Well-posed Quasi-linear Second-order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity

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Contents

Existence, uniqueness and well-posedness for a general class of quasi-linear evolution equations on a short time interval are established. These results, generalizing those of [29], are applied to second-order quasi-linear hyperbolic systems on $\mathbb{R}^n$ whose solutions $(u(t), \dot{u}(t))$ lie in the Sobolev space $H^{s+1} \times H^s$. Our results improve existing theorems by lowering the required value of s to $s > (n/2) + 1$, or $s > n/2$ in case the coefficients of the highest order terms do not involve derivatives of the unknown, and by establishing continuous dependence on the initial data for these values. As consequences we obtain well-posedness of the equations of elastodynamics if $s > 2.5$ and of general relativity if $s > 1.5$; $s \geqq 3$ was the best known previous value for systems of the type occuring in general relativity ([12], [16], [23]).

1. Introduction

Let X and Y be Banach spaces, with Y densely and continuously included in X. Let $W \subset Y$ be open, let $T > 0$ and let G: $[0, T] \times W \to X$ be a given mapping. A nonlinear evolution equation has the form

$$\dot{u}(t) = G(t, u(t)), \qquad \dot{u} = \frac{du}{dt}. \tag{1.1}$$

If $s \in [0, T)$ and $\phi \in W$ are given, a solution curve (or integral curve) of G with value ϕ at s is a map $u(\cdot) \in C^0([s, T], W) \cap C^1([s, T], X)$ such that (1.1) holds on $[s, T]$ and $u(s) = \phi$.

If these solution curves exist and are unique for ϕ in an open set $U \subset W$, we can define evolution operators $F_{t,s}$: $U \to W$ that map $u(s)=\phi$ to $u(t)$. We say equation (1.1) is *well-posed* if $F_{t,s}$ is continuous (in the Y-topology on U and W) for each t, s satisfying $0 \leqq s \leqq t \leqq T$.

We remark that joint continuity of $F_{t,s}(\phi)$ in (t, s, ϕ) follows under general hypotheses [3]. Furthermore, if one has well-posedness for short time intervals, it is easy to obtain it for the maximally extended flow (see [3], [13] for general discussions on nonlinear evolution equations).

Well-posedness can be difficult to establish in specific examples, especially for "hyperbolic" ones. The continuity of $F_{t,s}$ from Y to Y cannot in general be replaced by stronger smoothness conditions such as Lipschitz or even Hölder continuity; a simple example showing this, namely $\dot{u}+uu_x=0$ in $Y=H^{s+1}$, $X=H^s$ on $\mathbb{R}$, is given in [28]; see [13] for a discussion of these smoothness questions.

The most thoroughly studied nonlinear evolution equations are those giving rise to nonlinear contraction semigroups generated by monotone operators [1]. These sometimes have evolution operators defined on all of X. This is not typical of hyperbolic problems, where $F_{t,s}$ may be defined only in Y, may be continuous from Y to Y, be differentiable from Y to X, and be Y-locally Lipschitz from X to X, without being X-locally Lipschitz from X to X or Y-locally Lipschitz from Y to Y, as is shown by the above example.

Section 2 gives general criteria for the well-posedness of quasi-linear evolution equations. The theorems generalize those of [29] and like them rely on recently obtained estimates for time-dependent linear evolution equations [26, 27].

These results are applied in Section 3 to quasi-linear second-order hyperbolic *systems* of the form

$$\text{(QH)} \qquad a_{00}\frac{\partial^2 \psi}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} (a_{0i}+a_{i0})\frac{\partial^2 \psi}{\partial t\, \partial x_i} + b,$$

where the unknown $\psi=(\psi_1, \ldots, \psi_N)$ is an N-vector valued function of $t \in [0, T]$ and of $x=(x_1, \ldots, x_n) \in \mathbb{R}^n$, where $\{a_{ij} | i, j=1, \ldots, n\}$ is a collection of $(N \times N)$-matrix valued functions of the suppressed arguments $t, x, \psi, \frac{\partial \psi}{\partial t}, \nabla \psi$, and where b is an N-vector valued function of these same arguments. Here $\nabla \psi$ denotes the collection of first order derivatives of ψ with respect to x.

We make various hypotheses on the functions a_{ij} and b which are spelled out in Section 3. The equations are shown to be well-posed in $H^{s+1} \times H^s = Y$ (with $X=H^s \times H^{s-1}$) if $s>(n/2)+1$, or $s>n/2$ if a_{ij} does not depend on derivatives of ψ. In particular, the solution $\psi(t)$ is in H^{s+1} if $\psi(0) \in H^{s+1}$ and $\dot{\psi}(0) \in H^s$.

Sections 4 and 5 show how these results apply to elastodynamics and general relativity, respectively. Comparisons with theorems in the literature are given.

Most nonlinear hyperbolic systems do not allow smooth solutions for all time because of the presence of shocks. Exceptions are certain semi-linear equations (see, e.g., [3], [39]). At present there are no theorems ensuring the existence of unique global weak or strong solutions for an interesting class of equations other than those depending on one spatial variable (see [36] and remarks in Section 3). In studying these problems, it is desirable to have solutions with s as low as possible.

In the future we hope to consider equations (QH) with boundary conditions suitable for application to elastodynamics. This generalization may require X and Y to be manifolds, but the general results of Section 2 should still be applicable.

2. Quasi-linear Equations of Evolution

In this section we consider the abstract Cauchy problem

(Q) $$du/dt + A(t,u)\,u = f(t,u), \qquad 0 \leqq t \leqq T, \qquad u(0) = \phi,$$

where the unknown u takes values in a Banach space and $A(t,u)$ is a linear (in general unbounded) operator depending on t and u.

Equations of the form (Q) are considered in [29]. The results of [29] are useful in applications to a number of differential equations in mathematical physics, but are not strong enough for certain applications (elastodynamics, for example). It is the purpose of this section to generalize these results in several directions. Since we do not want to repeat the same proofs, we shall refer to [29] and use the same notation as far as possible. Also we shall refer freely to [26, 27] for the theory of linear evolution equations.

In [29] two Banach spaces $Y \subset X$ were used, with Y required to contain $u(t)$ and with X required to contain $\dot{u}(t)$ and be such that $-A(t,u)$ generates a quasi-contractive semigroup on X for each $u \in Y$. For each $\phi \in Y$, solutions of (Q) were sought for small t.

In the present generalization, we shall split these roles of X and assign them to two spaces, X and Z, so that $\dot{u}(t) \in X$ while $-A(t,u)$ generates a C_0-semigroup on Z. Moreover, we shall allow this semigroup to be quasi-contractive with respect to an equivalent norm $N(t,u)$ on Z depending on t and u. This dependence on u is assumed to be smooth in the X-norm. Furthermore, we introduce another Banach space Z' to describe the dependence of $A(t,u)$ on t and u. The case considered in [29] corresponds to the case when $X = Z = Z'$, and $N(t,u)$ is constant.

2.1. Assumptions

We start from four real Banach spaces

(2.1) $$Y \subset X \subset Z' \subset Z,$$

with all the spaces reflexive and separable and the inclusions continuous and dense. We assume that

(Z') Z' is an interpolation space between Y and Z (thus if $U \in B(Y) \cap B(Z)$, then $U \in B(Z')$ with $\|U\|_{Z'} \leqq c \max\{\|U\|_Y, \|U\|_Z\}$; $B(Y)$ denotes bounded operators on Y.)

Let $N(Z)$ be the set of all norms in Z equivalent to the given one $\|\ \|_Z$. $N(Z)$ is metrized by the distance function

(2.2) $$d(\|\ \|_\mu, \|\ \|_\nu) = \log \max\{\sup_{0 \neq z \in Z} \|z\|_\mu/\|z\|_\nu,\ \sup_{0 \neq z \in Z} \|z\|_\nu/\|z\|_\mu\}.$$

We now introduce four functions, A, N, S and f on $[0, T] \times W$, where $T > 0$ and W is an open set in Y, with the following properties:

For all $t, t', \ldots \in [0, T]$ for all $w, w', \ldots \in W$, there is a real number β and there are positive numbers $\lambda_N, \mu_N, \ldots$ such that the following conditions hold:

(N) $N(t, w) \in N(Z)$, with

$$d(N(t,w), \| \;\|_Z) \leqq \lambda_N,$$

$$d(N(t', w'), N(t, w)) \leqq \mu_N(|t'-t| + \|w'-w\|_X).$$

(S) $S(t, w)$ is an isomorphism of Y onto Z, with

$$\|S(t,w)\|_{Y,Z} \leqq \lambda_S, \qquad \|S(t,w)^{-1}\|_{Z,Y} \leqq \lambda'_S,$$

$$\|S(t', w') - S(t, w)\|_{Y,Z} \leqq \mu_S(|t'-t| + \|w'-w\|_X).$$

(A1) $A(t, w) \in G(Z_{N(t,w)}, 1, \beta)$, where $Z_{N(t,w)}$ denotes the Banach space Z with norm $N(t, w)$. (This means that $-A(t, w)$ is a C_0-generator in Z such that $\|e^{-\tau A(t,w)} z\| \leqq e^{\beta\tau} \|z\|$ for all $\tau \geqq 0$ and $z \in Z$.)

