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SOME APPLICATIONS OF GEOMETRY IN CONTINUUM MECHANICS*

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Some contemporary ideas from differential geometry are applied to continuum mechanics. The Lie derivative is used to clarify the notion of "objective rates", an intrinsic treatment of Piola transformations is described, a simplified proof of Vainberg's theorem for potential operators is given by way of the Poincaré lemma on infinite dimensional manifolds, and a new derivation of the basic equations of continuum mechanics is presented which is valid in a general Riemannian manifold setting.

ntroduction

A few years ago we undertook the project of systematically examining the mathematical foundations of continuum mechanics, with the aim of applying differential geometry, global and nonlinear analysis in a unified fashion. Some of our results on the analytical end appear elsewhere ([2], [3]), and others on topics including bifurcation theory, finite element analysis and elasticity as a Hamiltonian system will appear in [1] and in other references in preparation.

Our aim here is to discuss some of the applications of purely geometrical ideas to continuum mechanics. Although geometry is used to some extent in several recent references, there remains much room for systematic application of geometrical concepts. We give some of these in this note.

1. Notation

Everything is assumed C^∞ for convenience. Let N be a Riemannian manifold with metric denoted by g or $\langle\ ,\ \rangle$. Let $M \subset N$ be a submanifold which, for simplicity, we take to be open, and let the metric tensor G on M be that induced from N.

A *configuration* of M is a (smooth) map

$$\varphi\colon M \to N$$

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and a *motion* of M in N is a curve φ_t of configurations. We shall assume here that φ_t is regular; i.e., $\varphi_t: M \to \varphi_t(M)$ is a diffeomorphism.

The *material velocity* of a motion is the map which, at each t, is given by

$$V_t = \frac{\partial \varphi_t}{\partial t}.$$

Explicitly, if $X \in M$, then

$$V_t(X) = \frac{d}{dt}\varphi_t(X) \in T_{\varphi_t(X)}N,$$

the tangent space to N at $\varphi_t(X)$. We shall often write $\varphi_t(X) = \varphi(t, X) = x$. Thus V_t is a vector field covering the map φ_t:

$$\begin{array}{ccc} M & \xrightarrow{V_t} & TN \\ & {\scriptstyle\varphi_t}\searrow \quad \swarrow{\scriptstyle \text{projection}} & \\ & N & \end{array}$$

The vector field $v_t = V_t \circ \varphi_t^{-1}$ is called the *spatial velocity.*

The material and spatial accelerations A_t, $a_t = A_t \circ \varphi_t^{-1}$ are defined analogously. An easy calculation (as in [4]) shows that

$$a_t = \frac{\partial v_t}{\partial t} + \nabla_v v = \dot{v},$$

the material time derivative.

The tangent of φ, $F = T\varphi$: $TM \to TN$, is called the *deformation gradient*, and may be regarded as a *two point tensor* over φ. (Two point tensors are sections of a bundle $T^r_s(M) \otimes \varphi^* T^{r'}_{s'}(N)$ where $T^r_s(M)$ is the bundle of tensors over M of type $\binom{r}{s}$ and φ^* denotes the pull-back bundle; these objects may be covariantly differentiated, etc. The reader can relate the formulas obtained to those in the continuum mechanics textbooks, as [5]; see [1] for details.)

Let $C = \varphi_t^*(g)$, the metric g pulled back to M, classically known as the (*Green*) *deformation tensor*. Other classical deformation tensors can be obtained in an analogous invariant manner.

The *rate of deformation* tensor d is given by

$$d = \frac{1}{2} L_v g = \frac{1}{2} \varphi_* \left(\frac{d}{dt} C \right)$$

where L_v denotes Lie differentiation (including, in general, a $\dfrac{\partial}{\partial t}$ term, since tensorial quantities below may depend explicitly on time) and φ_* is the push-forward.

