LECTURE I

ATTEMPTS TO RELATE THE NAVIER-STOKES EQUATIONS TO TURBULENCE

Jerry Marsden

The present talk is designed as a survey, is slanted to my personal tastes, but I hope it is still representative. My intention is to keep the whole discussion pretty elementary by touching large numbers of topics and avoiding details as well as technical difficulties in any one of them. Subsequent talks will go deeper into some of the subjects we discuss today.

We start with the law of motion of an incompressible viscous fluid. This is given by the Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v - (v\cdot\nabla)v = -\nabla p + f \\ \operatorname{div} v = 0 \\ v = \begin{cases} 0 \text{ or} \\ \text{prescribed} \end{cases} \text{ on } \partial\Omega \end{cases}$$

where Ω is a region containing the fluid, v the velocity field of the fluid, p the pressure and f the external forces. ν represents here the kinematic viscosity, or, in the way we wrote our equations $1/Re$, where Re is the Reynolds number. The derivation of these equations can be found in any book on hydrodynamics, such as Landau and Lifschitz [1], K. O. Friedrichs and R. von Mises [1], and Hughes and Marsden [1]. We note here that the relevance of the incompressibility condition $\text{div } v = 0$ for turbulence is a matter for debate, but the general agreement today seems to be that compressible phenomena are not a necessary factor in turbulence; they start to be necessary only at very high speeds of the fluid.

Turbulence is the chaotic motion of a fluid. Our goal in this talk is to try to relate this universally accepted physical definition to the dynamics of the Navier-Stokes equations. There have been at least three attempts to explain the nature of turbulence, each attempt offering a model which will be briefly discussed below:

(a) The Leray picture (1934). Since the existence theorems for the solutions of the Navier-Stokes equations in three dimensions give only local semiflows (i.e., existence and uniqueness only for small intervals of time), this picture assumes that turbulence corresponds to a breakdown of the equations after a certain interval of time; in other words, one assumes that the time of existence of the solutions is <u>really</u> finite. Schaffer [1] looked at those t for which the equations break down and found that this set is of Hausdorff measure $\leq 1/2$. It is hard to imagine realistic physical situations for which the Navier-Stokes equations break down.

(b) <u>The E. Hopf-Landau-Lifschitz picture</u>. This is extensively discussed in Landau-Lifschitz [1] and consists of the idea that the solutions exist even for large t, but that they become quasi-periodic. Loosely speaking, this means that as time goes by, the solutions pick up more and more secondary oscillations so that their form becomes, eventually,

$$v(t) = f(\omega_1 t, \ldots, \omega_k t)$$

with the frequencies irrationally related. For k big, such a solution is supposed to be so complicated that it gives rise to chaotic movement of the fluid.

(c) <u>The Ruelle-Takens picture</u> (1971) assumes that the dynamics are inherently chaotic.

In the usual engineering point of view, the "nature" of turbulence is not speculated upon, but rather its statistical or random nature is merely assumed and studied.

Having this picture, a main goal would be to link up the statistics, entropy, correlation functions, etc., in the engineering side with a "nice" mathematical model of turbulence. More than that, such a model must be born out of the Navier-Stokes equations. Note that in this model we believe, but do not assume, that the solutions of the Navier-Stokes equations exist for large t and that the information on the chaoticness of the fluid motion is already in the flow. Needless to say, today we are very far away from this goal. This last picture is interesting and has some experimental support (J. P. Gollub, H. L. Swinney, R. Fenstermacher [1], [2]) which seems to contradict the Landau picture. There are "nice" mathematical models intrinsically chaotic strongly related to the Navier-Stokes equations. These are the Lorentz equations obtained as a truncation of the Navier-Stokes equations for the Benard problem and whose dynamics are chaotic.

The rest of the talk is devoted to a survey of the pros and cons of these models. All the details on these will be made by means of a series of remarks.

Remark 1. In two dimensions the Navier-Stokes equations and also the Euler equations (set $\nu=0$ in the Navier-Stokes equations, which corresponds to a non-viscous fluid) have global t-solutions. Hence, the Leray picture cannot happen in two dimensions! (Leray [1], Wolibner [1], Kato [1], Judovich [1]).

In three dimensions, the problem is open. There are no theorems and no counterexamples. However, there is some very inconclusive numerical evidence which indicates that

(a) for many turbulent or near turbulent flows, the Navier-Stokes equations do not break down.

(b) for the Euler equations with specific initial data on T^3 (the Taylor - Green vortex):

$$\begin{cases} v_1 = \cos x \sin y \sin z \\ v_2 = -\sin x \cos y \sin z \\ v_3 = 0 \end{cases}$$

the equations might break down after a finite time. Specifically, after $T \approx 3$, the algorithm used breaks down. This may be due to truncation errors or to the actual equations breaking down, quite probably the former. We only mention that this whole analysis requires the examination of convergence of the algorithms as well as their relation to the exact equations; see the numerical studies of Chorin [1,2], Orszag [1] and Herring, Orszag, Kraichnan and Fox [1], Chorin etal [1], and references therein.