(A2) $S(t, w)\, A(t, w)\, S(t, w)^{-1} = A(t, w) + B(t, w)$, where

$$B(t, w) \in B(Z), \qquad \|B(t, w)\|_Z \leqq \lambda_B.$$

(A3) $A(t, w) \in B(Y, X)$, with $\|A(t, w)\|_{Y,X} \leqq \lambda_A$ and

$$\|A(t, w') - A(t, w)\|_{Y,Z'} \leqq \mu_A \|w'-w\|_{Z'}$$

and with $t \mapsto A(t, w) \in B(Y, Z)$ continuous in norm.

(A4) There is an element $y_0 \in W$ such that

$$A(t, w)\, y_0 \in Y, \qquad \|A(t, w)\, y_0\|_Y \leqq \lambda_0.$$

(f1) $f(t, w) \in Y, \qquad \|f(t, w)\|_Y \leqq \lambda_f, \qquad \|f(t, w') - f(t, w)\|_{Z'} \leqq \mu_f \|w'-w\|_{Z'},$

and $t \mapsto f(t, w) \in Z$ is continuous.

Remarks. *1.* If $N(t, w) = \text{const} = \| \;\|_Z$, condition (N) is redundant. If $S(t, w) = \text{const} = S$, condition (S) is trivial. If both are assumed, and $X = Z' = Z$, we have the case of [29].

2. In most applications we can choose $Z' = Z$ and/or $Z' = X$.

2.2 Statement of theorems

Theorem I. *Let* (Z'), (N), (S), (Al) *to* (A4), *and* (f1) *be satisfied. Then there are positive constants ρ' and $T' \leqq T$ such that if $\phi \in Y$ with $\|\phi - y_0\|_Y \leqq \rho'$, then* (Q) *has a unique solution u on $[0, T']$ with*

$$u \in C^0([0, T']; W) \cap C^1([0, T']; X). \tag{2.3}$$

Here ρ' depends only on λ_N, λ_S, λ'_S, and $R = \operatorname{dist}(y_0, Y \setminus W)$, while T' may depend on all the constants β, λ_N, μ_N, … and R. When ϕ varies in Y subject to $\|\phi - y_0\|_Y \leqq \rho'$, the map $\phi \mapsto u(t)$ is Lipschitz continuous in the Z'-norm, uniformly in $t \in [0, T']$.

Remarks. *1.* ρ' may be chosen as any positive number not exceeding κR, where $\kappa = e^{-2\lambda_N}/2\lambda_S \lambda'_S$ (see (2.16) below).

2. In most applications there is a dense set of y_0 in W satisfying (A4). In such a case, given any $\phi \in W$ we can apply Theorem I by choosing an appropriate y_0 such that $\|\phi - y_0\|_Y \leqq \rho'$. To find y_0, let $R_\phi = \operatorname{dist}(\phi, Y \setminus W)$ and choose a y_0 with $\|y_0 - \phi\|_Y \leqq \kappa(1+\kappa)^{-1} R_\phi$. Then

$$R = \operatorname{dist}(y_0, Y \setminus W) \geqq (1+\kappa)^{-1} R_\phi,$$

so that we may choose $\rho' = \kappa(1+\kappa)^{-1} R_\phi$ by Remark 1 above. This gives $\|\phi - y_0\|_Y \leqq \rho'$ as required.

To establish well-posedness as defined in Section 1, we have to strengthen some of the assumptions. We assume for simplicity that $S(t, w)$ does not depend on w, and we introduce the following conditions:

(A5) $\quad \|B(t, w') - B(t, w)\|_Z \leqq \mu_B \|w' - w\|_Y$.

(f2) $\quad \|f(t, w') - f(t, w)\|_Y \leqq \mu'_f \|w' - w\|_Y$.

Theorem II. *Let (Z'), (N), (S), (A1) to (A5), (f1) and (f2) be satisfied, where $S(t, w)$ is assumed to be independent of w. Then there is a positive constant $T'' \leqq T'$ such that when ϕ varies in Y subject to $\|\phi - y\|_Y \leqq \rho'$, the map $\phi \mapsto u(t)$ given by Theorem I is continuous in the Y-norm, uniformly in $t \in [0, T'']$.*

Remark. As in [29] we can prove a similar continuity theorem when not only the initial value ϕ but also the functions N, A and f are varied, i.e., the solution is "stable" when the equations themselves are varied. It appears, on the other hand, that the variation of S is rather difficult to handle.

2.3 Proof of Theorem I

Let $R = \operatorname{dist}(y_0, Y \setminus W)$. Let E be the set of functions v: $[0, T'] \to Y$ such that

(2.4) $$\|v(t) - y_0\|_Y \leqq (3/4) R \qquad (\text{so that } v(t) \in W)$$

and

(2.5) $$\|v(t') - v(t)\|_X \leqq L' |t' - t|,$$

where $T' \leqq T$ and L' are to be determined later.

For each $v \in E$, set

$$\begin{aligned} N^v(t) &= N(t, v(t)), & S^v(t) &= S(t, v(t)), \\ A^v(t) &= A(t, v(t)), & B^v(t) &= B(t, v(t)), \\ f^v(t) &= f(t, v(t)), & &\text{where } t \in [0, T'], \end{aligned}$$

and consider the linear Cauchy problem

$$(\mathrm{L}^v) \quad du/dt + A^v(t) u = f^v(t), \qquad 0 \leqq t \leqq T',$$
$$u(0) = \phi.$$

Lemma 2.1. *The family $\{A^v(t)\} \subset G(Z)$ is stable on any subinterval of J, with stability constants β and*

$$M = \exp\{2\lambda_N + 2\mu_N(1 + L')|J|\},$$

where $|J|$ is the length of J. That is, for any finite family $\{t_j\}$ with $0 \leqq t_1 \leqq \ldots \leqq t_k \leqq T$, we have

$$\left\| \prod_{j=1}^{k} \exp\{-s_j A^v(t_j)\} \right\| \leqq M e^{\beta(s_1 + \cdots + s_k)}$$

for all $s_j \geqq 0$; here the product is time ordered in the sense that a factor with a larger t_j stands to the left of those with a smaller t_j.

Proof. (A1) implies that

$$A^v(t) \in G(Z_{N^v(t)}, 1, \beta)$$

and (N) implies that

$$d(N^v(t'), N^v(t)) \leqq \mu_N(|t'-t| + \|v(t') - v(t)\|_X) \leqq \mu_N(1+L')\,|t'-t|$$

by (2.5). Thus Lemma 2.1 follows from [26, Proposition 3.4]. The factor $e^{2\lambda_N}$ in M comes from the necessity to relate $N^v(0)$ to $\| \; \|_Z$.

Lemma 2.2. *$t \mapsto A^v(t) \in B(Y, Z)$ is continuous in norm.*

The proof is similar to that of Lemma 9.1 of [29] and is therefore omitted. The reader wishing to write out the proof should note that by (A3) the map $w \mapsto A(t, w)$ is Lipschitz continuous from Z' to $B(Y, Z')$ and hence *a fortiori* from X to $B(Y, Z)$.

Lemma 2.3. *We have*

$$\|S^v(t)\|_{Y,Z} \leqq \lambda_S, \qquad \|S^v(t)^{-1}\|_{Z,Y} \leqq \lambda'_S,$$
$$\|S^v(t') - S^v(t)\|_{Y,Z} \leqq \mu_S(1+L')\,|t'-t|.$$

Furthermore, $S^v(\cdot)$ is a strong indefinite integral of a strongly integrable function $\dot{S}^v(\cdot)$ on $[0, T']$ such that $\|\dot{S}^v(t)\|_{Y,Z} \leqq \mu_S(1+L')$ a.e.

Proof. The first three inequalities follow from (S) and (2.5). The last assertion then follows from a theorem of KŌMURA [33], which implies that a Lipschitz-continuous Z-valued function is an indefinite integral of a bounded function (note that Z is reflexive and separable).