Here is a simple example of the insight geometry provides: let $N = R^n$ and let K be the curvature tensor formed by pretending C is the metric. Then $K = 0$. (These are called the *compatibility conditions.*)

The proof does *not* require computation; it follows from these two facts: the curvature of R^n is zero, and the curvature of a pulled-back metric is the pull-back of the curvature.

2. Objective rates; an application of Lie derivatives

The subjects of objectivity and objective rates have been controversial in mechanics. Here we consider them as an application of Lie derivatives. Let $\boldsymbol{v}$, as above, be a given spatial velocity field and let $\boldsymbol{t}$ be a given symmetric contravariant two tensor (the stress tensor, say). Let $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\boldsymbol{t}_3$ be the three associated tensors with indices raised by the metric g and let $\boldsymbol{t}_4 = \boldsymbol{t}$. In coordinates $\{x^a\}$ on N,

$$\boldsymbol{t}_1 = (t^{ab}), \quad \boldsymbol{t}_2 = (t_a{}^b), \quad \boldsymbol{t}_3 = (t^a{}_b), \quad \boldsymbol{t}_4 = (t_{ab}).$$

Noting that the Lie derivative does not commute with raising or lowering indices, we get four different formulas:

$$\begin{aligned}
(L_v \boldsymbol{t}_1)^{ab} &= \dot{t}^{ab} - t^{cb} v^a{}_{|c} - t^{ac} v^b{}_{|c}, \\
g^{ac}(L_v \boldsymbol{t}_2)_c{}^b &= \dot{t}^{ab} - t^{ad} v^b{}_{|d} + t^{ab} v_{d|}{}^a, \\
(L_v \boldsymbol{t}_3)^a{}_c g^{cb} &= \dot{t}^{ab} - t^{ab} v^a{}_{|d} + t^{ad} v_{d|}{}^b, \\
g^{ac}(L_v \boldsymbol{t}_4)_{cd} g^{db} &= \dot{t}^{ab} + t^{cb} v_{c|}{}^a + t^{ac} v_c{}^{|b},
\end{aligned}$$

and for the density $\boldsymbol{t}_1 \otimes dv$, where dv is the volume element for the metric g,

$$(L_v(\boldsymbol{t}_1 \otimes dv))^{ab} = ((L_v \boldsymbol{t}_1)^{ab} + t^{ab} \operatorname{div} \boldsymbol{v}) dv.$$

We shall show shortly that it is not an accident that the so-called "objective fluxes" (the right-hand sides above) turn out to be Lie derivatives with respect to the velocity.

The tensor $L_v \boldsymbol{t}_1$ has been associated with the name Oldroyd and $L_v(\boldsymbol{t}_1 \otimes dv)$ with the name Truesdell. We see that all of these tensors are different manifestations of the Lie derivative of $\boldsymbol{t}$. Therefore, preference of one over the other is mostly a matter of taste and convenience; see, however, the proviso below. (Workers in continuum mechanics are quite insistent on their tastes!)

Any linear combination of the preceding formulas will also qualify as an "objective flux", e.g.,

$$\tfrac{1}{2}\left((L_v \boldsymbol{t}_3)^a{}_c g^{cb} + g^{ac}(L_v \boldsymbol{t}_2)_c{}^b\right) = \dot{t}^{ab} + t^{ad}\omega_d{}^b - t^{db}\omega^a{}_d,$$

where $\omega^a{}_b$ are associated components of the spin $2\omega_{ab} = v_{a|b} - v_{b|a}$; this tensor is associated with the name Jaumann.

We note in passing that, like the Lie derivative in general, the right-hand sides may be expressed *without* using covariant derivatives. For example,

$$(L_v \boldsymbol{t}_1)^{ab} = \frac{\partial t^{ab}}{\partial t} + \frac{\partial t^{ab}}{\partial x^c} v^c - t^{cb}\frac{\partial v^a}{\partial x^c} - t^{ac}\frac{\partial v^b}{\partial x^c}$$

and if $t^{ab} = t^{ba}$, then

$$(L_v \boldsymbol{t}_1)^{ab} = \frac{\partial t^{ab}}{\partial t} + \frac{\partial t^{ab}}{\partial x^c} v^c - \text{twice symmetric part of } \left(t^{ac}\frac{\partial v^b}{\partial x^c}\right).$$

One seems never to see this in practice, but it could yield savings in numerical computations.

