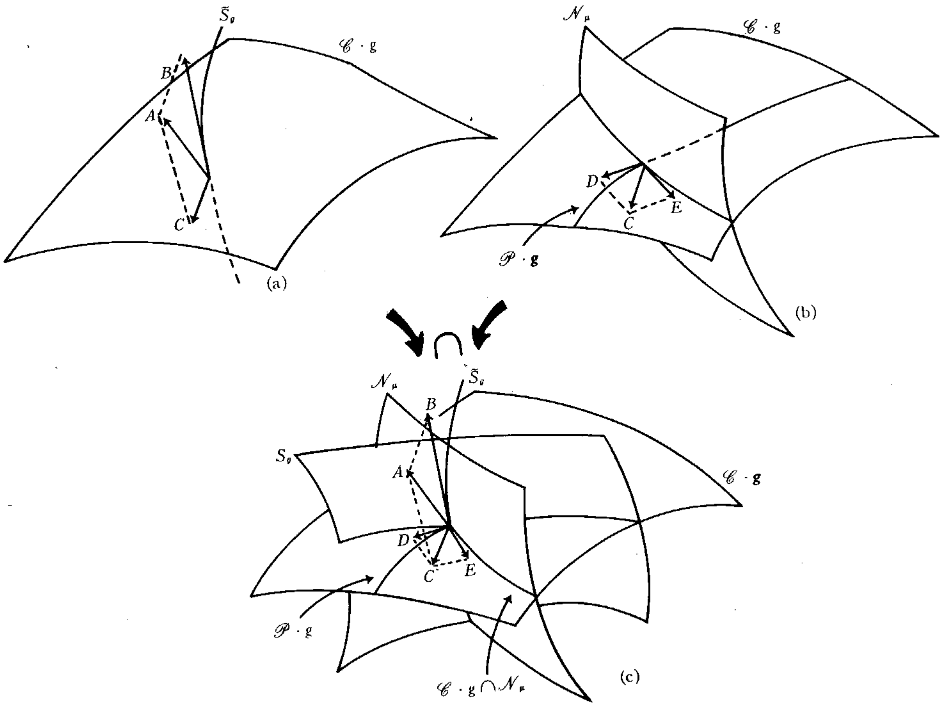

FIGURE 1

A schematic representation of the decomposition 2.5.

(a) $A = B + C$; $A \in T_g\mathscr{M}$, $B \in T_g\tilde{S}_g$, $C \in T_g(\mathscr{C} \cdot g)$; the splitting $S_2 = \text{Ker } \tau_g^* \oplus$ Range τ_g.

(b) $C = D + E$; $D \in T_g(\mathscr{P} \cdot g)$, $E \in T_g((\mathscr{C} \cdot g) \cap \mathscr{N}_\mu)$; the splitting in 2.3.

(c) $A = B + D + E \in T_g\tilde{S}_g \oplus T_g(\mathscr{P} \cdot g) \oplus T_g((\mathscr{C} \cdot g) \cap \mathscr{N}_\mu)$; the splitting in 2.5 where $\tilde{S}_g = S_g \cap \mathscr{N}_\mu$.

(3) The above splitting has the feature that there are no curvature restrictions on g. The analogous Barbance-Deser-Berger-Ebin splitting requires $\rho =$ constant. Setting $\mathscr{M}_\rho = \{g | R(g) = \rho\}$, we have

$$T_g\mathscr{M} = \text{Range } \gamma_g^* \oplus T_g(\mathscr{M}_\rho \cap S_g) \oplus T_g\mathscr{O}_g$$

where $\gamma_g^*(f) = g\Delta f + \text{Hess } f - f \cdot \text{Ric }(g)$. If g is Einstein, $T_g(\mathscr{M}_\rho \cap S_g)$ consists of transverse, trace-constant tensors so $\mathscr{M}_\rho \cap S_g$ is analogous to $\tilde{S}_g$. See Fischer-Marsden [**13**] for details.