Remark 2. The Landau picture predicts Gaussian statistics. This is not verified in practice. The model with chaotic dynamics does not predict such a statistic (see Ruelle [2], Gollub and Swinney [1]).

Remark 3. The Landau picture is unstable with respect to small perturbations of the equations. The Ruelle-Takens picture is, in some sense, a stabilization of the Hopf-Landau-Lifschitz picture. However, as Arnold has pointed out, strange attractors may form a small open set and still the quasi-periodic motions may be observed with higher probability.

Remark 4. Chaotic dynamics is not necessarily born from complicated equations. The Navier-Stokes equations are complicated enough to give rise to very complicated dynamics, eventually leading to a chaotic flow. The reason for this is that simple ordinary differential equations lead to chaotic dynamics (see below) and "any" bifurcation theorem for ordinary differential equations can work for Navier-Stokes equations, cf. Marsden-McCracken [1]. We do not want to go into the details here of this statement and we merely say that we look at the Navier-Stokes equations as giving rise to a vector field on a certain function space, we prove the local smoothness of the semi-flow and verify all conditions required for a bifurcation theorem; in this way we are able to discuss how a fixed point of this vector field splits into two other fixed points, or a closed orbit, and discuss via a certain algorithm their stability. Later talks with clarify and give exact statements of the theorems involved; we have in mind here the Hopf bifurcation theorem and its extension to semi-flows (see Marsden [2], Marsden and McCracken [1] and the appendix following).

Remark 5. As we mentioned earlier, the global t- existence theorem for the solutions of the Navier-Stokes equations is completely open in three dimensions. It is not necessary in the Ruelle-Takens picture of turbulence to assume this global t-existence. If one gets an attractor which is bounded, global t-solutions will follow.

Remark 6. There are other "simpler" partial differential equations where complex bifurcations have been classified:

(a) Chow, Hale, Malet-Paret [1] discuss the von Karmen equations. (This seems to be a highly nontrivial application of ideas of catastrophe theory.)

(b) P. Holmes [1] fits the bifurcation problem for a fluttering pipe into Taken's normal form.

Remark 7. There are at least two physically interacting real mathematical models with chaotic dynamics:

(a) Lorentz equations

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \quad . \qquad \text{(Note the symmetry } x \mapsto -x,\ y \mapsto -y,\ z \mapsto z.\text{)}$$

They represent a modal truncation of the Navier-Stokes equations in the Benard problem. It is customary to set $\sigma = 10$, $b = 8/3$; r is a parameter and represents the Rayleigh number. We shall come back to these equations in Remark 9.

(b) Rikitake dynamo. This model consists of two dynamos which are both viewed as generators, and as motors in interaction; it is a model for the Earth's magnetohydrodynamic dynamo. It has also chaotic dynamics. See Cook and Roberts [1]. The equations are:

$$\dot{x} = -\mu x + zy$$

$$\dot{y} = -\mu y - \alpha x + xz$$

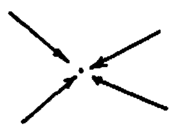
$$\dot{z} = 1 - xy$$

(c) A model of mixing salt with fresh water in the presence of temperature gradients. This was communicated to me personally by H. Huppert at Cambridge.

Remark 8. In many cases, existence of center manifolds of dimension k justify a modal or other truncation to give a k-dimensional system, i.e., all the complexity really takes place in a finite dimensional invariant manifold. (Exact statements will be given in one of the next talks.)

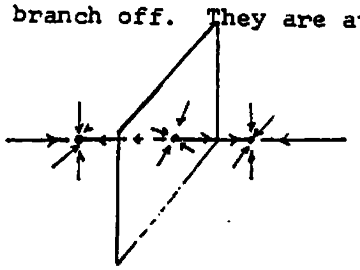
Remark 9. For the actual Navier-Stokes equations we do not know any solutions which are turbulent, or even that they exist. In any specific turbulent flow we don't know what the chaotic attractor might look like, or how one might form. However, we do know how this works (or think we do) for the Lorenz model. It is true that there are many objections to my drawing conclusions about the turbulence stemming from the Navier-Stokes equations by working with a truncation; it is argued that truncation throws turbulence away, too. However, I think that the model of Lorenz equations, though a truncation, can give some insight on what may happen in the much more complicated situation of the Navier-Stokes equations. I want to present here briefly the bifurcation for the Lorenz model when r (the Rayleigh number) varies. The picture presented below is due to J. Yorke, J. Guckenheimer, and O. Lanford. I am indebted to them and to N. Kopell for explaining the results. (See Kaplan and Yorke [1] and Guckenheimer's article in Marsden and McCracken [1] as well as William's lecture below.)