Lemma 2.4. *We have*

$$S^v(t)\, A^v(t)\, S^v(t)^{-1} = A^v(t) + B^v(t).$$

Moreover the map $t \mapsto B(t) \in B(Z)$ is weakly continuous (and hence strongly measurable).

The proof is essentially the same as that of Lemma 9.2 of [29]; we need only use Lemmas 2.2 and 2.3 above.

Lemma 2.5. *We have $\|f^v(t)\|_Y \leqq \lambda_f$. Moreover the map $t \mapsto f^v(t)$ is continuous in the Z-norm and weakly continuous (and hence strongly measurable) in the Y-norm.*

The proof is the same as that of Lemma 9.3 of [29]; note that $Z' \subset Z$.

According to Lemmas 2.1, 2.2, and 2.4, we can construct an evolution operator $\{U^v(t, s)\}$ associated with the family $\{A^v(t)\}$ of generators, such that $U^v(t, s) \in B(Z) \cap B(Y)$; see [27].

Lemma 2.6. *We have the estimates*

(2.6) $\|U^v(t,s)\|_Z \leqq \exp\{2\lambda_N + [2\mu_N(1+L') + \beta](t-s)\}$,

(2.7) $\|U^v(t,s)\|_Y \leqq \lambda_S \lambda'_S \exp\{2\lambda_N + e^{2\lambda_N}[2\mu_N + \lambda'_S \mu_S)(1+L') + \beta + \lambda_B](t-s)\}$.

Proof. (2.6) follows directly from Lemma 2.1 and [26, Theorem 4.1]. To prove (2.7), we note that $\|U^v(t,s)\|_Y \leqq \lambda_S \lambda'_S \|W^v(t,s)\|_Z$, where

$$W^v = S^v U^v (S^v)^{-1} = \sum_{p=0}^{\infty} [-U^v(B^v - C^v)]^p U^v,$$

$$C^v(t) = \dot{S}^v(t) S^v(t)^{-1},$$

in the symbolic notation used in [27, Section 5] (change X of [27] into Z). Since

$$\|B^v(t) - C^v(t)\|_Z \leqq \lambda + \lambda'_S \mu_S (1+L')$$

by (A2) and Lemma 2.3, it is easy to deduce (2.7).

Now we can solve (L^v), noting Lemma 2.5. As in [29, (9.12)] the solution u is given by

(2.8) $$u(t) - y_0 = U^v(t,0)(\phi - y_0) + \int_0^t U^v(t,s)[f^v(s) - A^v(s) y_0]\, ds.$$

Using (2.7), we obtain for $0 \leqq t \leqq T'$,

(2.9) $$\|u(t) - y_0\|_Y \leqq \lambda_S \lambda'_S e^{2\lambda_N + \gamma T'}[\|\phi - y_0\|_Y + (\lambda_f + \lambda_0) T'],$$

where

(2.10) $$\gamma = e^{2\lambda_N}[(2\mu_N + \lambda'_S \mu_S)(1+L') + \beta + \lambda_B].$$

From $du/dt = f^v(t) - A^v(t) u(t)$ we then deduce

(2.11) $$\|du(t)/dt\|_X \leqq c\lambda_f + \lambda_A(\|y_0\|_Y + [\text{right member of (2.9)}],$$

where c is a constant such that $\|\ \|_X \leqq c\|\ \|_Y$; note that $\|A^v(t)\|_{Y,X} \leqq \lambda_A$ by (A3).

Lemma 2.7. *L' and T' can be chosen to be independent of v and such that u is in E.*

Proof. In view of (2.4) and (2.5), this will be the case if the right members of (2.9) and (2.11) are smaller than $3/4R$ and L', respectively. This can be achieved by a proper choice of L' and T', provided that

(2.12) $$\lambda_S \lambda'_S e^{2\lambda_N} \|\phi - y_0\|_Y \leqq R/2.$$

Indeed, first choose L' such that

(2.13) $$c\lambda_f + \lambda_A(\|y_0\|_Y + \tfrac{1}{2}R) = L'/2.$$

Then the required inequalities hold for $T' = 0$ and hence for sufficiently small $T' > 0$.

With this choice of L' and T', we have defined a map $\Phi: v \mapsto u = \Phi v$ of E into itself.

As in [29], we then introduce a metric in E by

$$d(v, w) = \sup_{0 \leqq t \leqq T'} \|v(t) - w(t)\|_{Z'} \tag{2.14}$$

and note that E becomes a complete metric space (here we have replaced the X-norm used in [29] by the Z'-norm).

Lemma 2.8. *If T' is sufficiently small, Φ is a contraction map of E into itself.*

Proof. First we note that the factor $\lambda_S \lambda'_S e^{2\lambda_N + \gamma T'} = K$ used in (2.9) majorizes both (2.6) and (2.7), since $\lambda_S \lambda'_S \geqq 1$. Hence by condition ($Z'$) it also majorizes $\|U^v(t, s)\|_{Z'}$ up to a constant factor. With this remark it is easy to argue as in [29] to obtain

$$d(\Phi w, \Phi v) \leqq c K T' [\mu_f + \mu_A(\|y_0\|_Y + R)]\, d(w, v). \tag{2.15}$$

Hence Φ is a contraction if T' is chosen sufficiently small.

It follows that Φ has a unique fixed point, which is obviously a unique solution in E of (Q). This completes the proof of the first part of Theorem I, where ρ' may be chosen as

$$\rho' = e^{-2\lambda_N} R / 2\lambda_S \lambda'_S. \tag{2.16}$$

(For a proof of uniqueness for evolution equations under general assumptions, which includes the case here, see [13], Theorem 6.13.)

The proof of the Lipschitz continuity in Z' of the map $\phi \mapsto u(t)$ is essentially contained in the proof of [29], Lemma 10.1. Indeed, if ϕ' is another initial value and if Φ' is the associated map of E into E, we have

$$d(\Phi' v, \Phi v) = \sup_t \|U^v(t, 0)(\phi' - \phi)\|_{Z'} \leqq c K \|\phi' - \phi\|_{Z'},$$

from which the assertion follows together with $\Phi' u' = u'$, $\Phi u = u$. (Again, this sort of result can be proved for general evolution equations under hypotheses including those here; cf. [13].)

2.4. *Proof of Theorem* II

The proof of Theorem II is essentially the same as that of Theorem 7 of [29]; here it is simpler since we are not varying the functions A and f.

Suppose ϕ^n is a sequence such that $\|\phi^n - y_0\|_Y \leqq \rho'$, $\|\phi^n - \phi\|_Y \to 0$ as $n \to \infty$. Then we can apply the proof of Theorem I given above (with the same function space E) to (Q) with the initial value ϕ^n, to construct the corresponding solution u^n on $[0, T']$.

Then we prove that $\|u^n(t) - u(t)\|_Y \to 0$ uniformly on a subinterval $[0, T'']$. The proof is almost the same as that of [29, Theorem 7] (replace X by Z). It is essential here that we have assumed that $S(t, w) = S(t)$ does not depend on w; the t-dependence causes no difficulty. We shall omit the details.

3. Quasi-linear Hyperbolic Systems

In this section we shall apply Theorems I and II of Section 2 to the equations (QH). This can be done by the standard procedure of writing (QH) as a first order system in t by introducing $\dot{\psi}(t)$ as a new variable and writing $u(t)=(\psi(t), \dot{\psi}(t))$.

3.1. Assumptions and statement of results

The equations (QH) will, in general, be hyperbolic only if ψ, $\dot{\psi}$, $\nabla\psi$ are confined to take values in some open set Ω. In other words, the type of the equation may depend on the solution itself.

Thus, let $\Omega\subset\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{nN}$ be an open set containing the origin and which is contractible to the origin, and let $a_{ij}, i,j=0,1,\dots,n,$ and b be defined on $[0,T]\times\mathbb{R}^n\times\Omega$. These variables will be denoted by $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\Omega$.

Let $H^s(\mathbb{R}^n,\mathbb{R}^N)$ be the usual Sobolev space of functions whose derivatives up to order s are in L_2 (see [19]; spaces with non-integer s are obtained by interpolation). Let $C_b^s(\mathbb{R}^n\times\Omega,\mathbb{R}^m)$ denote the functions of class C^s in x and p whose x-derivatives up to order s are bounded.

Regarding the functions a_{ij} and b, we make the following hypotheses*:

(a1) $a_{ij}\in\mathrm{Lip}([0,T], C_b^{s+1}(\mathbb{R}^n\times\Omega,\mathbb{R}^{N^2})),\ i,j=0,1,\dots,n$

$$b\in C^0([0,T], C_b^{s+1}(\mathbb{R}^n\times\Omega,\mathbb{R}^N)),$$

and

$$b(\cdot,\cdot,0)\in C^0([0,T], H^s(\mathbb{R}^n,\mathbb{R}^N)).$$

(a2) $a_{ij}^*=a_{ji}$, where a_{ij}^* is the transpose of the $N\times N$ matrix a_{ij}.

