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WELL-POSEDNESS OF THE EQUATIONS OF A NON-HOMOGENEOUS PERFECT FLUID

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Abstract

The Euler equations for a non-homogeneous, non-viscous incompressible fluid are shown to be well-posed for a short time interval, using techniques of infinite dimensional geometry and a weighted Hodge theorem. Regularity and other properties of these solutions are pointed out as well.

1. Introduction.

In [2], D. Ebin and the author introduced a technique for solving the Euler equations for a perfect (homogeneous, non-viscous, incompressible) fluid based on the use of the group $\mathcal{D}_\mu^s$ of Sobolev class H^s (or $W^{s,p}$ or Hölder class $C^{k+\alpha}$) volume preserving diffeomorphisms. This method originated in an idea of V. Arnold [1].

In the present paper we shall use similar methods to solve the equations with the additional complication of non-homogeneity. These equations are (on a Riemannian manifold M), cf. [3]:

$$
(\underline{1}) \quad \begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v = \dfrac{\text{grad } p}{\rho} & (\nabla = \text{covariant derivative}) \\ \dfrac{\partial \rho}{\partial t} + \text{grad } \rho \cdot v = 0 & \\ \text{div } v = 0 & \\ v \text{ parallel to } \partial M = \text{boundary of } M & \end{cases}
$$

where v is the velocity field of the fluid, $\rho(x,t) > 0$ is the mass density and $p(x,t)$ is the pressure. One is given $v(x,0)$, $\rho(x,0)$ and the problem is to find $v(x,t)$, $p(x,t)$, $\rho(x,t)$ satisfying ($\underline{1}$) .

The key differences with the case of constant ρ are: first, ($\underline{1}$) is a coupled system between ρ and v and second, the corresponding equations on $\mathcal{D}^s_\mu$ are no longer right invariant (as we shall see below). Therefore the equations ($\underline{1}$) are not derivable using the methods of Arnold (cf. [1,4]), although the methods of [2] do apply when suitably modified.

One also gets, as in [2], solutions for non-homogeneous viscous flow (in case of manifolds with no boundary) and strong convergence in H^s or $W^{s,p}$ of the solutions to solutions of ($\underline{1}$) as the viscosity $\nu \to 0$.

2. Statement of the Results.

Concerning equations (1) we have the following which is our main result:

Theorem. *Let* M *be a compact,* C^∞ *, n-dimensional manifold possibly with* (C^∞) *boundary, oriented and having a* (C^∞) *Riemannian metric.*

Let $H^s(TM)$ *[resp.* $W^{s,p}(TM)$, $C^{k+\alpha}(TM)$*] denote the vector fields of Sobolev class* H^s *[resp.* $W^{s,p}$ *, Hölder class* $C^{k+\alpha}$*] and suppose* $s > \frac{n}{2} + 1$ *[resp.* $s > \frac{n}{p} + 1$, $k \geq 1$, $0 < \alpha < 1$*]. Similarly define* $H^s(M) = H^s(\Lambda^\circ(M))$ *to be the real valued functions of class* H^s .

For $v_0 \in H^s(TM)$ *,* $\rho_0 \in H^s(M)$ *[resp.* $v_0 \in W^{s,p}(TM)\ C^{k+\alpha}(TM)$ *,* $\rho \in W^{s,p}(M)\ C^{k+\alpha}(M)$*] with* $\operatorname{div} v_0 = 0$ *and* v_0 *parallel to* ∂M *, and* $\rho_0(x) > 0$ *for all* $x \in M$ *, there is an* $\varepsilon > 0$ *and a unique solution*

$$v(x,t) = v_t(x) \in H^s(TM) \quad [\textit{resp.}\ W^{s,p}(M)\ C^{k+\alpha}(TM)]$$

$$\rho(x,t) = \rho_t(x) \in H^s(M) \quad [\textit{resp.}\ W^{s,p}(M)\ C^{k+\alpha}(M)]$$

of the equations (1) for $-\varepsilon < t < \varepsilon$ *;* v *and* ρ *are at least* C^1 *jointly in* t,x *so the solution is a classical solution. The pressure* p *is* H^{s+1} *[resp.* $W^{s+1,p}$ *,* $C^{k+\alpha}$*] . Also, the* *equations* (1) *define on* $\{v,\rho \in H^s(TM) \times H^s(M) | \text{div } v=0$, v *parallel to* ∂M *and* $\rho(x) > 0$ *for all* $x \in M\}$ [*resp.* $W^{s,p}(TM)\times W^{s,p}(M)$, $C^{k+\alpha}(TM)\times C^{k+\alpha}(M)$] *a strongly continuous (non-linear) local one parameter group (this domain of definition is an open set in a Banach space).*