<u>$r < 1$</u> : Then the origin is a global sink:



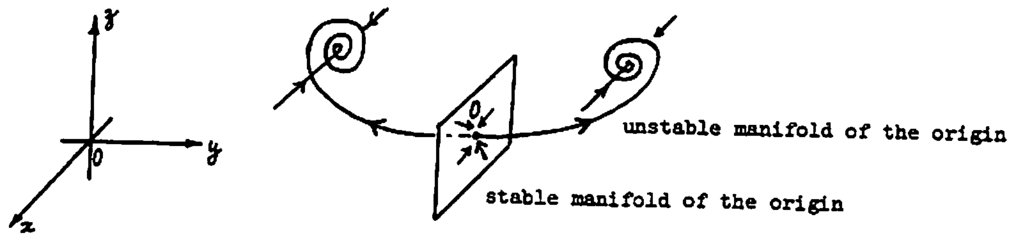
(all eigenvalues are real and negative for $1 > r > (4\sigma-(\sigma+1)^2/4\sigma$ i.e. $1 > r > -2.025$).

<u>$r = 1$ and $1+\varepsilon$</u> : At this value the first bifurcation occurs. One real eigenvalue for the linearization at zero crosses the imaginary axis travelling at nonzero speed on the real axis, for the origin a fixed point. Two stable fixed points branch off. They are at $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$.



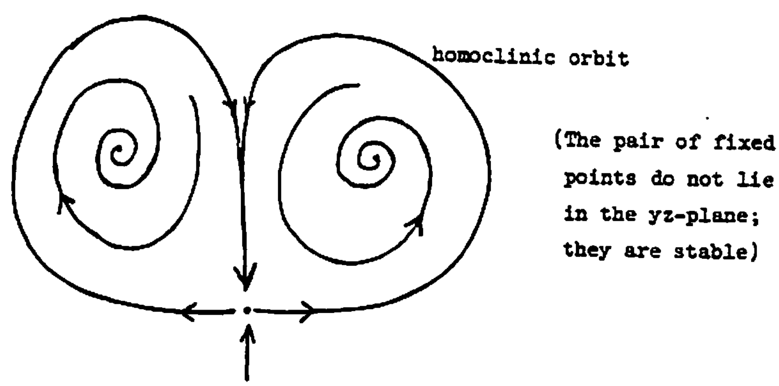
This is a standard and elementary bifurcation resulting in a loss of stability by the origin.

As r increases the two stable fixed points develop two complex conjugate and one negative real eigenvalues. The picture now looks like (z-axis is oriented upwards and the plane is the x 0 z plane):

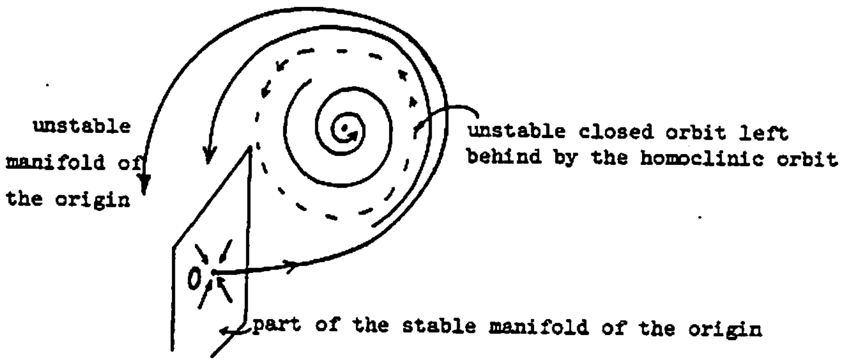


As r increases, the "snails" become more and more inflated.

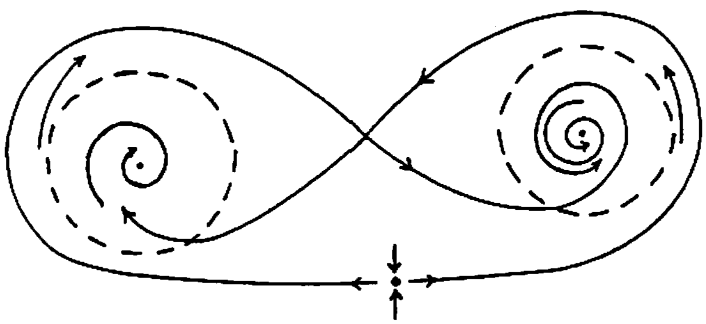
r ≅ 13.926: At around this value (found only by numerical methods) the "snails" are so big that they will enter the stable manifold of the origin. Stable and unstable manifold become identical; the origin is a homoclinic point. Another bifurcation now takes place. The picture is, looking in along the x-axis.



r > 13.926: The two orbits with infinite period "starting" and "ending" in the origin "cross over". The "snails" still inflate and by doing this, the homoclinic orbits leave behind unstable closed periodic orbits. The picture of the right hand side is:

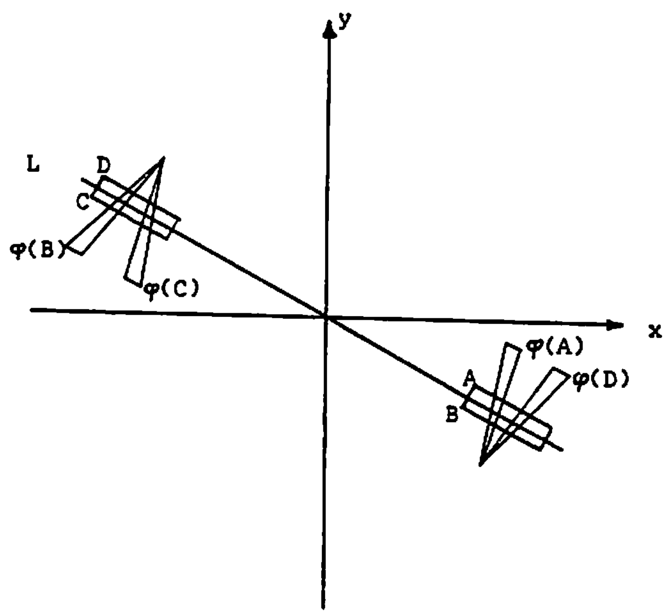


The unstable manifold of the origin gets attracted to the opposite fixed point for these values of r.



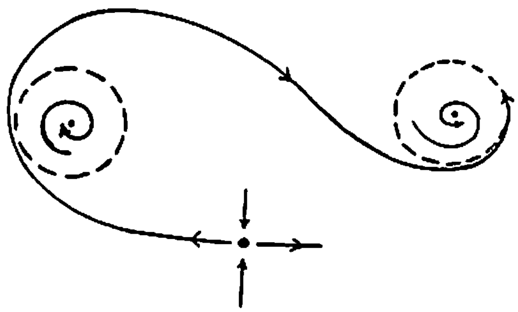
At this stage, which Yorke calls "preturbulent," there is a horseshoe strung out between the attracting fixed points. There are infinitely many periodic orbits, but eventually most orbits go to one of the attracting fixed points. There is no strange attractor, but rather a "meta-stable" invariant set; points near it eventually leave it in a sort of probabilistic way to one of the attracting fixed points.

To study this situation, one looks at the plane $z = r-1$ and the Poincaré, or once return map φ for the plane. On this plane one draws L, the stable manifold of the origin intersected with the plane.

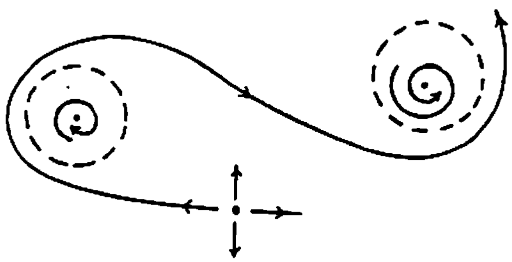


The images of the four regions A, B, C, D are shown. If one compares this picture with Smale's horseshoe example (Smale [1]) one sees that a horseshoe must be present. As r increases, eventually the images of the rectangles above will be inside themselves and an attractor will be born. This is the bifurcation to the Lorenz attractor. Viewing the dynamical system as a whole, we see the following (only one half is drawn for clarity).

$r = 24.06$:



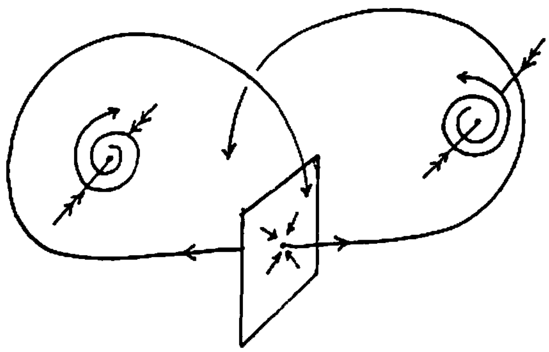
$r > 24.06$:



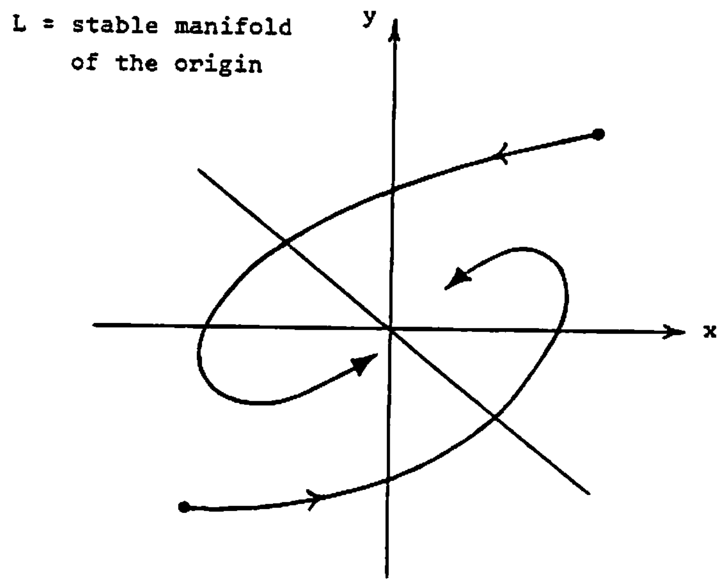
Now, between the two periodic orbits a "strange" attractor, called Lorenz attractor, is appearing. This attractor traps all the orbits that cross over the small piece of the stable manifold of the origin and throws them on the other side. Imagine we put a plane somewhere not far away from the origin, perpendicular to the drawn stable manifold and we would like to find out the points through which a specific orbit is going, travelling from one unstable closed orbit to another, and repelled by these each time; the result would be a random distribution of points in this "transveral cut" through the Lorenz attractor. For the nature of this attractor, see the talk of R. Williams in these notes, and the paper by J. Guckenheimer forming Section 12 of Marsden-McCracken [1]. We note that this attractor is nonstandard since it has two fixed points replaced by closed orbits in the "standard" Lorenz attractor. As r increases, this nonstandard Lorenz attractor grows from its initial shape and the unstable closed orbits shrink.