(a3) There is an $\varepsilon>0$ such that

$$a_{00}(t,x,p)\geqq\varepsilon I$$

and

$$\sum_{i,j=1}^{n}\xi_i\xi_j a_{ij}(t,x,p)\geqq\varepsilon\left(\sum_{i=1}^{n}\xi_i^2\right)I$$

for all $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\Omega$ and all $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$. (These are matrix inequalities with I denoting the unit matrix.)

Remark. Without loss of generality, we could assume that a_{ij} is symmetric by replacing it with $(a_{ij}+a_{ji})/2$. However, the more primitive assumption (a2) is the form relevant for elastodynamics, so we shall leave it in that form.

We consider the Cauchy problem for (QH) with $\dot{\psi}=\partial\psi/\partial t$ and the initial condition

$$\begin{aligned}\psi(0,\cdot)&=\psi_0\in H^{s+1}(\mathbb{R}^n,\mathbb{R}^N),\\ \dot{\psi}(0,\cdot)&=\dot{\psi}_0\in H^s(\mathbb{R}^n,\mathbb{R}^N),\end{aligned}\tag{3.1}$$

where it is assumed that

$$(\psi_0(x),\dot{\psi}_0(x),\nabla\psi_0(x))\in\Omega\quad\text{for all } x\in\mathbb{R}^n\tag{3.2}$$

so that the right member of (QH) makes sense at $t=0$.

* If s is not an integer we require a_{ij} and b to be sufficiently smooth functions of their arguments. For example in (a1) $s+1$ can be replaced by $[s]+2$.

Theorem III. *Assume $s>n/2+1$ and that* (a1)–(a3) *hold. (If the a_{ij} do not depend on the derivatives of ψ, only $s>n/2$ need be assumed.) Then the Cauchy problem for* (QH) *is well-posed in the following sense. Given ψ_0 and $\dot{\psi}_0$ satisfying* (3.1) *and* (3.2), *there is a neighborhood V of $(\psi_0, \dot{\psi}_0)$ in $H^{s+1}\times H^s$ and a positive number $T'\leqq T$ such that, for any initial condition in V,* (QH) *has a unique solution $\psi(t,\cdot)$ for $t\in[0,T']$ satisfying* (3.1) *and* (3.2) *(with 0 replaced by t). Moreover, $\psi\in C^r([0,T'], H^{s+1-r})$, $0\leqq r\leqq s$, and the map $(\psi_0,\dot{\psi}_0)\mapsto(\psi(t,\cdot),\dot{\psi}(t,\cdot))$ is continuous in the topology of $H^{s+1}\times H^s$, uniformly in $t\in[0,T']$.*

Remark. The system (QH) is reversible so we could work on $[-T', T']$ just as well.

System of the type (QH) have been considered by a number of authors. See, for example, [4, 12, 18, 20, 34, 37, 43, 44, 45]. However, none had considered such a low value of s. Well-posedness in the sense of continuous data dependence from $H^{s+1}\times H^s$ to *itself* (i.e., in the sense of dynamical systems) is especially delicate for these low values of s. For larger s this well-posedness may be implicit in the proofs of [12] (see also [8]).

By differentiating the equation (as in [16]) it is not hard to show that regularity holds in the sense that if the hypotheses hold for larger s then T' can be chosen independent of s. This allows one to conclude that if the initial data is C^∞, so is the solution.

The asymptotic conditions implicit in the spaces $H^{s+1}(\mathbb{R}^n,\mathbb{R}^N)$ are not always appropriate. Although the hyperbolicity of the equation ([11]) diminishes the importance of this point, one can contemplate building in other asymptotic properties. For example, in relativity ψ should be $O(1/r)$ at ∞. One can imagine $\psi_\infty(t,x)$ given and ask that $\psi-\psi_\infty=\alpha$ be of class H^{s+1} and examine the equation satisfied by α. Often it is again of type (QH); this happens in relativity, for instance, as is shown in Section 5.

3.2. *Proof of Theorem* III

For the proof we shall need some properties of Sobolev spaces H^s and uniformly local Sobolev spaces H^s_{ul}; proofs can be found in [19], [28], [42].

1. If $s>\frac{n}{2}+k$, k a non-negative integer, then

$$H^s(\mathbb{R}^n,\mathbb{R}^m)\subset H^s_{ul}(\mathbb{R}^n,\mathbb{R}^m)\subset C^k_b(\mathbb{R}^n,\mathbb{R}^m)$$

and the inclusions are continuous.

2. If $s>\frac{n}{2}$, then pointwise multiplication induces continuous bilinear maps

$$H^{s-l}(\mathbb{R}^n,\mathbb{R}^m)\times H^{k+l}(\mathbb{R}^n,\mathbb{R})\to H^k(\mathbb{R}^n,\mathbb{R}^m)$$

and

$$H^{s-l}_{ul}(\mathbb{R}^n,\mathbb{R}^m)\times H^{k+l}(\mathbb{R}^n,\mathbb{R})\to H^k(\mathbb{R}^n,\mathbb{R}^m)$$

for $0\leqq l\leqq s$, $0\leqq k\leqq s-l$.

3. (*a*) Assume that $\Omega\subset\mathbb{R}^m$ is open, $s>\frac{n}{2}$, $\phi\in H^s(\mathbb{R}^n,\mathbb{R}^m)$ takes values in Ω, $W\subset H^s(\mathbb{R}^n,\mathbb{R}^m)$ is a ball centered at ϕ with radius chosen small enough so any

$u \in W$ takes values in a compact set $C \subset \Omega$ (this is possible by property 1), and that $F: \mathbb{R}^m \times \Omega \to \mathbb{R}^p$ is C_b^s (again "b" referring to the first variable). Then there is a constant C_1 such that for all $u \in W$,

$$\|F(\cdot, u(\cdot))\|_{H^s_{u\ell}} \leqq C_1(1 + \|u\|^s_{H^s}). \tag{3.3 a}$$

(*b*) If F is of class C_b^{s+1}, there is a constant C_2 such that

$$\|F(\cdot, u(\cdot)) - F(\cdot, v(\cdot))\|_{H^r} \leqq C_2 \|u - v\|_{H^r} \tag{3.3 b}$$

for $u, v \in W$ and $0 \leqq r \leqq s$.

(*c*) If in addition $0 \in \Omega$, if Ω is contractible to 0 and if $F(\cdot, 0) \in H^r$, then $F(\cdot, u(\cdot)) \in H^r$ for $u \in W$

Remarks. *1.* Parts (*a*) and (*b*) are proved in [28]. The estimate (3.3 b) uses the convexity of W and the mean value theorem along the line joining v and w.

2. To prove (*c*) we use contractibility of Ω to join ϕ to 0 by a smooth curve $\phi_\lambda \in H^s(\mathbb{R}^n, \mathbb{R}^m)$ which take values in Ω. (If $H(\lambda, x)$ is the C^∞ contracting homotopy, $H(1, x) = x$, $H(0, x) = 0$, and $H(\lambda, x) \in \Omega$ for $x \in \Omega$, $0 \leqq \lambda \leqq 1$, set $\phi_\lambda(x) = H(\lambda, \phi(x))$.) Along the curve ϕ_λ construct a finite covering by balls $W_1, W_2, \ldots, W_q = W$ such that $u \in W_i \subset H^s(\mathbb{R}^n, \mathbb{R}^m)$ implies u takes values in Ω; this is possible by property 1 and the compactness of $\{\phi_\lambda \in H^s(\mathbb{R}^n, \mathbb{R}^m) \mid \lambda \in [0, 1]\}$. Now by a finite number of applications of (3.3 b) starting at $v = 0$ we obtain our result (*c*).

In order to verify the hypotheses of Theorems I and II, we shall need to set up some further notation and establish a number of lemmas. Let

$$X = H^s(\mathbb{R}^n, \mathbb{R}^N) \times H^{s-1}(\mathbb{R}^n, \mathbb{R}^N),$$

$$Y = H^{s+1}(\mathbb{R}^n, \mathbb{R}^N) \times H^s(\mathbb{R}^n, \mathbb{R}^N),$$

and

$$Z = Z' = H^1(\mathbb{R}^n, \mathbb{R}^N) \times H^0(\mathbb{R}^n, \mathbb{R}^N).$$

Also let $W \subset Y$ be a ball centered at ϕ with radius small enough so that $u \in W$ satisfies (3.2).