We conclude this section with a general discussion of what we mean by objective.

DEFINITION. Let t be a tensor field (or tensor density) on a manifold N and ψ a diffeomorphism of N to N'. We say that the push-forward $t' = \psi_* t$ is the *objective transformation* of t, i.e., t transforms in the usual way under the map ψ.

PROPOSITION 1. *Let φ_t be a regular motion of M in N with velocity field v_t (spatial velocity). Let ψ_t be a motion of N in N' and let $\varphi'_t = \psi_t \circ \varphi_t$ be the superposed motion of M in N'.*

Let t be a given time-dependent tensor field on N and let

$$t' = \psi_* t$$

i.e., transform t objectively.

Let v' be the velocity field of φ'_t. Then

$$L_{v'} t' = \psi_*(L_v t),$$

i.e., "objective tensors (or tensor densities) have objective Lie derivatives".

Proof: We first note that

$$v'_t = w_t + \psi_{t*} v_t$$

where w_t is the spatial velocity of ψ_t. This follows by differentiating $\varphi'_t(X) = \psi_t(\varphi_t(X))$ in t. (As can be seen v is *not* objective.)

Now we compute, writing $L_v = \dfrac{\partial}{\partial t} + \mathscr{L}_v$, and letting $\psi_{r,s} = \psi_r \circ \psi_s^{-1}$ be the time dependent flow of w:

$$
\begin{aligned}
L_{v'} t' &= L_{w+\psi_* v}(\psi_* t) = \mathscr{L}_{w+\psi_* v}(\psi_* t) + \frac{\partial}{\partial t}(\psi_* t) \\
&= \psi_*(\mathscr{L}_v t) + \mathscr{L}_w(\psi_* t) + \frac{\partial}{\partial t}(\psi_* t) = \psi_*(\mathscr{L}_v t) + L_w(\psi_* t) \\
&= \psi_*(\mathscr{L}_v t) + \frac{d}{dr}\psi^*_{r,t}(\psi_{r*} t_r)\Big|_{r=t} = \psi_*(\mathscr{L}_v t) + \frac{d}{dr}(\psi_r \circ \psi_t^{-1})^*(\psi_{r*} t_r)\Big|_{r=t} \\
&= \psi_*(\mathscr{L}_v t) + \frac{d}{dr}\psi_{t*} t_r\Big|_{r=t} = \psi_*\left(\mathscr{L}_v t + \frac{d}{dr} t_r\Big|_{r=t}\right) = \psi_*(L_v t). \quad \square
\end{aligned}
$$

As a corollary, the "objective fluxes" discussed earlier are objective tensors with this proviso: if the expressions involving g_{ab} or g^{ab} explicitly are to transform like tensors with the *same* g_{ab} resulting after the transformation, ψ must be an isometry at the point of interest. (If we also transform the metric tensor, this proviso is unnecessary.)

3. The Piola transform: an application of differential forms

The Piola transformation is a fundamental operation relating the material and spatial descriptions in continuum mechanics. Here we show how it may be given an intrinsic and simple treatment using operations on differential forms.

We begin by defining the Piola transform of vector fields.

DEFINITION. Let y be a vector field on N and $\varphi: M \to N$ an orientation preserving diffeomorphism. The *Piola transform* of y is the vector field Y on M given by

$$Y = J\varphi^* y$$

where J is the Jacobian of φ. (In coordinates, $\{x^a\}$ on N and $\{X^A\}$ on M,

$$Y^A = J(F^{-1})^A{}_b y^b$$

where $J(t, X) = \dfrac{\sqrt{\det g}}{\sqrt{\det G}} \dfrac{\partial(\varphi^1, \ldots, \varphi^n)}{\partial(X^1, \ldots, X^n)}$ and $F^a{}_A = \dfrac{\partial \varphi^a}{\partial X^A}$.)