r ≈ 24.74 = $\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$: It is proved (Marsden and McCracken [1]) that a subcritical Hopf bifurcation occurs. The two closed "ghost" orbits shrink down to the fixed points which become in this way unstable.

r > 24.74: We now have a "standard" Lorenz attractor. The picture is:



r ≥ 50. The situation for larger r is somewhat complicated and not totally settled. According to some calculations of Lanford, the following seems to happen. If we look at the once return map φ on the plane $z = r-1$, as above, then the unstable manifold of the two symmetrical fixed points develop a fold. See the following figure. When this happens, stable large amplitude closed orbits seem to bifurcate off. This folding is probably because these two fixed points are becoming stronger repellers and tend to push away the other unstable manifold.



The situation is analogous to the bifurcations for the map $y = ax(1-x)$ which occurs in population dynamics.

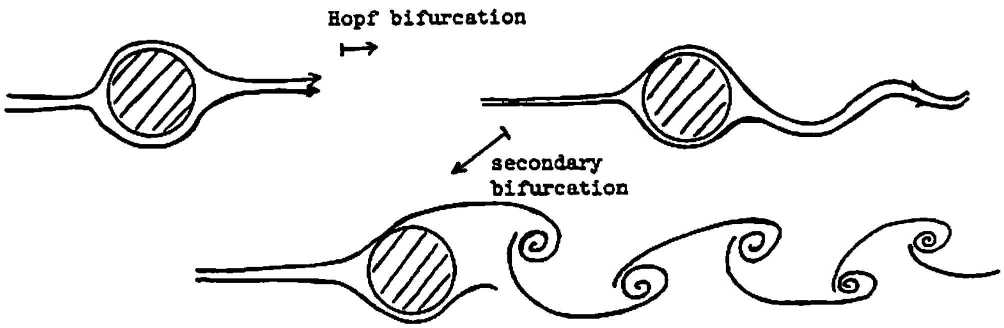
One can, of course vary the other parameters in the Lorenz model, or vary more than one. For example, Lorenz himself in recent numerical work has looked at bifurcations for small b (which is supposed to resemble large r).

<u>Research projects</u>: 1) Figure out the qualitative dynamics and bifurcation of the Rikitake two-disc dynamo.+

2) Real "pure" fluid models are needed; one might try getting a model for:

a) Couette Flow; see Coles [1] for many good remarks on this flow, and Stuart [1].

b) Flow behind a cylinder:



Here the symmetry will play a central role. Note that the third picture still represents a periodic solution in the space of divergence-free vector fields. My conjecture would be that the secondary Hopf bifurcation is illusory and what happens is that the original closed orbit produced by the Hopf bifurcation gets twisted somehow in the appropriate function space.

As A. Chorin has suggested, one should remember that the Lorenz model is global in some sense. The choas is associated

+ Some progress gas been made on this problem recently by P. Holmes and D. Rand.

with large scale motions. One would like a model with chaotic dynamics which is made up of a few interacting vortices and a mechanism for vortex production. "Real turbulence" seems to be more like this.

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APPENDIX TO LECTURE I: BIFURCATIONS, SEMIFLOWS, AND NAVIER-STOKES EQUATIONS

Tudor Ratiu

As was pointed out in J. Marsden's talk, the Ruelle-Takens picture for turbulence assumes that the motion of the fluid is inherently chaotic, that the flow obtained for $Re = 0$ (solutions of the Stokes equations) gets more and more complicated as the Reynolds number Re increases, due to bifurcation phenomena until it eventually gets trapped into a "strange" attractor which has chaoticness as one of its main features. In this talk I shall summarize the mathematical results involved in this machinery, trying to back up with exact statements of theorems many exciting ideas presented in Marsden's exposition. The main source of this talk is Marsden-McCracken [1].

The leading idea is to obtain a model born out of the Navier-Stokes equations for homogeneous, incompressible, viscous fluids:

$$\begin{cases} \frac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v = -\text{grad } p + f , \quad \nu = 1/Re \\ \text{div } v = 0 \\ v = \text{prescribed on } \partial M \text{, possibly depending on } \nu \end{cases}$$

Everything takes place in a compact Riemannian manifold M with smooth boundary ∂M, v representing the velocity field of the fluid, p the pressure and f the external force exercised on the moving fluid. As already mentioned, Euler's equations for an ideal fluid are obtained by setting $\nu = 0$ in the above equations; it is a theorem that the solutions to the Euler equations are obtained as a strong limit in the H^s-topology for $s > (\dim M)/2+1$ (see Ebin-Marsden [1]). Also notice that in Euler's equations we have to change the boundary conditions to $v \parallel \partial M$. The intuitive reason why this is so is that our fluid, being ideal, has no friction at all on the walls; however, a much more subtle mathematical analysis of the above described limit process yields formally the same result, cf. Marsden [2], Ebin-Marsden [1].