Further, the (time dependent) flow η_t *of* v_t *is an* H^s [*resp.* $W^{s,p}$, $C^{k+\alpha}$] *diffeomorphism of* M *and we have*

(2) $$\rho(x,t) = \rho_0(\eta_t^{-1}(x))$$

Further properties of the solutions to equations (1) are as follows:

(i) (Conservation of energy):

$\int_M \langle v_t, v_t\rangle \rho_t \, d\mu$ is independent of t (where μ is the measure on M)

(ii) (Conservation of angular momentum): if Y is a vector field on M whose flow F_t is an isometry of M (i.e., $L_Y g = 0$ where g is the given metric on M and L_Y is the Lie derivative), and if ρ_0 is invariant under F_t, then

$$\int_M \langle v_t, Y\rangle \rho_t \, d\mu \text{ is independent of } t$$

(iii) (Covering Theorem): there is an H^s [resp. $W^{s,p}$, $C^{k+\alpha}$] neighborhood U of the identity diffeomorphism and an H^s [resp. $W^{s,p}$, $C^{k+\alpha}$] neighborhood V of 0 in $H^s(TM)$ [resp. $W^{s,p}(TM)$, $C^{k+\alpha}(TM)$] such that for $\eta \in U$ there is a unique $v_0 \in V$ such that if η_t is the flow for the solution v_t of (1) then $\eta_1 = \eta$.

(iv) (Variational Theorem): for U,V as in (iii), and any curve ζ_t of volume preserving diffeomorphisms joining the identity to η and $\zeta_t \in U$, then

$$\int_0^1 \int_M \langle v_t, v_t \rangle \rho_t \, d\mu dt \leq \int_0^1 \int_M \left\langle \frac{d\zeta_t}{dt} \circ \zeta_t^{-1}, \frac{d\zeta_t}{dt} \circ \zeta_t^{-1} \right\rangle \rho_t \circ \zeta_t^{-1} d\mu dt$$

Remarks. 1. In (iv) we only have minimizing among curves in a sufficiently small H^s neighborhood U. (Examples of Hilbert manifolds with weak Riemannian metrics are known such that curves of shorter length than the geodesic curves are obtained if these curves are allowed to go outside such a neighborhood.)

2. One can solve equations (1) if a force term $f(x,t)$ is added. This force $f_t(x)$ should be in $H^{s+\ell}(TM)$ for $\ell \geq 1$ and must be divergence free and parallel to ∂M. We can also allow $\ell = 0$ although the proof is harder (requiring appendix B of [2]).

3. All of the results (i)-(iv) are easily proved from our methods below and the results of [2], [4], [5].

4. In [5], a priori estimates are established which enable one to give an alternative Galerkin type proof of existence and uniqueness for the Euler equations (see [7]

for details). A similar situation holds here. However, the present method has the advantage that the other properties of the solution are obtained automatically and naturally; these require considerably more effort using the method of [7]. This is the case, in particular, for the well-posedness. We note that the solutions are continuous but probably not locally Lipschitz functions of the initial data. (This is true only in Lagrangian coordinates.)

In §3, 4 below we will outline the proof of the main theorem. The idea is similar to that used in [2]: We replace the problem with one of finding geodesics with respect to a weak Riemannian metric on an infinite dimensional manifold of diffeomorphisms. We stick to the H^s case, $W^{s,p}$ and $C^{k+\alpha}$ being similar.

3. A Generalized Hodge Decomposition.

See §7 of [2] and Chapter 7 of [6] for the usual Hodge decomposition. We use the notations of [2].

Let $\sigma \in H^s(M)$ and $\sigma(x) > 0$ (if s is not larger than $n/2 + 1$ a Hodge theorem is also valid but we must suppose σ is continuous in that case).

Define the operator $d^\sigma: H^s(\Lambda^k) \to H^{s-1}(\Lambda^{k+1})$ by

$$d^\sigma(\alpha) = \frac{d(\sigma\alpha)}{\sigma}$$

($H^s(\Lambda^k)$ stands for the k-forms of class H^s.)

Then it is easy to see that

$$(d^\sigma)^2 = 0$$

and that relative to the inner product

$$(\alpha,\beta)_\sigma = \int(\alpha \wedge *\beta)\sigma$$

that δ and d^σ are duals:

$$(\alpha,\delta\beta)_\sigma = (d^\sigma\alpha,\beta)_\sigma$$

A form α is called σ-closed if $d^\sigma\alpha = 0$ and is σ-harmonic if $d^\sigma\alpha = 0$ and $\delta\alpha = 0$. The notions of rallel to ∂M and normal to ∂M are the same as in [2]. Since there exists constants M , $\delta > 0$, such that $\delta \leq \sigma(x) \leq M$, we see that the norm of $(\, , \,)_\sigma$ is equivalent to the usual H^0 norm (with $\sigma = 1$).

