#47

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A New Hamiltonian Structure for the Dynamics of General Relativity[1]

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Abstract

A new compact form of the dynamical equations of relativity is proposed. The new form clarifies the covariance of the equations under coordinate transformations of the space-time. On a deeper level, we obtain new insight into the infinite-dimensional symplectic geometry behind the dynamical equations, the decompositions of gravitational perturbations, and the space of gravitational degrees of freedom. Prospects for these results in studying fields coupled to gravity and the quantization of gravity are outlined.

It has been over 15 years since Arnowitt Deser, and Misner laid down the basic formulation of Einstein's equations as dynamical equations for an evolving spatial universe [3]. This procedure is basic for studying the evolution equations from a geometric point of view [11], for establishing the stability of space-times [7, 13, 14, 18, 20] and the positivity of their energy content [5, 8], and for approaching many other important questions.

On the other hand, there have been significant developments in Hamiltonian mechanics and symplectic geometry (i.e., Poisson bracket structures) during the same period. Both classical mechanics and field theories have been successfully put into the general context of symplectic geometry [1, 9]. From this geometrization and unification spring new insights and methods. Specifically, there is now a satisfactory general procedure for eliminating the symmetries of a given Hamiltonian system. Previously, this was well understood in the classical literature only for commutative symmetry groups (i.e., for the relatively rare occurrence of first integrals in involution) and in special cases, such as rotational sym-

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metry (see [16, 21]). This generalization for eliminating the symmetries of a Hamiltonian system is important in relativity since its gauge group is noncommutative and is infinite dimensional.

It is tempting to apply this new elimination procedure to the space of solutions to Einstein's equations and thereby remove the group of coordinate transformations of a space-time. The resulting space of four-geometries is called the space of gravitational degrees of freedom. Some important conformal representations of this space have been constructed by O'Murchadha and York [20] and York [22, 23]. However, we desire a construction that is natural with respect to the dynamics and wish to prove that the space is a smooth infinite-dimensional manifold and carries a Poisson bracket structure.

In the meantime, Moncrief [19] has published an important new decomposition of gravitational perturbations, one piece of which represents the direction of the space of gravitational degrees of freedom. This decomposition unifies and extends several previous decompositions in geometry and relativity due to Berger and Ebin [4] and Deser [10]. However, only Moncrief's formulation reveals explicitly the symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the identity operator.

The above developments suggest that between these otherwise diverse approaches some beautiful connections can be made. However, there is an immediate and basic obstacle. The dynamical equations of Arnowitt et al. are not written in a form that makes explicit use of the symplectic structure, and consequently it is not clear how to use the symplectic geometry alluded to above. Part of the problem is that the equations contain terms involving the lapse function N and the shift vector field X. These terms are necessary because of the arbitrary way the space-time can be sliced into space and time or, in other words, because of the coordinate invariance of the space-time.

The remainder of the essay will explain how the authors have solved this problem using a new form for the evolution equations and how the solution might be useful for future research.

We begin with some notation that will enable us to state the equations. Let M be a 3-manifold and let Q be an open subset of a linear space of C^∞ tensor fields of some specified type on M and let Q^* be the space of dual tensors. For instance, if Q consists of the symmetric covariant two-tensors $\phi = \phi_{ij}$, Q^* is the set of contravariant symmetric two-tensor densities $\pi = \pi^{ij}$. If $L: Q_1 \to Q_2$ is a linear differential operator, $L^*: Q_2^* \to Q_1^*$ denotes its adjoint obtained in the usual manner by integration by parts. If $T: Q_1 \to Q_2$ is a nonlinear differential operator, $DT(\phi)$ denotes its linearization (= Fréchet derivative, or functional derivative) at $\phi \in Q_1$, so $DT(\phi)$ is a linear differential operator from Q_1 to Q_2. We let J denote the symplectic matrix on $Q \times Q^*$ as defined above and let $P = Q \times Q^*$ denote the *phase space*.