For $w = (\sigma, \dot\sigma) \in W$ define the operator $A(t, w)$ by

$$A(t, w) = -\begin{pmatrix} 0 & I \\ a_{00}^{-1} \sum\limits_{i,j=1}^{n} a_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j} & a_{00}^{-1} \sum\limits_{i=1}^{n} (a_{0i} + a_{i0}) \dfrac{\partial}{\partial x_i} \end{pmatrix}$$

where a_{ij} stands for $a_{ij}(t, x, \sigma, \dot\sigma, \nabla \sigma)$, $i, j = 0, 1, \ldots, n$. Also, define $f(t, w)$ by

$$f(t, w) = \big(0, b(t, x, \sigma, \dot\sigma, \nabla \sigma)\big).$$

Lemma 3.1. *There are constants λ_f and λ_A such that for $t \in [0, T]$, $w \in W$, we have*

$$A(t, w) \in B(Y, X), \qquad f(t, w) \in Y$$

and

$$\|A(t, w)\|_{Y, X} \leqq \lambda_A, \qquad \|f(t, w)\|_Y \leqq \lambda_f.$$

Proof. We have

$$A(t, w)(\psi, \dot\psi) = -\left(\dot\psi, a_{00}^{-1}\left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\right) + a_{00}^{-1} \sum_{i=1}^{n} (a_{0i} + a_{i0}) \frac{\partial \dot\psi}{\partial x_i}\right).$$

By the properties of Sobolev spaces above, $a_{00}^{-1} a_{ij}$ and $a_{00}^{-1}(a_{0i}+a_{i0})$ are uniformly bounded in $H_{ul}^s(\mathbb{R}^n, \mathbb{R}^{N^2})$ for $w \in W$, $t \in [0, T]$. Therefore, for $(\psi, \dot\psi) \in Y$,

$$\begin{aligned}\|A(t,w)(\psi,\dot\psi)\|_X &\leqq \|\dot\psi\|_{H^s} + \sum_{i,j=1}^n \left\| a_{00}^{-1} a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\|_{H^{s-1}} \\ &\quad + \sum_{i=0}^n \left\| a_{00}^{-1}(a_{0i}+a_{i0}) \frac{\partial \dot\psi}{\partial x_i} \right\|_{H^{s-1}} \\ &\leqq C_1 \Big(\|\dot\psi\|_{H^s} + \sum_{i,j=1}^n \|a_{00}^{-1} a_{ij}\|_{H_{ul}^s} \|\psi\|_{H^{s+1}} \\ &\quad + \sum_{i=1}^n \|a_{00}^{-1}(a_{0i}+a_{i0})\|_{H_{ul}^s} \|\dot\psi\|_{H^s} \Big) \\ &\leqq C_2 \|(\psi,\dot\psi)\|_Y.\end{aligned}$$

Here we have used the fact that $H_{ul}^s \cdot H^{s-1} \subset H^{s-1}$ for $s > \frac{n}{2}$. Thus we can take $\lambda_A = C_2$.

Similarly, since $b(t, \cdot, 0) \in H^s$ and $b(t, \cdot, \cdot) \in C_b^{s+1}$, it is clear that $b(t, x, \sigma, \dot\sigma, \nabla\sigma)$ is uniformly bounded in H^s for $t \in [0, T]$, $w = (\sigma, \dot\sigma) \in W$ (see property 3 of Sobolev spaces). Therefore $f(t, w)$ is uniformly bounded in Y.

Lemma 3.2. *There are constants $M, c_0, d_0 > 0$ such that*

$$\begin{aligned}|B(t,w;\psi_1,\psi_2)| &\leqq M \|\psi_1\|_{H^1} \|\psi_2\|_{H^1}, \\ B(t,w;\psi,\psi) &\geqq c_0 \|\psi\|_{H^1}^2 - d_0 \|\psi\|_0^2\end{aligned} \tag{3.4}$$

for all $\psi \in H^1(\mathbb{R}^n, \mathbb{R}^N)$, $t \in [0, T]$ and $w \in W$, where

$$B(t,w;\psi_1,\psi_2) = \sum_{i,j=1}^n \left(a_{ij} \frac{\partial \psi_1}{\partial x_i}, \frac{\partial \psi_2}{\partial x_j} \right)_0;$$

here $(\ ,\ )_0$ denotes the L^2 inner-product on $\mathbb{R}^n$ and a_{ij} stands for $a_{ij}(t, x, \sigma, \dot\sigma, \nabla\sigma)$ if $w = (\sigma, \dot\sigma)$.

This is GÅRDING's inequality, which follows from (a1) and (a3); see [15], [41] or [50]. (Note: Reference [41] considers *systems*, which we are concerned with here; the version of GÅRDING's inequality presented there (p. 253) replaces $\mathbb{R}^n$ with a bounded domain G. However, using the uniform continuity of a_{ij}, we can readily adapt the proof given there to our context. If $a_{ij}^{\alpha\beta}$ are the matrix elements of a_{ij} in the notation of [41], then the second inequality of (a3) is exactly

$$\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{ij}^{\alpha\beta} \lambda_\alpha \lambda_\beta \xi_i \xi_j \geqq \varepsilon \|\lambda\|^2 \|\xi\|^2,$$

which is the strong ellipticity hypothesis of [41].)

For $t \in [0, T]$ and $w \in W$ define a norm $N(t, w)$ on Z by

$$\|(\psi,\dot\psi)\|_{N(t,w)} = B(t,w;\psi,\psi) + d_0 \|\psi\|_0^2 + (a_{00}\dot\psi, \dot\psi)_0.$$

From (3.4) it follows that $N(t, w) \in N(Z)$. The corresponding bilinear form will be denoted $(\ ,\ )_{N(t,w)}$.

Lemma 3.3. *There is a constant $\mu_1>0$ such that for $t, t' \in [0, T]$ and $w, w' \in W$, $w=(\sigma, \dot{\sigma})$, $w'=(\sigma', \dot{\sigma}')$, we have*

$$\|a_{ij}-a'_{ij}\|_{H^r}=\|a_{ij}(t, x, \sigma, \dot{\sigma}, \nabla\sigma)-a_{ij}(t', x, \sigma', \dot{\sigma}', \nabla\sigma')\|_{H^r} \leqq \mu_1(|t-t'|+\|w-w'\|_{H^{r+1}\times H^r}) \tag{3.5}$$

for $0\leqq r\leqq s$, $i, j=0, 1, \ldots, n$; moreover there is a constant μ_2 such that

$$\|b(t, x, \sigma, \dot{\sigma}, \nabla\sigma)-b(t, x, \sigma', \dot{\sigma}', \nabla\sigma')\|_{H^r}\leqq\mu_2\|w-w'\|_{H^{r+1}\times H^r}. \tag{3.6}$$

Proof. (3.6) results at once from (3.3) and the hypotheses (a1) on b. Similarly (3.5) is proved by using the Lipschitz estimate on a_{ij}, adding and subtracting the term $a_{ij}(t', x, \sigma, \dot{\sigma}, \nabla\sigma)$, and using the fact that W is a bounded subset of Y.

From (3.3) we can derive a number of estimates. For example, from (3.5) with $r=s-1$, we get

$$|B(t, w; \psi, \dot{\psi})-B(t', w'; \psi, \dot{\psi})|\leqq\mu_3(|t-t'|+\|w-w'\|_X) \tag{3.7}$$

if $s-1>n/2$. If a_{ij} is independent of derivatives of σ, then H^r on the left side of (3.5) may be replaced by H^{r+1}, so (3.7) remains valid assuming only $s>n/2$.

By the same proof the estimates (3.5) are also valid when a_{ij} is replaced by $a_{00}^{-1}a_{ij}$. From this we can get an estimate on $A(t, w)$.