3. The action of $\mathscr{C}$ on $T^*\mathscr{M}$. Let $T^*\mathscr{M}^{s,p}$ be the L_2-cotangent bundle of $\mathscr{M}^{s,p}$; i.e. $T_g^*\mathscr{M}^{s,p} = \{\pi = \pi' \otimes \mu(g) | \pi'$ is a 2-contravariant symmetric tensor of class $W^{s,p}$; i.e. $(\pi') \in (S^2)^{s,p}\}$. We have a natural L_2 pairing between $T_g^*\mathcal{M}^{s,p}$ and $T_g\mathcal{M}^{s,p}$ given by

$$\langle \pi, h \rangle = \int_M \pi' \cdot h \, d\mu(g).$$

As usual, $T^*\mathcal{M}$ carries a canonical (weak) symplectic structure (see Chernoff-Marsden [7]). The action of $\mathscr{C}$ on $\mathcal{M}$ lifts naturally to a symplectic action of $\mathscr{C}$ on $T^*\mathcal{M}$ given by

$$(\eta, p) : (g, \pi) \mapsto (p\eta^* g, p^{-1}\eta^*\pi)$$

where $\eta^*\pi = \eta^*\pi' \otimes \eta^*\mu(g)$ and where $\eta^*\pi'$ is the pullback of contravariant tensors.

Indeed, by the general formula for the lifted action (see Chernoff-Marsden [7]), if G acts on M by mappings $\Phi_g : M \to M$, the lifted action on T^*M is

$$\Phi_g^*\alpha \in T_{\Phi_g(x)}{}^*M \quad \text{if } \alpha \in T_x^*M$$
$$\Phi_g^*(\alpha) \cdot v = \langle \alpha, (T_x\Phi_g)^{-1} \cdot v \rangle.$$

In our case, π regarded as a linear form on $T_g^*\mathcal{M}$ is transformed to

$$\begin{aligned} \pi_{(\varphi,p)} \cdot h &= \langle \pi, (\varphi^{-1})^* p^{-1} h \rangle \\ &= \int \pi \cdot (\varphi^{-1})^* p^{-1} h \\ &= \int (\varphi^*\pi) \cdot p^{-1} h \quad \text{(change of variables)} \\ &= \int (p^{-1}\varphi^*\pi) \cdot h \end{aligned}$$

so π is transformed, as a tensor density, to $p^{-1}\varphi^*\pi$.

Note that this action preserves the L_2 metric on $T^*\mathcal{M}$, as does the action of $\mathscr{D}$.

If G acts on a manifold M and hence on T^*M, the *moment*, or *conserved quantity* for the action is a map ψ

$$\psi : T^*M \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G, given by

$$\psi(\alpha) \cdot \xi = \langle \alpha, \xi_M(n) \rangle, \quad \alpha \in T_x^*M, \xi \in \mathfrak{g}.$$

Here ξ_M denotes the infinitesimal generator of the action (see, for example, Chernoff-Marsden [7]). We now work out ψ for the action of $\mathscr{C}$ on $\mathcal{M}$.

3.1 LEMMA. *The moment for the action of $\mathscr{C}$ on $\mathcal{M}$ is given by:*

$$\psi : T^*\mathcal{M} \to \mathscr{X}^* \oplus (C^\infty)^* = \Lambda_d^1 \oplus C_d^\infty$$
$$\psi(g, \pi) = (2(\delta\pi)^\flat, \operatorname{tr} \pi)$$

where $\mathscr{X}^* = \Lambda_d{}^1$ *are the* 1*-form densities, and* $C_d{}^\infty$ *the scalar densities (i.e.* Λ^n*, the space of n-forms), and* $(\delta\pi)^b$ *is the* 1*-form density associated with the vector density* $\delta\pi$*.*

Proof. The infinitesimal generator of the action of $\mathscr{C}$ on $\mathscr{M}$ is given by

$$\xi_{\mathscr{M}}(g) = L_X g + fg \in T_g\mathscr{M},$$

where $\xi = (X, f) \in \mathscr{X} \oplus C^\infty$. Therefore,

$$\begin{aligned}\psi(g, \pi) \cdot \xi &= \langle \pi, \xi_{\mathscr{M}}(g)\rangle \\ &= \int \pi \cdot \xi_{\mathscr{M}}(g) \\ &= \int \pi \cdot L_X g + \int \pi \cdot (fg) \\ &= 2\int X \cdot \delta\pi + \int f \operatorname{tr} \pi.\end{aligned}$$

Thus the corresponding L_2 dual object is $\psi(g, \pi) = (2(\delta\pi)^b, \operatorname{tr} \pi)$.