We can phrase this in another useful way:

PROPOSITION 2. *Y is the Piola transform of y if and only if*

$$\varphi^*(i_y dv) = i_Y dV,$$

where $i_y dv$ is the interior product, dv is the volume element on N and dV that on M.

Proof: Notice that $n-1$ forms and vector fields are in one-to-one correspondence via $Y \mapsto i_Y dV = *Y$. But

$$\varphi^*(i_y dv) = i_{\varphi^* y}\varphi^* dv = i_{\varphi^* y} J dV = i_{J\varphi^* y} dV$$

so the assertion follows. □

PROPOSITION 3 (Piola Identity). *If Y is the Piola transform of y, then*

$$\mathrm{DIV}\, Y = J(\mathrm{div}\, y) \circ \varphi,$$

where DIV *is the divergence on M and* div *is the divergence on N.*

Proof: Let $U \subset M$ be an open set with smooth boundary ∂U. By the change of variables theorem, and the above proposition,

$$\int_{\partial U} i_Y dV = \int_{\partial \varphi_t(U)} i_y dv.$$

By Gauss' theorem, noting $i_Y dV = \langle Y, N \rangle dA$ (N is the unit out ard normal and dA is the area element on ∂U),

$$\int_U \mathrm{DIV}\, Y dV = \int_{\varphi_t(U)} \mathrm{div}\, y dv = \int_U J(\mathrm{div}\, y) \circ \varphi dV.$$

Since U is arbitrary, the assertion follows. □

Another way of expressing the Piola transformation is as follows:

$$\langle Y, N \rangle dA = \langle y, n \rangle da.$$

Since $Y^A = J(F^{-1})^A{}_a y^a$, the Piola identity can be written this way: $\mathrm{DIV}(JF^{-1}) = 0$, where $\mathrm{DIV}(JF^{-1})_a \equiv \dfrac{1}{\sqrt{\det G}} \dfrac{\partial}{\partial X^A}(\sqrt{\det G}\, J(F^{-1})^A{}_a)$.

We can also make a Piola transformation on any index of a tensor. For example, let t^{ab} be a given two tensor, the Cauchy stress tensor. If we make a Piola transformation on the last index, we get the Piola–Kirchhoff stress tensor: $T^{aB} = J(F^{-1})^B{}_b t^{ab}$, a two-point tensor. The Piola identity, then, tells us that

$$\mathrm{DIV}\, T = J(\mathrm{div}\, t) \circ \varphi,$$

i.e.,

$$T^{aB}{}_{|B} = J t^{ab}{}_{|b}.$$

4. Vainberg's theorem: an application of the Poincaré lemma

In [6], Vainberg has given a well-known criterion for when equations are derivable from a variational principle, i.e., are Euler–Lagrange equations. We give here a simple proof of the abstract form of this result using the Poincaré lemma for differential forms.

Let E be a Banach space, $\langle\ ,\ \rangle$ be a continuous bilinear form on E and let $A: E \to E$ be a given (nonlinear) operator. The Fréchet derivative of A at x is denoted $DA(x)$.

DEFINITION. We say A is a *potential operator* if there is a function $L: E \to R$ such that

$$dL(x) \cdot v = \langle A(x), v \rangle$$

for all x in E and $v \in E$.

The equation $A(x) = 0$ represents the Euler–Lagrange equations for $x \in E$, in abstract form.

PROPOSITION 4. *A given operator A is a potential operator if and only if for each $x \in E$, v_1 and $v_2 \in E$,*

$$\langle DA(x) \cdot v_1, v_2 \rangle = \langle DA(x) \cdot v_2, v_1 \rangle.$$

If $\langle\ ,\ \rangle$ is symmetric, this is equivalent to saying $DA(x)$ is a symmetric linear operator on E.