Lemma 3.4. *There is a constant $\mu_4>0$ such that for $t, t'\in[0, T]$ and $w, w'\in W$ we have*

$$\|A(t, w)-A(t', w')\|_{Y, X}\leqq\mu_4(|t-t'|+\|w-w'\|_X) \tag{3.8}$$

and similarly

$$\|A(t, w)-A(t', w')\|_{Y, Z'}\leqq\mu_5(|t-t'|+\|w-w'\|_{Z'}). \tag{3.9}$$

Proof. First take the case in which $s>\frac{n}{2}+1$ and $0\leqq r\leqq s-1$; then

$$\begin{aligned}
&\|(A(t, w)-A(t', w'))(\psi, \dot{\psi})\|_{H^{r+1}\times H^r}\\
&=\left\|\sum_{i,j=1}^{n}(a_{00}^{-1}a_{ij}-(a_{00}^{-1}a_{ij})')\frac{\partial^2\psi}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n}(a_{00}^{-1}(a_{0i}+a_{i0})-a_{00}^{-1}(a_{0i}+a_{i0})')\frac{\partial\dot{\psi}}{\partial x_i}\right\|_{H^r}\\
&\leqq c\sum_{i,j=1}^{n}\|a_{00}^{-1}a_{ij}-(a_{00}^{-1}a_{ij})'\|_{H^r}\|\psi\|_{H^{s+1}}\\
&\quad+c\sum_{i=1}^{n}\|a_{00}^{-1}(a_{0i}+a_{i0})-a_{00}^{-1}(a_{0i}+a_{i0})'\|_{H^r}\|\dot{\psi}\|_{H^s}\\
&\leqq\mu_4(|t-t'|+\|w-w'\|_{H^{r+1}\times H^r})\|(\psi, \dot{\psi})\|_Y.
\end{aligned}$$

If $s>n/2$ and if a_{ij} does not depend on derivatives of σ, then we can use (3.5) with H^r on the left replaced by H^{r+1} and replace the use of $H^{s-1}\cdot H^r\subset H^r$ above by $H^{s-1}\cdot H^{r+1}\subset H^r$ (see property 2 of Sobolev spaces above) to obtain the same estimate.

We get (3.8) and (3.9) by taking $r=s-1$ and $r=0$ respectively.

In the same way we can prove the following estimates for f:

$$\|f(t,w)-f(t,w')\|_Y \leqq \mu_6 \|w-w'\|_Y \tag{3.10}$$

and

$$\|f(t,w)-f(t,w')\|_{Z'} \leqq \mu_7 \|w-w'\|_{Z'} \tag{3.11}$$

for $0 \leqq t \leqq T$ and $w, w' \in W$. Although f need not be Lipschitz in t, its Z-continuity in t follows from the t-continuity of b.

From (3.4) and (3.7) we can derive estimates for $N(t,w)$, namely:

$$\mu_8 \|u\|_Z \leqq \|u\|_{N(t,w)} \leqq \mu_8^{-1} \|u\|_Z \tag{3.12}$$

and

$$\left| \|u\|_{N(t,w)} - \|u\|_{N(t',w')} \right| \leqq \mu_9 (|t-t'| + \|w-w'\|_X) \|u\|_Z \tag{3.13}$$

for $t, t' \in [0,T]$ and $w, w' \in W$, $u \in Z$.

Lemma 3.5. *There is a constant $\beta > 0$ such that for $t \in [0,T]$, $w \in W$, $u \in Z$, and $\lambda > \beta$ we have*

$$\|A(t,w)u + \lambda u\|_{N(t,w)} \geqq (\lambda - \beta) \|u\|_{N(t,w)}. \tag{3.14}$$

Proof. Let $w = (\sigma, \dot\sigma)$ and $u = (\psi, \dot\psi)$, and let a_{ij} stand for $a_{ij}(t, x, \sigma, \dot\sigma, \nabla\sigma)$. By using the Schwarz inequality, integration by parts, and the symmetry of $a_{0i} + a_{i0}$, we obtain

$$\begin{aligned}
&\|A(t,w)u + \lambda u\|_{N(t,w)} \|u\|_{N(t,w)} \\
&\quad \geqq (A(t,w)u + \lambda u, u)_{N(t,w)} \\
&\quad = B(t,w; -\dot\psi + \lambda\psi, \psi) + d_0(-\dot\psi + \lambda\psi, \psi)_0 + \lambda(\dot\psi, \dot\psi)_0 \\
&\qquad + \left(a_{00} \left(-a_{00}^{-1} \sum_{i,j=1}^n a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - a_{00}^{-1} \sum_{i=1}^n (a_{0i} + a_{i0}) \frac{\partial \dot\psi}{\partial x_i} \right), \dot\psi \right)_0 \\
&\quad = \sum_{i,j=1}^n \left(a_{ij} \left(\frac{-\partial \dot\psi}{\partial x_i} + \frac{\lambda \partial \psi}{\partial x_i} \right), \frac{\partial \psi}{\partial x_j} \right)_0 + d_0(-\dot\psi + \lambda\psi, \psi)_0 \\
&\qquad - \sum_{i,j=1}^n \left(a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j}, \dot\psi \right)_0 - \sum_{i=1}^n \left((a_{0i} + a_{i0}) \frac{\partial \dot\psi}{\partial x_i}, \dot\psi \right)_0 + \lambda(\dot\psi, \dot\psi)_0 \\
&\quad = \sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial x_i} \dot\psi, \frac{\partial \psi}{\partial x_j} \right)_0 + \lambda B(t,w; \psi, \psi) + d_0(-\dot\psi + \lambda\psi, \psi)_0 \\
&\qquad + \tfrac{1}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} (a_{0i} + a_{i0}) \dot\psi, \dot\psi \right)_0 + \lambda(\dot\psi, \dot\psi)_0.
\end{aligned}$$

Since $\dfrac{\partial a_{ij}}{\partial x_k}$ is uniformly bounded, the first and fourth terms are bounded by $C_1 \|\dot\psi\|_0 \|\psi\|_{H^1}$ and $C_2 \|\dot\psi\|_0^2$. These two together are bounded by $C_3(\|\dot\psi\|_0^2 + \|\psi\|_{H^1}^2)$. Thus by (3.12), the sum of the first and fourth terms exeeds $-C_4 \|u\|_{N(t,w)}^2$.

Therefore,

$$\begin{aligned}\|A(t,w)\,u+\lambda u\|_{N(t,w)}\,\|u\|_{N(t,w)} & \\ \geqq -C_4\,\|u\|^2_{N(t,w)}+\lambda\,\|u\|^2_{N(t,w)}+d_0(\dot\psi,\psi)_0 & \\ \geqq(\lambda-\beta)\,\|u\|^2_{N(t,w)}, &\end{aligned}$$

where $\beta=C_4+d_0\,\mu_8/2$. Thus (3.14) holds.

Lemma 3.6. *If β is sufficiently large, then the map $A(t,w)+\lambda I\colon H^2\times H^1\to Z$ is invertible for $\lambda>\beta$, and by* (3.14),

$$\|(A(t,w)+\lambda I)^{-1}\|_{N(t,w)}\leqq 1/(\lambda-\beta).$$

Since the proof is similar to that in YOSIDA [50, XIV, 3] we just sketch the idea. One first shows that

$$\tilde A_\lambda(t,w)=\sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2}{\partial x_i\,\partial x_j}+\lambda(a_{0i}+a_{i0})\,\frac{\partial}{\partial x_i}$$

satisfies the elliptic estimates

$$\|\psi\|^2_{H^2}\leqq C_\lambda(\|\tilde A(t,w)\,\psi\|_0^2+\|\psi\|_0^2)$$

and that any weak solution of $\tilde A_\lambda(t,w)\,\psi+\lambda^2\psi=0$ is actually a strong solution. (This is done for systems in [41, Chapter 6], again for bounded regions, but the same proof works on $\mathbb{R}^n$.) From this and GÅRDING's inequality it follows that $\tilde A_\lambda(t,w)+\lambda^2 I$ is one-to-one with closed range. Since $\tilde A^*_\lambda(t,w)+\lambda^2 I$ is also one-to-one (for β large enough) for similar reasons, it follows that $\tilde A_\lambda(t,w)+\lambda^2 I$ has dense range and so is invertible. Finally, invertibility of $A(t,w)+\lambda I$ can then be reduced to that of $\tilde A_\lambda(t,w)+\lambda^2 I$: the solution of $(A(t,w)+\lambda I)(\psi,\dot\psi)=(f,\dot f)$ is

$$\psi=(\tilde A_\lambda(t,w)+\lambda^2 I)^{-1}\,a_{00}\,\dot f,\qquad \dot\psi=f-\lambda\psi.$$

Let $S\colon Y\to Z$ be defined by

$$S=\begin{pmatrix}(1-\Delta)^{s/2} & 0\\ 0 & (1-\Delta)^{s/2}\end{pmatrix},$$

so that S is an isomorphism between Y and Z. For $t\in[0,T]$ and $w\in W$, let

$$B(t,w)=\begin{pmatrix}0 & 0\\ [(1-\Delta)^{s/2},A_1](1-\Delta)^{-s/2} & [(1-\Delta)^{s/2},A_2](1-\Delta)^{-s/2}\end{pmatrix},$$

where

$$\begin{aligned}A_1&=-\sum_{i,j=1}^{n} a_{00}^{-1}\,a_{ij}\,\frac{\partial^2}{\partial x_i\,\partial x_j},\\ A_2&=-\sum_{i=1}^{n} a_{00}^{-1}\,(a_{0i}+a_{i0})\,\frac{\partial}{\partial x_i},\end{aligned}$$

and $[\,,]$ denotes the commutator.