Lie differentiation $L_X \phi$ of fields ϕ by vector fields X is a first-order differential operator in X. Its negative adjoint is called the *flux density* $\mathcal{J}$; it may be regarded as a map of P to Λ^1_d, the one form densities (dual to vector fields), and is explicitly defined by

$$\int X \cdot \mathcal{J}(\phi, \pi) = -\int \pi \cdot L_X \phi$$

Let $\mathcal{H}: P \to C^\infty_d$ (scalar densities) be a given Hamiltonian density and define

$$\Phi(\phi, \pi) = (\mathcal{H}(\phi, \pi), \mathcal{J}(\phi, \pi))$$

For general relativity, Q is the space of Riemannian metrics g_{ij} on M, Q^* the symmetric two tensor densities π^{ij}, and (see [3, 4, 11])

$$\mathcal{H}(g, \pi) = [\pi' \cdot \pi' - \tfrac{1}{2}(\mathrm{Tr}\pi')^2 - R(g)]\sqrt{\det g_{ij}}$$

where π' denotes the tensor part of $\pi = \pi' \otimes \mu_g$, $\cdot$ denotes contraction to scalars, tr is trace, and $R(g)$ is the scalar curvature. One calculates that $(g, \pi) = 2\pi_i{}^j{}_{|j}$, twice the covariant divergence of π.

Let us next recall the meaning of the lapse and shift functions of Wheeler (see [17]). Let V be a space-time with a Lorentz metric $^{(4)}g$. Let i_λ be a *slicing* of V by M; i.e., for each number λ, i_λ is an embedding of M to a spacelike hypersurface of V (and these embedded manifolds fill out an open set in V). The λ-derivative of i_λ is a vector field on V defined along the embedded hypersurfaces. The negative of the normal and tangential components, regarded as scalar and vector functions on M, are called the *lapse* N and *shift* X. They depend, of course, on the slicing of the space-time and in fact characterize the slicing.

For vacuum space-times, Einstein's equations state that the Ricci tensor of $^{(4)}g$ vanishes. Misner et al. showed that these equations are equivalent to certain complicated-looking evolution equations and constraint equations (see [3], p. 236 or [17], p. 525) for the 3-metric g_{ij} induced on M by a slicing and its corresponding conjugate momentum π^{ij} (defined to be $[(\mathrm{tr}k)g^{ij} - k^{ij}]\sqrt{\det g_{lm}}$ where k_{ij} is the second fundamental form or extrinsic curvature of the embedded hypersurface regarded as a two-tensor on M).

Our first main point is that these equations can in fact be written in the following compact way (the slicing parameter λ is often denoted t, but it need not be a timelike direction, so we use λ):

$$(E) \quad \frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \end{pmatrix} = J \circ D\Phi(g, \pi)^* \begin{pmatrix} N \\ X \end{pmatrix} \quad \text{[evolution equations]}$$

$$(C) \quad \Phi(g, \pi) = 0 \quad \text{[constraint equations]}$$

Computing the adjoint $D\Phi(g, \pi)^*$ (see [13]) shows that these equations are equivalent to the Arnowitt-Deser-Misner equations.

This new way of writing the equations is of intrinsic interest in itself. However, we claim something more profound: We assert that equations of the same form also apply if there are general (nongravitational) tensor fields present (for example, electromagnetic or matter fields) which are nonderivatively coupled to the gravitational field; i.e., the Hamiltonian for the nongravitational fields does not depend on the spatial derivatives of the metric tensor nor on π. These new equations are

$$(E_T)\quad \frac{\partial}{\partial\lambda}\begin{pmatrix} g, & \phi \\ \pi, & \pi_\phi \end{pmatrix} = J \circ D\Phi_T((g,\phi),(\pi,\pi_\phi))^* \begin{pmatrix} N \\ X \\ \psi \end{pmatrix}$$

$$(C_T)\quad \Phi_T((g,\phi),(\pi,\pi_\phi)) = 0$$

where ϕ represents all dynamical fields other than the metric g, π_ϕ is the conjugate momentum of ϕ, and ψ represents all degenerate fields (other then the lapse N and shift X) which are freely specifiable, $\Phi_T = (\mathcal{H}_{\text{relativity}} + \mathcal{H}_{\text{fields}}, \mathcal{J}_{\text{relativity}} + \mathcal{J}_{\text{fields}}, \mathcal{C})$ is the total Hamiltonian and flux density for the coupled system, and $\mathcal{C}$ represents the constraints associated with the degenerate fields ψ. These equations give a unified Hamiltonian formulation of general field theories coupled to gravity!

We shall now make a series of remarks intended to show the geometric and analytic utility of this new formulation of Hamiltonian equations for field theories.

First of all, the above form of the evolution equations shows explicitly how the dynamical equations are generated by the linearized constraints, and shows explicitly how the equations depend on the slicing. Moreover, the above form shows explicitly the role the symplectic structure plays, i.e., the equations are of the Hamiltonian type (see [9]) and that the symplectic structure J is independent of the slicing.