Let $\mathcal{H}^s$ denote the space of σ-harmonic fields of class H^s . We have:

<u>Generalized Hodge Decomposition</u>:

(3) $$H^s(\Lambda^k) = d^\sigma\left(\frac{H_n^{s+1}(\Lambda^{k-1})}{\sigma}\right) \oplus \delta(H_t^{s+1}(\Lambda^{k+1})) \oplus \mathcal{H}_\sigma^s$$

where $\oplus$ *denotes direct orthogonal sum in the inner product* $(\, , \,)_\sigma$. *Here* H_n^s *denotes the space of* H^s

forms which are normal to ∂M . *Also,*

$(\mathcal{H}^s_\sigma)_n$ *and* $(\mathcal{H}^s_\sigma)_t$ *are finite dimensional.*

There is a similar result for $W^{s,p}$ *and* $C^{k+\alpha}$.

We also have the decomposition

$$H^s(\Lambda^k) = d^\alpha \frac{H^{s-1}(\Lambda^{k-1})}{\sigma} \oplus \mathcal{C}^s_t(\Lambda^k) \tag{4}$$

where $\mathcal{C}^s_t$ *denotes the coclosed forms* $(\delta\alpha = 0)$ *tangent to* ∂M .

The proof of this follows from the methods of [6], Chapter 7. (The proof of the differentiability of the members of the decomposition follows from the fact that

$$d^\sigma\left(\frac{H^{s+1}_n}{\sigma}\right) = \frac{1}{\sigma}\, d(H^{s+1}_n)$$

is closed in the H^0 topology on H^s as is $\delta(H^{s+1}_t)$, by the usual Hodge theorem.) We shall omit the detailed proof.

As a corollary of decomposition (4), note that any H^s vector field X can be written $X = Y + \frac{\text{grad } p}{\sigma}$ where $\delta Y = 0$, and Y is parallel to ∂M .

4. Outline of the Proof of the Main Result.

We shall consider the H^s case; that for $C^{k+\alpha}$ is similar. Let $\mathcal{D}^s_\mu$ denote the H^s diffeomorphisms of M

which preserve the standard volume element μ . This is a Hilbert manifold, and a topological group; right multiplication is C^∞ . Define the following weak metric on $\mathcal{D}^s_\mu$:

$$(X,Y)_{\rho_0} = \int_M \langle X(x),Y(x)\rangle_{\eta(x)} \rho_0(x)\, d\mu(x)$$

where $X,Y \in T_\eta \mathcal{D}^s_\mu$, the tangent space at η which can be identified with H^s sections of TM covering $\eta \in \mathcal{D}^s_\mu$. That $(\ ,\ )_{\rho_0}$ is smooth in η follows because the metric on M is C^∞ as in [2]. However, the metric $(\ ,\ )_{\rho_0}$ is not right invariant (unless ρ_0 is constant).

Let $\tilde{M}$ be the double of M and consider the manifold $H^s(M,\tilde{M})$ and the map

$$\mathcal{Z}\colon TH^s(M,\tilde{M})|\mathcal{D}^s_\mu = H^s(M,TM)|\mathcal{D}^s_\mu \to T(H^s(M,TM)|\mathcal{D}^s_\mu)$$

defined by $X \mapsto Z\circ X$ where $Z\colon TM \to T^2M$ is the spray derived from the given metric on M . Then $\mathcal{Z}$ is C^∞ .

Now define a map

$$P^{\rho_0}\colon H^s(M,TM)|\mathcal{D}^s_\mu \to T\mathcal{D}^s_\mu$$

by its action $P^{\rho_0}_\eta$ on the fiber over $\eta \in \mathcal{D}^s_\mu$, as

$$P^{\rho_0}_\eta(X) = R_\eta \circ P_e^{\rho_0\circ\eta^{-1}} \circ R_{\eta^{-1}}$$

where $P_e^{\rho_0 \circ \eta^{-1}}$ is the projection of a vector field onto its divergence free part parallel to the boundary with respect to $\sigma = \rho_0 \circ \eta^{-1}$ (in the generalized Hodge decomposition described in §3 above) and where $R_\eta Y = Y \circ \eta$ is right translation by η .