Lemma 3.7. *We have $B(t,w)\in B(Z)$. Moreover there exist constants μ_9 and μ_{10} such that*

$$\|B(t,w)\|_Z\leqq\mu_9 \tag{3.15}$$

and

$$\|B(t, w) - B(t, w')\|_Z \leqq \mu_{10} \|w - w'\|_Y \tag{3.16}$$

for $t \in [0, T]$ *and* $w, w' \in W$.

Proof. We have

$$\|B(t, w)(\psi, \dot{\psi})\|_Z = \|[(1-\Delta)^{s/2}, A_1](1-\Delta)^{-s/2}\psi + [(1-\Delta)^{s/2}, A_2](1-\Delta)^{-s/2}\dot{\psi}\|_0.$$

Write

$$[(1-\Delta)^{s/2}, A_1](1-\Delta)^{-s/2}\psi$$
$$= -\sum_{i,j=1}^{n} [(1-\Delta)^{s/2}, a_{00}^{-1} a_{ij}](1-\Delta)^{-\left(\frac{s-1}{2}\right)} \frac{\partial^2}{\partial x_i \partial x_j} (1-\Delta)^{-1/2}\psi.$$

Now clearly $\left\| \frac{\partial^2}{\partial x_i \partial x_j} (1-\Delta)^{-1/2}\psi \right\|_0 \leqq C_1 \|\psi\|_{H^1}$, and from [29, Lemma A2] the commutator with a multiplication operator can be estimated as follows:

$$\|[(1-\Delta)^{s/2}, a_{00}^{-1} a_{ij}](1-\Delta)^{-\left(\frac{s-1}{2}\right)}\|_{L^2, L^2} \leqq C_2 \|\operatorname{grad} a_{00}^{-1} a_{ij}\|_{s-1}.$$

Using properties 2 and 3 of Sobolev spaces and the chain rule to expand the gradient, we see that the right-hand side is uniformly bounded. Hence, after estimating the second terms in a similar manner, we find

$$\|B(t, w)(\psi, \dot{\psi})\|_Z \leqq C_3 \|(\psi, -\dot{\psi})\|_Z,$$

i.e., (3.15) holds. Starting from (3.5) with $r = s$, that is, from

$$\|a_{00}^{-1} a_{ij} - (a_{00}^{-1} a_{ij})'\|_{H^s} \leqq \mu_1 \|w - w'\|_Y$$

(here $t = t'$), the estimate (3.16) we obtain by similar reasoning.

Now we shall verify the hypotheses from Section 2.

(Z'): Since $Z' = Z$, this condition is trivial.

(N): If we write

$$\frac{\|u\|_{N(t,w)}}{\|u\|_{N(t',w')}} = \frac{\|u\|_{N(t,w)} - \|u\|_{N(t',w')}}{\|u\|_{N(t',w')}} + 1,$$

use estimates (3.12), (3.13), and the fact that $\log(1+x) \leqq (\text{const})x$ for $0 < \delta \leqq x + 1 < \infty$, we get condition (N) easily.

(S): Since S is independent of t and w, this is trivial.

(A1): This follows from Lemmas 3.5 and 3.6 and the HILLE-YOSIDA theorem ([50], Ch. IX).

(A2): That $SAS^{-1} = A + B$; i.e. that $B = [S, A] S^{-1}$ for A and B as given above, is a straightforward algebraic computation. The estimate $\|B(t, w)\|_Z \leqq \lambda_B$ is given in Lemma 3.7.

(A3): This condition is verified in Lemma 3.1 and estimate (3.9).

(A4): Using Remark 2 following Theorem I, we need only find a dense set of y_0 satisfying (A4). Such a dense set is

$$W \cap (H^{s+2}(\mathbb{R}^n, \mathbb{R}^N) \times H^{s+1}(\mathbb{R}^n, \mathbb{R}^N)).$$

Indeed, if $y_0=(\psi_0, \dot\psi_0)$ lies in this set, we have the estimate

$$\begin{aligned}
&\|A(t,w)(\psi_0,\dot\psi_0)\|_Y \\
&\leqq \|\dot\psi_0\|_{H^{s+1}} + \sum_{i,j=1}^{n} \left\| a_{00}^{-1} a_{ij} \frac{\partial^2 \psi_0}{\partial x_i \partial x_j} \right\|_{H^s} + \sum_{i=1}^{n} \left\| a_{00}^{-1}(a_{0i}+a_{i0}) \frac{\partial \dot\psi_0}{\partial x_i} \right\|_{H^s} \\
&\leqq \|\dot\psi_0\|_{H^{s+1}} + \sum_{i,j=1}^{n} C \|a_{00}^{-1} a_{ij}\|_{H^s_{ul}} \|\psi_0\|_{H^{s+2}} + \sum_{i=1}^{n} C \|a_{00}^{-1}(a_{0i}+a_{i0})\|_{H^s_{ul}} \|\dot\psi_0\|_{H^{s+1}} \\
&\leqq C_1(\|\dot\psi_0\|_{H^{s+1}} + \|\psi_0\|_{H^{s+2}}) = \lambda_0,
\end{aligned}$$

since, as earlier, the H^s_{ul} norms are uniformly bounded by property 3(a) of Sobolev spaces.

(f1): See Lemma 3.1 and estimate (3.11), together with the remark following (3.11).

(A5): See estimate (3.16).

(f2): See estimate (3.10).

Remark. In order to utilize available energy estimates or other conservation laws to investigate global properties in time, it is important to have s as low as possible. One often has a bound of the form

$$\|(\psi,\dot\psi)\|_{H^1\times H^0} \leqq \text{Const} \tag{3.17}$$

(independent of t) for a solution of (QH) derived from an energy inequality (such is often the case in elastodynamics and general relativity). In a number of interesting semilinear cases with a_{ij} independent of ψ, $\partial\psi/\partial t$ and $\nabla\psi$ this is enough to obtain global solutions, as is well known (see references in [39]). In the quasilinear case, this is not so due to shocks. We remark, however, that in case a_{ij} does not depend on derivatives of ψ, then (3.17) is enough to guarantee global solutions in one dimension. This requires an examination of the proof, though we shall not give the details. The remark is relevant for general relativity; see [40].

4. Applications to Nonlinear Elastodynamics

Theorem III enables us to establish the well-posedness of the equations of nonlinear ealstodynamics. Here we consider the case in which the body is all of $\mathbb{R}^3$; more interesting cases involving boundaries require extensions of the previous results and will be pursued in future work.

Suppose $\rho: \mathbb{R}^3 \to \mathbb{R}$, the density of the initial configuration. $\hat{B}: [0,T]\times\mathbb{R}^3 \times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^9 \to \mathbb{R}^3$, the extrinsic body force, and $\hat{P}: \mathbb{R}^3\times\Omega_1 \to \mathbb{R}^9$, the first Piola-Kirchhoff stress tensor, are given and that $\Omega_1 \subset \mathbb{R}^9$ is an open set contractible to the origin.

If $U(t,X)$, $X=(X_1,X_2,X_3)\in\mathbb{R}^3$, is the displacement, we write

$$B(t,X) = \hat{B}(t, X, U(t,X), \dot{U}(t,X), \nabla U(t,X)),$$

(although ∇U and $\dot U$ may normally not be present as arguments of $\hat B$) and

$$P(t,X) = \hat{P}(X, \nabla U(t,X)).$$

Remark. With no change in the proof, we can allow $\hat{P}$ to depend on t, U or $\dot{U}$ as well, cf. Theorem III.

The equations of nonlinear (finite) elastodynamics may be written

$$\rho\ddot{U}=\rho B+\mathrm{DIV}\,P, \quad \text{where } (\mathrm{DIV}\,P)^\alpha=\frac{\partial}{\partial X_i}P_i^\alpha. \tag{4.1}$$

The Cauchy, or initial value problem, consists of finding a function $U(t,X)$ satisfying (4.1) subject to given initial conditions

$$U(0,X)=U_0(X), \quad \dot{U}(0,X)=\dot{U}_0(X). \tag{4.2}$$

Define the fourth-order tensor $A_{ij}^{\alpha\beta}=\dfrac{\partial\hat{P}_i^\alpha}{\partial(\partial_j U_\beta)}$, and let a_{ij} denote the matrix whose $\alpha\beta^{\text{th}}$ entry is $A_{ij}^{\alpha\beta}$. We make the following assumptions:

(e1) $\hat{P}$ is a C_b^{s+2} (or $C_b^{[s]+3}$ if s is not an integer) function* of X, ∇U; $\hat{B}$, ρ are C_b^{s+1} (or $C_b^{[s]+2}$ if s is not an integer) functions, and

$$\hat{B}(\cdot,\cdot,0,0,0)\in C^0([0,T],H^s(\mathbb{R}^3,\mathbb{R}^3)).$$

(e2) $A_{ij}^{\alpha\beta}=A_{ji}^{\beta\alpha}$.