Proof: Consider the one form $\alpha(x) \cdot v = \langle A(x), v \rangle$. Then A is a potential operator if and only if α is exact. By the Poincaré lemma, this is the case if and only if $d\alpha = 0$. But by the coordinate formula for exterior derivative (the "curl" in this case),

$$d\alpha(x) \cdot (v_1, v_2) = \langle DA(x) \cdot v_1, v_2 \rangle - \langle DA(x) \cdot v_2, v_1 \rangle$$

so the result follows immediately. □

5. Balance of energy and the basic equations of continuum mechanics: an application of differential geometry

In this section we want to derive the equations of continuum mechanics in a Riemannian manifold setting, not for the sake of generality, but to reexamine the fundamental ideas involved in this process. In an R^3 setting the basic equations are often derived from scalar and vector integral balance hypotheses. This approach cannot be used in the manifold case since there is no way to give invariant meaning to the integration of vector fields. Another approach used in R^3 is to employ an energy balance as the basic postulate and assume the energy balance is form invariant under all time-dependent isometries of R^3. This approach also cannot be directly extended since a general manifold may not admit any isometries. However, manifolds do admit lots of "local" isometries and by way of these we may generalize the energy balance argument to manifolds. Since, to the best of our knowledge, the universe is not R^3 and does not admit isometries, our new derivation represents to us a significant philosophical improvement.

We shall need the following standard results. First, the *transport theorem*: Let $f(t, x)$ be a given real-valued function of t and $x \in \varphi_t(M)$, and let U be a (nice) open set in M. Then

$$\frac{d}{dt}\int_{\varphi_t(U)} f dv = \int_{\varphi_t(U)} (\dot{f}+f\operatorname{div}\boldsymbol{v})dv.$$

Of course, this follows easily on manifolds by changing variables.

We shall secondly need the following (for some technical points), whose proof is found in basic texts, *Cauchy's theorem*. Let $a(t, x)$, $b(t, x)$ and $c(t, x, \boldsymbol{n})$ be scalar functions defined for $t \in R$, $x \in \varphi_t(M)$ and unit vectors $\boldsymbol{n}$ at x. Assume that a, b, c satisfy the master balance law in the sense that for any nice open set $U \subset M$, we have

$$\frac{d}{dt}\int_{\varphi_t(U)} a dv = \int_{\varphi_t(U)} b dv + \int_{\partial\varphi_t(U)} c(t, x, \boldsymbol{n}) da,$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\varphi_t(U)$. Then there exists a unique vector field $\boldsymbol{c}(t, x)$ on $\varphi_t(M)$ such that

$$c(t, x, \boldsymbol{n}) = \langle \boldsymbol{c}(t, x), \boldsymbol{n}\rangle.$$

Let $\varrho(t, x)$, $\boldsymbol{b}(t, x)$, $h(t, x, \boldsymbol{n})$, $\hat{\boldsymbol{t}}(t, x, \boldsymbol{n})$, $e(t, x)$ and $r(t, p)$ be given functions on N; h and $\hat{\boldsymbol{t}}$ depend on a unit vector $\boldsymbol{n}$.

These functions are said to satisfy the *balance of energy* principle if, for all (nice) $U \subset M$,

$$\frac{d}{dt}\int_{\varphi_t(U)} \varrho(e+\tfrac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle)dv = \int_{\varphi_t(U)} \varrho(\langle \boldsymbol{b}, \boldsymbol{v}\rangle + r)dv + \int_{\partial\varphi_t(U)} (\langle \hat{\boldsymbol{t}}, \boldsymbol{v}\rangle + h)da$$

where $\hat{\boldsymbol{t}}$ and h are evaluated on the unit outward normal $\boldsymbol{n}$ of $\partial\varphi_t(U)$.