Note that $\psi : T^*\mathscr{M}^{s,p} \to (\mathscr{X}^{s-1,p})^* \oplus (W^{s,p})^*$.

If a group G acts symplectically on a symplectic manifold P and $\psi : P \to \mathfrak{g}^*$ is a moment, then to obtain a new symplectic manifold in which the symmetries have been divided out, we form the reduced phase space:

$$P_\mu = \psi^{-1}(\mu)/G_\mu.$$

Here $\mu \in \mathfrak{g}^*$ and $G_\mu \subset G$ is the isotropy subgroup for the co-adjoint action of G on $\mathfrak{g}^*$; (see Marsden-Weinstein [**20**]).

In our case, $P = T^*\mathscr{M}$, $G = \mathscr{C}$, $\mu = 0$ (so $G_\mu = G$) and ψ is given by (3.1). Since

$$\psi^{-1}(0) = \mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}}$$

where $\mathscr{C}_\delta = \{(g, \pi) | \delta\pi = 0\}$ and $\mathscr{C}_{\mathrm{tr}} = \{(g, \pi) | \operatorname{tr} \pi = 0\}$, we obtain:

3.2 Theorem. *The reduce phase space for the action of* $\mathscr{C}$ *on* $T^*\mathscr{M}$ *is*

$$\mathscr{E} = \mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}}/\mathscr{C}.$$

Thus $\mathscr{E}$ is the space of (g, π) with π transverse-traceless and with (g_1, π_1) identified with (g_2, π_2) if there is a conformal transformation (φ, p) such that

$$(p\varphi^* g_1, p^{-1}\varphi^*\pi_1) = (g_2, \pi_2).$$

Although it follows automatically from the general theory that $\mathscr{C}$ leaves $\mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}}$ invariant, it is instructive to see it directly. Indeed, if

$$(\bar{g}, \bar{\pi}) = (p\varphi^* g, p^{-1}\varphi^*\pi)$$

then in general,

$$\operatorname{tr} \bar{\pi} = \varphi^*(\operatorname{tr} \pi)$$

and

$$\delta_{\bar{g}}\bar{\pi} = p^{-1}\varphi^*(\delta_g\pi) + (1/2p^2)\varphi^*(\operatorname{tr} \pi) \cdot \operatorname{grad} p.$$

Thus $\mathscr{C}_\delta \cap \mathscr{C}_{\operatorname{tr}}$ is mapped into itself by the action of $\mathscr{C}$.

Next we examine when $\mathscr{E}$ is a manifold. This is done in a series of steps.

3.3 PROPOSITION. *$\mathscr{C}_{\operatorname{tr}}{}^{s,p} \subset T^*\mathscr{M}^{s,p}$ is a smooth submanifold with tangent space*

$$T_{(g,\pi)}\mathscr{C}_{\operatorname{tr}}{}^{s,p} = \{(h, \omega) \in T_{(g,\pi)}{}^*\mathscr{M}^{s,p} \approx S_2{}^{s,p} \times (S_d{}^2)^{s,p} | h \cdot \pi + \operatorname{tr} \omega = 0\}$$

where $(S_d{}^2)^{s,p} = S_{s,p}{}^2 \otimes \mu(g)$.

Proof. Consider the map $\operatorname{tr} : T^*\mathscr{M}^{s,p} \to \Lambda_n{}^{s,p} = (W^{s,p})^*$, the n-forms of class $W^{s,p}$, given by $(g, \pi) \mapsto \operatorname{tr} \pi$, the g-trace of π. This is a smooth map with derivative

$$D \operatorname{tr} (g, \pi)(h, \omega) = h \cdot \pi + \operatorname{tr} \omega.$$

This is clearly surjective; to solve $h \cdot \pi + \operatorname{tr} \omega = \mu$, let $h = 0$, $\omega = g^\# \otimes \mu/n$ where $(g^\#)^{ij} = g^{ij}$, i.e. $^\#$ raises the indices. Since the kernel can be directly seen to split, we have a submersion and the result. (The splitting lemma 1.2, can also be used here).

Likewise, $\hat{\mathscr{C}}_{\operatorname{tr}} = \{(g, \pi) | \operatorname{tr} \pi = (\text{constant}) \cdot \mu(g)\}$ is a manifold, as is $\mathscr{C}_{\operatorname{tr}=\rho} = \{(g, \pi) | \operatorname{tr} \pi = \rho\, \mu(g)\}$ the manifold of (g, π) with prescribed trace.