Secondly, the form (E) allows one to see more easily relationships between properties of the space-time and corresponding conditions on solutions of the constraint equations. For example, the equations (E) simplify the calculations in the proof of Moncrief's criterion which relates linearization stability of solutions of the constraint equations, and hence of the space-time, to the absence of Killing fields on the space-time (see [13, 14, 18]). Very recently this idea has been used by Arms to successfully analyze the linearization stability of the coupled Einstein-Maxwell system.

Thirdly, the equations (E) give a unified picture of decomposition theorems used in relativity. Moncrief's basic decomposition theorem [19] states that the phase space can be decomposed as follows:

$$P = \text{range}\, J \cdot D\Phi(g,\pi)^* \oplus \{\text{kernel}\, D\Phi(g,\pi) \cap [\text{kernel}\, (D\Phi(g,\pi)\circ J)]^*\} \oplus [\text{range}\, D\Phi(g,\pi)^*]^*$$

$$= ① \oplus ② \oplus ③$$

The three summands in this decomposition are shown in Figure 1 below. This decomposition generalizes Deser's [10] classical decomposition of tensors into transverse-traceless and other pieces. In terms of the equations (E), each term

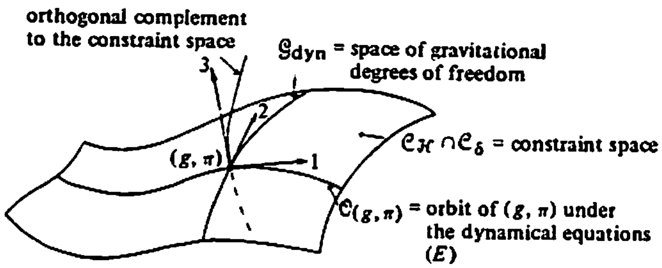

Fig. 1. Geometrical representation of the space of gravitational degrees of freedom.

in the above decomposition can be given a geometrical meaning (see Figure 1), and the decomposition itself becomes a special case of a general fact in symplectic geometry [2]. The present formulation is not merely a restatement; it also shows us how, with no extra effort, to explicitly decompose perturbations of general field theories coupled to gravity!

Finally, the form (E) enables us to give a representation of the space of gravitational degrees of freedom that is directly related to the dynamical equations. We let $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}$ denote the space of solutions of the constraint equations (C). It is known under what conditions $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}$ is a manifold near $(g, \pi) \in \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}$. If we identify all $(g, \pi) \in \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}$ that are induced on all spacelike hypersurfaces of a space-time $(V, {}^{(4)}g)$, or of an isometry class of spacetimes $[{}^{(4)}g]$, which satisfies Einstein's empty space field equations, we obtain a quotient space $\mathcal{G}_{dyn} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}/\sim$. The general theory of reduction of phase spaces with symmetry [16] shows that $\mathcal{G}_{dyn}$ is almost always a smooth symplectic manifold. Moreover, since coordinate transformations of the space-time yield all the different possible slicings, $\mathcal{G}_{dyn}$ represents the space of solutions of Einstein's equations where solutions which are related by a coordinate transformation are identified; i.e., with the space of gravitational degrees of freedom, or mathematically, with the space of isometry classes of solutions to the empty space field equations. The tangent space to $\mathcal{G}_{dyn}$ is exactly the second summand in the above decomposition, showing the natural relationship of the manifolds $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}$, $\mathcal{G}_{dyn} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_{\delta}/\sim$, and the equations (E).

Similar methods of symplectic geometry can be applied to give results for general field theories coupled to gravity. Our new formulation of these coupled systems allows for the organization of deep theorems concerning the structure of the spaces of degrees of freedom in a systematic and unified manner.

Future prospects for the methods described here are bright. There is every reason to believe that a more profound understanding of fields coupled to the purely gravitational field will result. In another direction, there is hope that it will help clarify the quantum gravity problem as well. Admittedly, the solution of the coordinate gauge problem is only a beginning, but its rigorous resolution is still a significant one, for we now have a well-defined symplectic space in which to quantize.

It is gratifying that methods of infinite-dimensional analysis have been so successful in recent years in the analysis of general relativity (see [4-15, 18-23]). It is now time to seriously use the additional machinery provided by the natural symplectic structure of the spaces involved.

Note Added in Proof. Recently, in a milestone series of papers, Kuchař [25, 26] has shown how to construct a generalized Hamiltonian formulation for a general Lagrangian field theory. His work provides the details of the ideal envisioned by Dirac (see [24] and the references therein) of constructing a dynamical formulation of a covariant field theory. Our work compliments his, inasmuch as we assume one has a Hamiltonian formulation of the system under study and begin from there to give a compact formulation of the evolution equations and derive the consequences of this formulation.

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