Now we would like to write our Euler and Navier-Stokes equations in the form of a system of evolution equations

$$\frac{dv}{dt} = X_\nu(v) \quad , \quad v(0) = \text{given} \quad ,$$

where X_ν is a densely defined nonlinear operator on a function space picked in such a way that our boundary conditions and $\operatorname{div} v = 0$ should be automatically satisfied. The answer to this question is given by the Hodge Decomposition Theorem.

Denote by $W^{s,p}$ the completion of the normed vector space of vector-valued C^∞-functions on M under the norm

$$\| f \|_{s,p} = \sum_{0 \le t \le s} \| D^t f \|_{L^p} ;$$

here $D^t f$ denotes the differential of f, $s \geq 0$ and $1 < p < \infty$. $W^{s,p}(M)$ is the set of vector fields of class $W^{s,p}$ on M. Note that a function is of class $W^{s,p}$ if and only if all its derivatives up to order s are in L^p.

Hodge Decomposition Theorem. Let M be a compact Riemannian manifold with boundary and $X \in W^{s,p}(M)$, $s \geq 0$, $1 < p < \infty$. Then X has a unique decomposition

$$X = Y + \text{grad } f$$

where $\text{div } Y = 0$, $Y \| \partial M$, $Y \in W^{s,p}(M)$ and f is of class $W^{s+1,p}$.

Denote $\tilde{W}^{s,p}(M) = \{X \in W^{s,p}(M) \mid \text{div } X = 0, X \| \partial M\}$. Apply now the Hodge Theorem and get a map $P: W^{s,p}(M) \to \tilde{W}^{s,p}(M)$ via $X \mapsto Y$. Let us now reformulate the Euler equations: suppose $s > n/p$; find $v: (a,b) \to \tilde{W}^{s+1,p}(M)$ such that

$$\frac{dv(t)}{dt} + P((v(t) \cdot \nabla) v(t)) = 0$$

(plus initial data). We need to assume $s > n/p$ in order to insure that the product of two elements of $W^{s,p}$ is in $W^{s,p}$ (see Adams [1], page 115). In this way, if $v \in \tilde{W}^{s+1,p}(M)$,

$(v\cdot\nabla)v \in W^{s,p}(M)$ and we can apply the Hodge Theorem. In doing this we tacitly assume that the external force is a gradient.

In order to be able to write in a similar way the Navier-Stokes equations, we change the function space to $\tilde{W}_0^{s,p} = \{X \in W^{s,p}(M) | \operatorname{div} X = 0, X|\partial M = 0\}$. Then the Navier-Stokes equations can be reformulated: find $v: (a,b) \to \tilde{W}_0^{s+1,p}$ such that

$$\frac{dv(t)}{dt} - \nu P(\Delta v(t)) + P((v(t)\cdot\nabla)v(t)) = 0 \quad .$$

The following theorem is proved in Section 9 of Marsden-McCracken.

<u>Theorem</u>. *The Navier-Stokes equations in dimensions* 2 *or* 3 *define a smooth local semiflow on* $\tilde{W}_0^{s,2}$ *, i.e., we have a collection of maps* $\{F_t^\nu\}$ *for* $t \geq 0$ *satisfying:*

(a) F_t^ν *is defined on an open subset of* $[0,\infty) \times \tilde{W}_0^{s,2}$ *;*

(b) $F_{t+s}^\nu = F_t^\nu \circ F_s^\nu$ *;*

(c) F_t^ν *is separately (hence, jointly)** *continuous;*

(d) for each fixed t,ν , F_t^ν *is a* C^∞*-map, i.e.,* $\{F_t^\nu\}$ *is a <u>smooth semigroup</u>. More, our semiflow* $\{F_t^\nu\}$ *satisfies the so called <u>continuation assumption</u>, namely, if* $F_t(x)$ *lies in a bounded set of* $\tilde{W}_0^{s,2}$ *for each fixed* x *and for all* t *for which* $F_t^\nu(x)$ *is defined, then* $F_t(x)$ *is defined for all* $t \geq 0$.

Also, $F_t^\nu(x)$ *is jointly smooth in* t,x,ν *for* $t>0$.

*See Chernoff-Marsden [1], Chapter 3, or Marsden-McCracken [1], Section 8A, for the proof of the fact that separate continuity ⇒ joint continuity.

This result which goes back to Ladyzhenskaya [1] encourages us to not work with the Navier-Stokes equations under their classical form, but rather with the evolution equations in $\tilde{W}_0^{s,2}$ which they define and to analyze more closely their semiflow which has such pleasant properties.