We notice that $\mathscr{C}_{\operatorname{tr}} = \bigcup_{\mu \in \mathscr{V}} T^*\mathscr{N}_\mu$ so that $\mathscr{C}_{\operatorname{tr}}$ is the union of the cotangent bundles to the manifold partition $\{\mathscr{N}_\mu : \mu \in \mathscr{V}\}$ of $\mathscr{M}$.

We now examine the set $\mathscr{C}_\delta$.

3.4 PROPOSITION. *Let $(g, \pi) \in \mathscr{C}_\delta{}^{s,p}$ satisfy the following condition:*

$$C_\delta : \textit{if } X \in \mathscr{X}^{s,p} \textit{ is such that } L_X g = 0 \textit{ and } L_X \pi = 0, \textit{ then } X = 0.$$

Then $\mathscr{C}_\delta{}^{s,p} \subset T^*\mathscr{M}^{s,p}$ is a manifold in a neighborhood of (g, π) with tangent space

$$T_{(g,\pi)}\mathscr{C}_\delta{}^{s,p} = \{(h, \omega) \in T^*{}_{(g,\pi)}\mathscr{M}^{s,p} | \delta\omega + \tfrac{1}{2}\pi^{lm}h_{lm}{}^{|i} - \pi^{lm}h^i{}_{l|m}\} = \operatorname{Ker} D\delta(g, \pi)$$

Proof. Consider the C^∞ map $\delta^b : T^*\mathscr{M}^{s,p} \to (\mathscr{X}^*)^{s-p,1} = (\Lambda_d{}^1)^{s-1,p}$ (one-form densities) given by $\delta^b(g, \pi) = (\delta\pi)^b$, the g-divergence of π.

We can compute

$$\beta_{(g,\pi)} = D\,\delta^b(g, \pi) : S_2{}^{s,p} \times (S_d{}^2)^{s,p} \to (\Lambda_d{}^1)^{s-1,p}$$

to be

$$\beta_{(g,\pi)}(h, \omega) = (\delta\omega)^b + \tfrac{1}{2}\pi^{lm}h_{lm|i} - \pi^{lm}h_{il|m} - \pi^{lm}{}_{|m}h_{il}.$$

Its natural† L_2-adjoint $\beta_{(g,\pi)}{}^* : \mathscr{X}^{s,p} \to (S_d{}^2)^{s-1,p} \times S_2{}^{s-1,p}$ can be computed

†This refers to the fact that we are using the natural pairing between tensors and densities, so that the Hermetian structure of the bundles is not needed (cf. Palais [23]). This is purely a matter of style at this point.

from this as $\beta_{(g,\pi)}^*(X) = \frac{1}{2}(-(L_X\pi), (L_Xg))$ but the following seems simpler:

$$\int \langle \beta_{(g,\pi)} \cdot (h, \omega), X\rangle$$

$$= \frac{d}{d\lambda} \int \langle \delta^b(g + \lambda h, \pi + \lambda\omega), X\rangle|_{\lambda=0}$$

$$= \frac{1}{2}\frac{d}{d\lambda} \int \langle L_X(g + \lambda h), (\pi + \lambda\omega)\rangle|_{\lambda=0}$$

(since $2\delta_g\pi$ and L_Xg are adjoints)

$$= \frac{1}{2} \int L_Xh \cdot \pi + L_Xg \cdot \omega$$

$$= \frac{1}{2} \int -h \cdot L_X\pi + L_Xg \cdot \omega.$$

(That we can write $\int L_Xh \cdot \pi = \int - h \cdot L_X\pi$ follows from the expression of the Lie derivative in terms of flows and the change of variables formula.) Thus the above expression for $\beta_{(g,\pi)}^*$ follows.

From this expression for $\beta_{(g,\pi)}^*$ and our assumption on (g, π) we see that the kernel of $\beta_{(g,\pi)}^*$ is trivial. Also, $\beta_{(g,\pi)}^*$ has injective symbol, as is easily checked. Thus from the splitting lemma, δ^b is a submersion and the result follows.

Remarks. 1) The condition C_δ says that the pair (g, π) has no simultaneous Killing vector fields, i.e. the group

$$I_g \cap I_\pi = \{\eta \in \mathscr{V} | \eta^*g = g \quad \text{and} \quad \eta^*\pi = \pi\}$$

is discrete.