(e3) There is an $\varepsilon>0$ such that $\rho(X)\geqq\varepsilon$ and

$$A_{ij}^{\alpha\beta}(X,p)\,\xi^i\xi^j\lambda_\alpha\lambda_\beta\geqq\varepsilon\,\|\xi\|^2\,\|\lambda\|^2$$

for all $(X,p)\in\mathbb{R}^3\times\Omega_1$, $\xi\in\mathbb{R}^3$, and $\lambda\in\mathbb{R}^3$.

In terms of $A_{ij}^{\alpha\beta}$, equations (4.1) read

$$\rho\frac{\partial^2U^\alpha}{\partial t^2}=A_{ij}^{\alpha\beta}\frac{\partial^2U_\beta}{\partial X_i\,\partial X_j}+b^\alpha \tag{4.3}$$

where $b^\alpha=\rho B^\alpha+\sum_{j=1}^{3}\partial\hat{P}_j^\alpha/\partial X_j$. These are precisely of the form (QH). The conditions (e1)–(e3) correspond to (a1)–(a3), so we obtain the following.

Theorem IV. *Let* (e1)–(e3) *hold and assume that* $\nabla U_0(X)\in\Omega_1$ *for all* $X\in\mathbb{R}^3$. *Given initial conditions* U_0, $\dot{U}_0$ *in* $H^{s+1}\times H^s$, *where* $s>2.5$, *there is a* $T'>0$ *and a neighborhood* V *of* $(U_0,\dot{U}_0)$ *such that for any initial data in* V, (4.1) *has a unique solution on* $[-T',T']$; *the solution is a continuous function in* $H^{s+1}\times H^s$ *of the initial data.*

Remarks. *1.* Assumption (e2), equivalent to postulating the existence of a stored energy function, means that the material in question is hyperelastic.

2. Since $H^s\subset C_b^k$ if $s>(n/2)+k$, the solutions are classical and can be made as smooth as we please by choosing s large (see Section 3 for regularity remarks).

3. In some examples the ellipticity condition (e3) fails for large displacement gradients†. This is one of the reasons for restricting to an open set Ω_1.

* Recall from Section 3 that the "b" refers to boundedness only of X-derivatives.

† This remark is based on a preprint of J. ABOUDI, *Two-dimensional Wave Propagation in a Nonlinearly Elastic Half Space.*

4. In two dimensions we require $s>2$ and in one dimension $s>1.5$. We cannot lower the requirement $s>(n/2)+1$ to $s>n/2$ because $A_{ij}^{\alpha\beta}$ will always depend on ∇U.

5. In the linear case there are no restrictions on s other than $s\geqq 0$, and solutions are defined for all t and depend continuously on the initial data. This is contained in the above work. In elastodynamics quite a bit is already known about the linear case ([14], [15], [22], [30], [31]), but little seems to have been known about the nonlinear case ([9], [47], [48]).

6. The equations are linearization stable in the sense that first-order perturbation analysis is valid (this is proved by using [17] and [13]).

5. Applications to General Relativity

The literature on existence and uniqueness of the Cauchy problem in general relativity is extensive; see [5], [12], [16], [23], [38], and the references cited therein. For this problem there has been some interest in obtaining the lowest possible value of s, namely $s=2$ (in our notation); see for instance [23], p. 251. The use of Theorem III allows this to be accomplished.

The Einstein equations for a Lorentz metric $g_{\alpha\beta}$ on $\mathbb{R}^4$, $0\leqq\alpha,\beta\leqq 3$, can be written in the form

$$-\tfrac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\beta}+H_{\mu\nu}\left(g_{\alpha\beta},\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right)=0 \tag{5.1}$$

after a transformation of variables to harmonic coordinates; $H_{\mu\nu}$ is an explicit rational function of $g_{\alpha\beta}$ and its first derivatives ([16], p. 22). It is customary to write $(x^\alpha)=(t,x^i)$, $1\leqq i\leqq 3$.

The asymptotic behavior of $g_{\alpha\beta}$ requires special attention. Let us fix on a set of initial data $g_{\alpha\beta}^0$, $\dot g_{\alpha\beta}^0$; we are concerned with choosing other initial data in a neighborhood of this one.

In the asymptotically flat case $g_{\alpha\beta}^0$ will differ from the Minkowski metric by a term which is $O(1/r)$ at spatial infinity (see, for example [10]). This extra term is not in H^s, but it is nevertheless important to include it in order to ensure non-trivial solutions to the constraint equations on the initial data. We shall take $\psi_{\alpha\beta}=g_{\alpha\beta}-g_{\alpha\beta}^0$.

Assume the following conditions

(g1)* $\qquad g_{\alpha\beta}^0\in C_b^{s+1}(\mathbb{R}^3,\mathbb{R}),\qquad \dot g_{\alpha\beta}^0\in H^s(\mathbb{R}^3,\mathbb{R})$

and

$$\frac{\partial g_{\alpha\beta}^0}{\partial x^i}\in H^s(\mathbb{R}^3,\mathbb{R}),\qquad 0\leqq\alpha,\beta\leqq 3,\qquad 1\leqq i\leqq 3.$$

(g2) $\Omega\subset\mathbb{R}^{10}\times\mathbb{R}^{10}\times\mathbb{R}^{30}$ is chosen as follows: $\Omega=\Omega_0\times\mathbb{R}^{10}\times\mathbb{R}^{30}$, where Ω_0 is a ball about 0 such that if $(g_{\alpha\beta}-g_{\alpha\beta}^0)(x)\in\Omega_0$ for any $x\in\mathbb{R}^3$, then $g_{\alpha\beta}$ is of Lorentz signature $(-,+,+,+)$.

* Again, if s is not an integer, we require the class C_b^{s+1} to be replaced by $C_b^{[s]+2}$.

In the variables $\psi_{\alpha\beta}$, the equations (5.1) are of the form (QH). Since $H_{\mu\nu}$ is quadratic in the derivatives of $g_{\alpha\beta}$, condition (g1) implies condition (a1).

The coefficients of the second-order terms do not involve derivatives of ψ, so only $s>n/2$ is required, as opposed to elastodynamics which requires $s>(n/2)+1$.

Let us write $H^s_{g^0_{\alpha\beta}}$ etc., for the space of $g_{\alpha\beta}$ such that $g_{\alpha\beta}-g^0_{\alpha\beta}\in H^s$, topologized accordingly. Then Theorem III yields.

Theorem V. *Let* (g1) *and* (g2) *hold. Then for* $s>1.5$ *and initial data in a neighborhood of* $(g^0_{\alpha\beta}, \dot{g}^0_{\alpha\beta})$ *in* $H^{s+1}_{g^0_{\alpha\beta}}\times H^s_{\dot{g}^0_{\alpha\beta}}$, *equations* (5.1) *have a unique solution in the same space for a time interval* $[0, T']$, $T'>0$. *The solution depends continuously on the initial data in this space (i.e. it is well-posed or "Cauchy stable").*

Thus, with the asymptotic conditions subtracted off, $H^3\times H^2$ initial data generates a piece of H^3 space-time in a way which depends continuously on the initial data. If Ω is chosen too large or T' is allowed to be large, the Lorentz character of $g_{\alpha\beta}$ could be lost or a singularity could develop.

Remarks. *1.* Uniqueness of the spacetime up to H^{s+2} (e.g. H^4) coordinate transformations of the spacetime can be proved as in [16].

2. Previously the best known existence and uniqueness result was for $H^4\times H^3$ Cauchy data (and for H^5 coordinate transformations in Remark 1), although existence without uniqueness in $H^3\times H^2$ is claimed without proof in [23]. (Well-posedness is implicit in [8], [16], and [28] only for $s>(n/2)+1$.)

3. The results can be formulated on manifolds in standard fashion, as in [5], [38].

4. In the asymptotically Euclidean case of Theorem V, spacetimes with different mass require different choices of $g^0_{\alpha\beta}$ since the coefficient of $1/r$ will be different (see [10]).

5. See the remarks at the end of Section 3 and in [40] regarding global solutions.

6. The results concerning (QH) together with those for symmetric hyperbolic quasi-linear systems ([16], [28]) cover a wide variety of field theories.

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