Following the idea of chaotic dynamics, we may try to show that turbulence occurs after successive bifurcations of the solutions of the Navier-Stokes equations. Hence a first question is how much of the classical bifurcation theory can be obtained for semiflows. The work of Marsden shows that almost everything works, if one mimics the conditions on the semiflow from those, one usually has for vector fields. We shall summarize these results below.

Hence we have to cope with a system of evolution equations of the general form

$$\frac{dx}{dt} = X_\mu(x) \quad , \; x(0) = \text{given} \quad ,$$

where X_μ is a nonlinear densely defined operator on an appropriate Banach space E , usually -- as we already saw -- a function space and μ is a parameter. We assume that our system defines unique local solutions generating a semiflow F_t^μ for $t \geq 0$. The assumptions made on the semiflow are (a), (b), (c) and (d) above. We also ask for the continuation assumption described before. It may seem that we force our assumptions on the semiflow such as to suit our particular problem. In reality it is exactly the other way around: one usually has these conditions satisfied and checks them for the Navier-Stokes equations -- and this is hard work involving a serious mathematical machinery (see Section 9 of Marsden-McCracken). It is true that the continuation assumption might seem strong; but it merely says that we have at our disposal a "good" local existence theorem, so "good" as to insure the fact that an orbit fails to be defined only if it tends to infinity in a finite time. That makes sense physically, looking at expected solutions of the governing equations of the law of motion of a fluid (Navier-Stokes): a solution fails to exist only if it "blows up". Another remark is of mathematical character and concerns the generator X_μ ; this is not a smooth map from E to E , hence we cannot expect smoothness of $F_t^\nu(x)$ in t . The fact is that the trouble is actually only at $t = 0$, as can be seen from the theorem on the Navier-Stokes semiflow from before, and exactly the derivative at $t = 0$ gives the generator. The next group of assumptions regards the spectrum of the linearized semiflow relevant for the Hopf bifurcation.

<u>Spectrum Hypotheses</u>. Let $F_t^\mu(x)$ be jointly continuous in t,μ,x for $t > 0$ and μ in an interval around $0 \in \mathbb{R}$. Suppose in addition that:

(i) 0 is a fixed point of F_t^μ , i.e., $F_t^\mu(0) = 0$, $\forall \mu,t$;

(ii) for $\mu < 0$, the spectrum of $G_t^\mu = DF_t^\mu(0)$ is contained inside the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$;

(iii) for $\mu = 0$ (resp. $\mu < 0$) the spectrum of G_1^μ at the origin has two isolated simple eigenvalues $\lambda(\mu)$ and

$\overline{\lambda(\mu)}$ with $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$) and the rest of the spectrum is in D and remains bounded away from the unit circle;

(iv) $\left.\frac{d|\lambda(\mu)|}{dt}\right|_{\mu=0} > 0$, i.e., the eigenvalues move steadily across the unit circle.

Sometimes we look at these hypotheses but with (iii) changed to:

(iii') for $\mu = 0$ (resp. $\mu < 0$) the spectrum of G_1^μ at the origin has one isolated simple real eigenvalue $\lambda(\mu) = 1$ (resp. $\lambda(\mu) > 1$) and the rest of the spectrum is in D and remains bounded away from the unit circle;

(v) for $\mu = 0$ the origin is asymptotically stable.

We won't go into the technical details of this last hypothesis here and say only that it involves an algorithm of checking if a certain displacement function obtained via Poincaré map has strictly negative third derivative.

Bifurcation to Periodic Orbits: *Under the above hypotheses (i)-(v) there is a fixed neighborhood* V *of* 0 *in* E *and an* $\varepsilon > 0$ *such that* $F_t^\mu(x)$ *is defined for all* $t \geq 0$ *for* $\mu \in [-\varepsilon,\varepsilon]$ *and* $x \in V$ *. There is a one-parameter family of closed orbits for* F_t^μ *for* $\mu > 0$ *, one for each* $\mu > 0$ *varying continuously with* μ *. They are locally attracting and* *hence stable. Solutions near them are defined for all* $t \geq 0$. *There is a neighborhood* U *of the origin such that any closed orbit in* U *is one of the above orbits.*

Bifurcation to Fixed Points: *Same hypothesis with (iii) and (iii') interchanged. Then the same result holds, replacing the words "closed orbit" with "two fixed points".*

I shall not go into the proof of these theorems but will give the two crucial facts behind the formal proof. One is the Center Manifold Theorem and the other is a theorem of Chernoff-Marsden regarding smooth semiflows on finite-dimensional manifolds. Coupling these two results reduces the whole problem to the classical Hopf Bifurcation Theorem in 2 dimensions, which is relatively simple and goes back to Poincaré. Here are the statements:

Center Manifold Theorem for Semiflows: *Let* Z *be a Banach space admitting a* C^∞*-norm away from zero, and let* F_t *be a continuous semiflow defined in a neighborhood of zero for* $0 \leq t \leq z$. *Assume* $F_t(0) = 0$ *and that for* $t > 0$, $F_t(x)$ *is jointly* C^{k+1} *in* t *and* x . *Assume that the spectrum of the linear semigroup* $DF_t(0): Z \to Z$ *is of the form* $e^{t(\sigma_1 \cup \sigma_2)}$ *where* $e^{t\sigma_1}$ *lies on the unit circle (i.e.,* σ_1 *lies on the imaginary axis) and* $e^{t\sigma_2}$ *lies in the unit circle at non-zero distance from it for* $t > 0$ *(i.e.,* σ_2 *is in the left half* *plane). Let* Y *be the generalized eigenspace corresponding to the spectrum on the unit circle; assume* $\dim Y = d < +\infty$. *Then there exists a neighborhood of* 0 *in* Z *and a* C^k*-submanifold* $M \subseteq V$ *of dimension* d *passing through* 0 *and tangent to* Y *at* 0 *such that:*

(a) Local Invariance: if $x \in M$, $t > 0$ *and* $F_t(x) \in V$, *then* $F_t(x) \in M$ *;*

(b) Local Attractivity: if $t > 0$ *and* $F_t^n(x)$ *remains defined and in* V *for all* $n = 0,1,2,\ldots$, *then* $F_t^n(x) \to M$ *as* $n \to \infty$.

This is applied to F_t^μ after suspending μ to obtain the semi-flow $F_t(x,\mu) = (F_t^\mu(x),\mu)$ on the original space x the parameter space

The version of this theorem for a C^{k+1} map is well known; however, this statement regarding semiflows -- although believable -- wasn't present in the literature before; the first time it appears is in Section 2 of Marsden-McCracken. Note that everything works out nicely in the theorem, even though the generator X of the semiflow is unbounded.

<u>Theorem (Chernoff-Marsden)</u>: *Let* F_t *be a local semiflow on a Banach manifold* N *jointly continuous and* C^k *in* $x \in N$. *Suppose that* F_t *leaves invariant a finite dimensional submanifolf* $M \subseteq N$. *Then on* M , F_t *is locally reversible, is jointly* C^k *in* t *and* x *and is generated by a* C^{k-1} *vector field on* M.

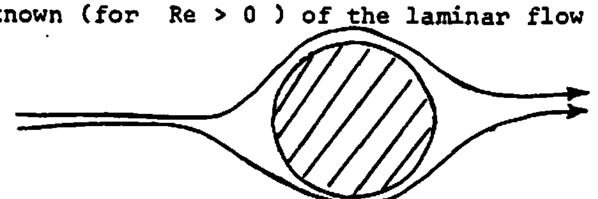
Some remarks are in order. Besides being one key factor in the proof of the bifurcation theorem, the center manifold theorem might justify some modal truncations of the Navier-Stokes equations to give a d-dimensional system (see Remark 8 of Lecture I by J. Marsden). Also, in Marsden-McCracken, Section 4A, an algorithm is described which enables us to check on the stability of the new born fixed points or closed orbits after bifuracations. Remark 4 of Lecture I hints toward that. The reduction to two dimensions appears as a corollary of the proof of the Bifurcation Theorem. The conclusion is that all the complexity in this case takes place only in a plane, even though we started off with an evolution equation on an infinite dimensional function space. This occurrence is characteristic when we work with semiflows; trying to prove a bifurcation, we reduce everything to a finite dimensional theorem for flows and this gives us then two things: the theorem itself and the reduction!

That's the way one approaches the next bifurcation to invariant tori. Here the Hopf Bifurcation Theorem for Diffeomorphisms will be needed and the idea of the proof is the same as before; one has to replace the argument of the Hopf Bifurcation Theorem in $\mathbb{R}^2$ with a similar argument using now the Hopf Bifurcation Theorem for Diffeomorphisms. I won't go into any technical details.

That would roughly solve the approach to the first two bifurcations. How about higher ones? The only leading idea is the Poincaré map, and the fact that something invariant for it, yields an invariant manifold of one higher dimension for the semiflow with the preservation of the attracting or repelling character: a fixed point -- attracting or repelling -- gave a closed orbit -- attracting or repelling -- a circle, an invariant torus, etc.

Let me mention that all these geometrical methods presented here are by no means the only ones with which one could attack bifurcation problems for the Navier-Stokes equations. An excellent reference is J. Sattinger [1], who in Chapters 4-7 does roughly the same thing, but using methods of eigenvalue problems, energy methods and Leray-Schauder degree theory. I prefer the above methods because I think they appeal more to one's geometrical intuition.

As a concluding remark, let me say that even if it seems that the first bifurcations can be attacked successfully with the above methods, the difficulties one faces might be very big. One has to start off with something known, namely a particular stationary solution, regard this as a fixed point of the generator of the semiflow and work his way through the conditions in the Bifurcation Theorem. In many cases we do not have even a stationary solution! In the research problem suggested in Lecture I about the flow behind a cylinder, the difficulty is exactly this one: there is no explicitly solution known (for $Re > 0$) of the laminar flow



in 2 or 3 dimensions, let alone of more complicated situations.

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