2) The space $\mathscr{C}_\delta$ also arises as one of the constraint spaces in the dynamics of general relativity; see Fischer-Marsden [**11**].

If we consider the intersection $\mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}}$, then the condition C_δ (on the absence of simultaneous Killing fields for (g, π)) changes to a stronger condition $C_{\delta,\mathrm{tr}}$ (on the absence of simultaneous conformal Killing fields).

Simultaneous conformal Killing fields (X, N) satisfy

$$L_Xg = -Ng, \quad L_X\pi = N\pi$$

and are elements of the Lie algebra of the Lie group

$$C_g \cap C_\pi = \{(\varphi, p) \in \mathscr{C} | p\varphi^*g = g, p^{-1}\varphi^*\pi = \pi\}$$

(that $C_g \cap C_\pi$ is a Lie group is proved the same way as for C_g, as was sketched in § 1).

3.5 THEOREM. *Let* $(g, \pi) \in \mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ *satisfy:*

$$C_{\delta,\mathrm{tr}} : \text{if } (X, N) \in \mathscr{X}^{s,p}\ W^{s+1,p} \text{ is such that } L_X g = -Ng \text{ and } L_X\pi = N\pi, \text{ then } X = 0,\ N = 0.$$

Then in a neighborhood of (g, π), $\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p} \subset T^*\mathscr{M}^{s,p}$ *is a smooth submanifold with tangent space*

$$T_{(g,\pi)}(\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}) = (T_{(g,\pi)}\mathscr{C}_\delta^{s,p}) \cap (T_{(g,\pi)}\mathscr{C}_{\mathrm{tr}}^{s,p})$$

given by 3.3 *and* 3.4.

Note. Condition $C_{\delta,\mathrm{tr}}$ implies $C_g \cap C_\pi$ is discrete (i.e. has trivial Lie algebra).

Proof. Let $\psi = (2\delta^b, \mathrm{tr}) : T^*\mathscr{M}^{s,p} \to (\mathscr{X}^{s-1,p})^* \oplus (W^{s,p})^*$ be given by $\psi(g, \pi) = (2(\delta\pi)^b, \mathrm{tr}\,\pi)$, as in (3.1). Thus

$$D\psi(g, \pi) \cdot (h, \omega) = (2D\delta^b(g, \pi) \cdot (h, \omega), D\,\mathrm{tr}(g, \pi) \cdot (h, \omega))$$

given by (3.3) and (3.4). The natural adjoint of $D\ (g, \pi)$ is

$$\begin{aligned} D\psi(g, \pi)^*(X, N) &= (2D\delta^b(g, \pi)^*(X) + D\,\mathrm{tr}(g, \pi)^*(N) \\ &= ((-L_X\pi), (L_Xg)) + (N\pi, N) = ((-L_X\pi + N\pi), (L_Xg + Ng)). \end{aligned}$$

Using the remarks following (1.2), $D\psi(g, \pi)^*$ has symbol

$$\sigma_\xi(Y, s) = (\pi^{ia}\xi_a Y^j + \pi^{ja}\xi_a Y^i + \pi^{ij}\xi_a Y^a + s\pi^{ij}, (Y_i\xi_j + \xi_i Y_j) + sg_{ij}).$$

If $\sigma_\xi(Y, s) = 0$ then $(Y_i\xi_j + \xi_iY_j) + sg_{ij} = 0$ in particular and so, as for τ_g (see the proof of (1.3)), $s = 0$ and $Y = 0$. Thus $D\psi(g, \pi)^*$ has injective symbol. To show $D\psi(g, \pi)^*$ is injective, note that

$$D\psi(g, \pi)^*(X, N) = 0$$

is equivalent to $L_Xg + Ng = 0$, and $L_X\pi - Ng = 0$. Thus $X = 0$, $N = 0$, by Condition $C_{\delta,\mathrm{tr}}$.

Thus, by (1.2), $D\psi(g, \pi)$ is surjective with splitting kernel, so ψ is a submersion and the result follows.