The next definition localizes this idea:

DEFINITION. The *balance of energy principle holds at* $x \in N$ *at time* t if for every sequence of nice open sets U_n converging to $X = \varphi_t^{-1}(x)$ (e.g.: diameter $U_n \to 0$) we have

$$\operatorname*{limit}_{n\to\infty}\left\{\frac{1}{\text{volume } \varphi_t(U_n)}\frac{d}{dt}\int_{\varphi_t(U_n)}\varrho(e+\tfrac{1}{2}\langle v,v\rangle)dv\right\}$$

$$=\operatorname*{limit}_{n\to\infty}\left\{\frac{1}{\text{volume } \varphi_t(U_n)}\left(\int_{\varphi_t(U_n)}\varrho(\langle b,v\rangle+r)dv+\int_{\partial\varphi_t(U_n)}(\langle\hat{t},v\rangle+h)da\right)\right\}.$$

DEFINITION. Let $\psi_t: N\to N$ be a regular motion and define *primed* quantities (depending on ψ_t) by:

$$x' = \psi_t(x),$$
$$\varrho'(t,x') = \varrho(t,x),\quad r'(t,x') = r(t,x),$$
$$n'(x') = T\psi_t(x)n,$$
$$h'(t,x',n') = h(t,x,n),\quad \hat{t}'(t,x',n') = T\psi_t(x)\hat{t}(t,x,n),$$
$$e'(t,x') = e(t,x),$$
$$\varphi_t' = \psi_t\circ\varphi_t,$$
$$v_t' = \text{velocity of } \varphi_t',$$
$$a_t' = \text{acceleration of } \varphi_t',$$
$$\varrho'b'-\varrho'a' = T\psi_t(\varrho b-\varrho a).$$

We speak of ψ_t as a *superposed motion.*

DEFINITION. Let w_t denote the spatial velocity of ψ_t. We say ψ_t is an infinitesimal isometry at x at time t if $(L_{w_t}g)(x) = 0$, i.e., $w_{a|b}+w_{b|a} = 0$ at x at time t.

DEFINITION. We say that our original system $\varphi_t, b, \varrho, \ldots$ is *mechanically covariant* if it satisfies balance of energy and, for every superposed motion ψ_t which is an infinitesimal isometry at x, the primed system satisfies balance of energy (all at time t).

The original system is *fully covariant* if $\varphi_t, \varrho, \ldots$ satisfies balance of energy and for *every* superposed motion ψ_t, the primed system satisfies balance of energy.

PROPOSITION 5. *Assume $\varphi_t, \varrho, b, \ldots$ is mechanically covariant. Then*

(i) $h(t,x,n) = \langle -q(t,x), n\rangle$ *for some vector* q,

(ii) $\hat{t}(t,x,n) = \langle t(t,x), n\rangle$ *for a* $\binom{1}{1}$ *tensor* t.

Moreover,

(A) *Conservation of mass holds*: $\dot{\varrho}+\varrho\operatorname{div}v = 0$,

(B) *The equations of motion hold*:

$$\varrho a = \varrho b+\operatorname{div}t,$$

(C) *Local energy balance holds*:

$$\varrho\dot{e}+\operatorname{div}q = t:d+\varrho r$$

and

(D) $t_{ab} = t_{ba}$.

Conversely, if these conditions hold, then the system is mechanically covariant.

PROPOSITION 6. *Assume $\varphi_t, \varrho, b, \ldots$ is fully covariant. Then it is mechanically covariant and $t_{ab} = 0$! (The converse is true as well.)*

The proof of Propositions 5 and 6 requires some preparatory lemmas.

LEMMA 1. $v_t' = w_t + \psi_{t*} v_t$.

This follows by differentiating $\varphi_t'(X) = \psi_t(\varphi_t(X))$.

LEMMA 2. *Let $x \in N$ and t be fixed. Let u be a given vector at x. Then there exists a superposed motion ψ_t which is an infinitesimal isometry at x, $u = \psi_t^* v_t'(x)$ and at this t, ψ_t = identity.*

Proof: This merely amounts to showing that w_t can be freely specified at x at a value z while maintaining $w_{a|b} + w_{b|a} = 0$ at x; e.g., in normal coordinates at x, set $w^i(x) = z^i + A_j^i x^j$ where A_j^i is an arbitrary skew matrix. Extend w (arbitrarily) to have compact support and let $\psi_t = F_{t-t_0}$ where F_t is the flow of w. □

LEMMA 3. $h(t, x, n) = \langle -q(t, x), n \rangle$ *for a vector field* q.