3.6 *Remarks*. 1) In the proof, $\mathrm{tr}\,\pi = 0$ was not used. Thus, for given ρ, the same proof shows $\mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}=\rho}$ is a manifold. If ρ is a non-zero constant then interestingly we can replace $C_{\delta,\mathrm{tr}}$ by C_δ. Indeed, taking the trace of $L_Xg + Ng = 0$ and $L_X\pi - N\pi = 0$ gives $-2\delta X + nN = 0$ and $X \cdot d\,\mathrm{tr}\,\pi' - \pi' \cdot L_Xg - (\delta X) \cdot (\mathrm{tr}\,\pi') - N\,\mathrm{tr}\,\pi' = 0$, or $X \cdot d\,\mathrm{tr}\,\pi' - \pi' \cdot (-Ng) - (n/2)N\,\mathrm{tr}\,\pi' - N\,\mathrm{tr}\,\pi' = X \cdot d\,\mathrm{tr}\,\pi' - (n/2)N\,\mathrm{tr}\,\pi' = 0$. Thus if $\mathrm{tr}\,\pi' =$ constant $\neq 0$, $N = 0$ and so from condition C_δ, $X = 0$.

2) In general, one can expect that if G acts on M and hence on T^*M, $\psi : T^*M \to \mathfrak{g}^*$ is the moment of the action, and if for $\alpha \in \psi^{-1}(0)$ the isotropy group of α is trivial or discrete, then $T\psi(\alpha)$ will be surjective. (In our case we had to check the ellipticity so that (1.2) could be used; in the finite dimensional case this would be automatic.)

We now examine the manifold structure of the quotient space $\mathscr{E} = \mathscr{C}_\delta \cap \mathscr{C}_{\mathrm{tr}}/\mathscr{C}$.

As with $\mathscr{M}^{s,p}$, note that the action of $\mathscr{C}^{s,p}$ on $T^*\mathscr{M}^{s,p}$ is a C^0 action, although translation by any $(\varphi, p) \in \mathscr{C}^{s,p}$ is a C^∞ map.

3.6 LEMMA. *If $(g, \pi) \in T^*\mathscr{M}^{s+r,p}$ then the orbit $\mathscr{C}^{s+1,p}(g, \pi)$ of (g, π) is a C^r submanifold with tangent space*

$$\begin{aligned} T_{(g,\pi)}(\mathscr{C}^{s+1,p} \cdot (g, \pi)) \\ = \{(L_X g + Ng, L_X\pi - N\pi) | X \in \mathscr{X}^{s+1,p}, N \in W^{s,p}\}. \end{aligned}$$

Also, $\mathscr{C}^{s+1,p} \cdot (g, \pi)$ projects to the orbit $\mathscr{C}^{s+1,p} \cdot g$ under the projection $T^*\mathscr{M}^{s,p} \to \mathscr{M}^{s,p}$.

Proof. Same as (1.5).

3.7 LEMMA. *Let $(g, \pi) \in \mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ satisfy condition $C_{\delta,\mathrm{tr}}$. Then there is a $\mathscr{C}^{s+1,p}$-invariant neighborhood U of $\mathscr{C}^{s+1,p} \cdot (g, \pi)$ such that $U \cap \mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ is a C^∞ submanifold of $T^*\mathscr{M}^{s,p}$.*

Proof. Since translation by each $(\varphi, p) \in \mathscr{C}^{s+1,p}$ is C^∞ and leaves $\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ invariant, the points at which $\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ is a C^∞ manifold can be merely translated.

3.8 LEMMA. *The action of $\mathscr{C}^{s+1,p}$ on $U \cap \mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}$ has a C^r slice $\tilde{S}_{(g,\pi)}$ at $(g, \pi) \in \mathscr{C}_\delta^{s+r,p} \cap \mathscr{C}_{\mathrm{tr}}^{s+r,p}$.*

Proof. Let $J : T_{(g,\pi)}{}^*\mathscr{M}^{s,p} \to T_{(g,\pi)}{}^*\mathscr{M}^{s,p}$ be given by $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, the usual symplectic form. Let

$$V = T_{(g,\pi)}(\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}) \cap J(T_{(g,\pi)}(\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p})).$$

A calculation like that in (3.5) shows that V is an L_2 orthogonal complement to $T_{(g,\pi)}(\mathscr{C}^{s+1,p} \cdot (g, \pi))$ in $T_{(g,\pi)}(\mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p})$. Our slice is then obtained by exponentiating V near (g, π) as in the proof of the slice theorem on $\mathscr{M}^{s,p}$ (see (1.6)).

The slice $\tilde{S}_{(g,\pi)}$ projects to the slice $\tilde{S}_g$ on $\mathscr{M}^{s,p}$ for $\mathscr{C}^{s+1,p}$ (see (1.6)) as is easily checked.

The appearance of the symplectic structure in this decomposition is not an accident. In fact York's decomposition and the Moncreif decomposition [**21**] (generalizing the Barbance-Deser-Berger-Ebin splitting) can be viewed as special cases of a general splitting for symplectic manifolds (see Arms-Fischer-Marsden [**2**]).

3.9 THEOREM. *Let $(g, \pi) \in \mathscr{C}_\delta^{s+r,p} \cap \mathscr{C}_{\mathrm{tr}}^{s+r,p}$, $r \geqq 1$ and suppose the isotropy group $C_g \cap C_\pi$ is trivial $= \{(\mathrm{id}, 1)\}$. Then in a neighborhood of $\mathscr{C}^{s+1,p} \cdot (g, \pi)$,*

$$\mathscr{E}^{s,p} = \mathscr{C}_\delta^{s,p} \cap \mathscr{C}_{\mathrm{tr}}^{s,p}/\mathscr{C}^{s,p}$$

is a C^∞ *manifold for which the canonical projection* $\pi : \mathscr{C}^{s,p} \cap \mathscr{C}_{\mathrm{tr}}{}^{s,p} \to \mathscr{E}^{s,p}$ *is a* C^r *submersion.*

Furthermore, $\mathscr{E}^{s,p}$ *carries naturally a (weak) symplectic (i.e. Poission bracket) structure.*

Proof. By (3.7) we can choose a neighborhood U such that $U \cap \mathscr{C}^{s,p} \cap \mathscr{C}_{\mathrm{tr}}{}^{s,p}$ is $\mathscr{C}^{s,p}$-invariant, so the quotient makes sense. Since the isotropy group of (g, π) is trivial, it is also trivial nearby since the action has a slice (3.8). This can be assumed to hold on U. Thus (on U), $\mathscr{E}^{s,p}$ is identifiable with $\tilde{S}_{(g,\pi)}$ and so has a C^∞ manifold structure. Using the fact that the orbit $\mathscr{C}^{s+1,p} \cdot (g, \pi)$ is C^r, we see that the projection onto the slice $\tilde{S}_{(g,\pi)}$ is C^r. The last result follows from general results about the symplectic structure on reduced phase spaces mentioned earlier (see Marsden-Weinstein [**20**]).

If (g, π) has discrete isotropy group, then $\mathscr{E}^{s,p}$ may be a manifold with identifications, namely $\tilde{S}_{(g,\pi)}/(C_g \cap C_\pi)$.

The kernel of $D\psi(g, \pi)^*$ is the Lie algebra of the isotropy group $C_g \cap C_\pi$. Thus, if we pass from a point (g, π) with trivial isotropy group to one with dimension $k > 0$, then we can expect $C_\delta \cap C_{\mathrm{tr}}$ to increase in dimension by k. The slice $\tilde{S}_{(g,\pi)}$ increases in dimension $2k$ (on a formal level, of course) since the space increases by dimension k and the orbit $\mathscr{C}^{s+1,p} \cdot (g, \pi)$ decreases by dimension k. Thus $\tilde{S}_{(g,\pi)}/C_g \cap C_\pi$ will increase in dimension by at least k. For these reasons, we expect $\mathscr{E}$ to contain genuine singularities when the conformal group of (g, π) undergoes a change in dimension.

The space $\mathscr{E}$ is one representation of the "space of gravitational degrees of freedom"; see [**14**; **22**].

In the non-compact case, where $M = \mathbf{R}^3$ and we deal with asymptotically flat metrics in $M_{s,\delta}{}^p$ spaces of Nirenberg-Walker-Cantor (cf. Cantor [**5**; **6**]), the situation in 3.9 is probably less pathological because the isotropy groups are always trivial. However, the analysis is complicated because there is no general splitting lemma and because it is necessary to include asymptotic conditions relevant for relativity (namely g_{ij} should be $\delta_{ij} + O(1/r)$ as $r \to \infty$). For a consideration of these results in the non-compact case, see Cantor [**6**] and for the study of the scalar curvature in the non-compact case, see Fischer-Marsden [**13**].

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