Proof: Choose ψ_t in Lemma 2 so $v_t' = 0$ at x. Balance of energy in primed quantities ... x shows that $h'(t, x', n')$ satisfies the conditions of Cauchy's theorem. Thus $h'(t, x', n') = \langle -q'(t, x'), n' \rangle$ for some q'. Since we have $\psi_t = \text{id}$, $q' = q$. □

LEMMA 4. $\hat{t}(t, x, n) = t(t, x)n$ *for a* $\binom{1}{1}$ *tensor* t.

Proof: In view of Lemma 3, for any ψ_t which is an infinitesimal isometry at x, $\langle \hat{t}', v' \rangle$ satisfies the conditions of Cauchy's theorem at x, i.e., $\langle \hat{t}', v' \rangle = \langle t'v', n' \rangle$ for some tensor t', (Here we again use Lemma 2; since $\psi_t = \text{id}$, $t = t'$ at x.) □

Now we have (i) and (ii) and are ready to start the actual proof.

The transport theorem and localization as usual prove the following.

LEMMA 5. *Balance of energy at x is equivalent to the following at x:*

$$\varrho(\dot{e} + \langle v, a \rangle) + (\dot{\varrho} + \varrho \operatorname{div} v)(e + \tfrac{1}{2}\langle v, v \rangle) = \varrho r + \varrho \langle b, v \rangle - \operatorname{div} q + \operatorname{div}(tv). \qquad (1)$$

By hypothesis, this holds for the primed quantities if ψ_t is an infinitesimal isometry at x.

Write

$$\operatorname{div}(t'v') = (\operatorname{div} t')v' + t' : d' + t' : \omega'$$

where $\omega_{ab}' = \frac{1}{2}(v_{a|b}' - v_{b|a}')$. By Lemma 1 and $\psi_t = \text{id}$,

$$d_{ab}' = \tfrac{1}{2}(v_{a|b}' + v_{b|a}') = \tfrac{1}{2}(v_{a|b} + v_{b|a}) = d_{ab}.$$

Thus, (1) for the primed quantities reads:

$$0 = \varrho'(\dot{e}' - \dot{r}') + \operatorname{div} q' - t' : d' + \langle v', \varrho'(a' - b') - \operatorname{div} t' \rangle + \\ + (\dot{\varrho}' + \varrho' \operatorname{div} v')(e' + \tfrac{1}{2}\langle v', v' \rangle) + t' : \omega'. \tag{2}$$

Keep working at t, x, use $\psi_t = \mathrm{id}$ and (1) together with the observation $(e')^{\cdot} = \dot{e}$ and $v' = w + v$ to write (2) as

$$0 = \{\langle w, \varrho(a-b) - \operatorname{div} t + (\dot{\varrho} + \varrho \operatorname{div} v)\, v \rangle\} + \\ + \{\tfrac{1}{2}(\dot{\varrho} + \varrho \operatorname{div} v)\langle w, w \rangle\} + t : \omega, \tag{3}$$

where $\omega_{ab} = (w_{a|b} - w_{b|a})/2$. Since (Lemma 2), we can choose w and ω as an arbitrary vector and skew matrix at x, respectively, we get $t{:}\omega = 0$, so t is symmetric, $\dot{\varrho} + \varrho \operatorname{div} v = 0$ and $\varrho(a-b) - \operatorname{div} t = 0$. This proves Proposition 5.

For Proposition 6, we observe that if we use the same derivation for general ψ_t we have

$$d'_{ab} = d_{ab} + \tfrac{1}{2}(w_{a|b} + w_{b|a}) = d_{ab} + k_{ab}$$

and (3) has an extra term $t{:}k$. But we know Proposition 5 holds, and so $t{:}k = 0$. Thus since k is an arbitrary symmetric matrix at x, in the fully covariant case, t must vanish identically. □

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