<u>Main Technical Lemma</u>. P^{ρ_0} *is a* C^∞ *mapping.*

This is proved as in [2], appendix A, with the following modifications. We define

$$\bar{d}^{\rho_0} : H^s(M, \Lambda^k) | \mathcal{D}^s \to H^{s-1}(M, \Lambda^{k+1}) | \mathcal{D}^s$$

by

$$\bar{d}_\eta^{\rho_0} = R_\eta \circ d^{\rho_0 \circ \eta^{-1}} \circ R_{\eta^{-1}}$$

(compare [2, appendix A, lemma 2]).
Then $\bar{d}^{\rho_0}$ is a C^∞ mapping (as is $\bar{\delta}_\eta = R_\eta \circ \delta \circ R_{\eta^{-1}}$) . Indeed, for $\alpha \in H^s(M, \Lambda^k)$ covering η ,

$$\bar{d}^{\rho_0}(\alpha) = R_\eta \left(\frac{d(\rho_0 \circ \eta^{-1} \cdot \alpha \circ \eta^{-1})}{\rho_0 \circ \eta^{-1}} \right) = \frac{\bar{d}(\rho_0 \alpha)}{\rho_0}$$

so $\bar{d}^{\rho_0}$ is smooth (by lemma 2 of [2]).

The proof of lemma 3 of [2] does not carry over to the present context because for one thing $(\mathcal{H}^s_{\rho_0\circ\eta^{-1}})_t$ does not consist of C^∞ elements, and depends on $\rho_0\circ\eta^{-1}$. However, we know from [2] that $\ker\bar{\delta}$ and $\operatorname{im}\bar{\delta}$ are subbundles in $H^s_t(M,\Lambda^k)|\mathcal{D}^s$. Also, $\operatorname{im}\bar{d}^{\rho_0} = \frac{\operatorname{im}\bar{d}^{\rho_0}}{\rho_0}$ so $\operatorname{im}\bar{d}^{\rho_0}$ is a subbundle. Hence ([2], lemma 1]) $\ker\bar{d}^{\rho_0}$ is also a subbundle. Therefore, $\ker\bar{d}^{\rho_0}\cap\ker\bar{\delta}$ is a subbundle since $\bar{d}^{\rho_0}$ restricted to $\ker\delta$ is still onto the subbundle $\operatorname{im}\bar{d}^{\rho_0}$ by the Hodge decomposition. Therefore the analogue of lemma 3 is valid.

With these modifications, the proof of the technical lemma is the same as [2, appendix A].

Now we define the map

$$S^{\rho_0}: T\mathcal{D}^s_\mu \to T^2\mathcal{D}^s_\mu$$

by

$$S^{\rho_0}(X) = TP^{\rho_0}(Z\circ X) = TP^{\rho_0}\circ\bar{Z}(X)$$

(where TP = tangent or derivative of P^{ρ_0}).

The above lemma shows that S^{ρ_0} is a C^∞ mapping; furthermore, by a straightforward modification of the proof in [2, §11], we find that S^{ρ_0} is the spray associated to the weak metric $(\ ,\ )_{\rho_0}$.

It follows immediately that S^{ρ_0} has (unique) integral curves $u_t \in T\mathcal{D}^s_\mu$ for any initial conditions $u_0 \in T_{\eta_0}\mathcal{D}^s_\mu$.

To prove the theorem, let u_t be the integral curve on $T\mathcal{D}^s_\mu$ with u_0 covering the identity and $u_0 = v_0$. Let

$$v_t = u_t \circ \eta_t^{-1}$$

where $u_t \in T_{\eta_t}\mathcal{D}^s_\mu$ so η_t is the time dependent flow of the vector field v_t , and η_0 = identity.

Also let (cf. equation (2))

$$\rho_t(x) = \rho_0 \circ \eta_t^{-1}(x)$$

We claim that v_t, ρ_t satisfy equations (1) (conversely given v_t, ρ_t we get η_t and hence u_t so solutions are unique; cf. [2, §15]). The result will follow from this.

By the same proof as [2, theorem 14.4] and the fact that $P_e^{\rho_0 \circ \eta^{-1}}$ is the identity on the divergence free vector fields, we have

$$\frac{dv_t}{dt} = -P_e^{\rho_0 \circ \eta_t^{-1}} (\nabla_{v_t} v_t)$$

$$= -P_e^{\rho_t} (\nabla_{v_t} v_t)$$

$$= -\nabla_{v_t} v_t + \frac{\operatorname{grad} \rho_t g_t}{\rho_t}$$

by the generalized Hodge decomposition. This is the equation for v_t if we set $p_t = \rho_t g_t$. That for ρ_t is seen as follows: Since

$$\frac{d}{dt} \eta_t^{-1} = -T\eta_t^{-1} \circ \frac{d\eta_t}{dt} \circ \eta_t^{-1}$$

and $\frac{d\eta_t}{\alpha t} = u_t$ (as S is a spray), we have

$$\frac{d}{dt} \rho_t = -d\rho_0 \cdot T\eta_t^{-1} \cdot v_t$$

$$= -d(\rho_0 \circ \eta_t^{-1}) \cdot v_t$$

$$= -d(\rho_t) \cdot v_t$$

where we have used the chain rule. This establishes the equation for ρ_t and completes the proof.

5. Regularity of Solutions.

Because we do not have right invariance on $\mathcal{D}_\mu^s$, the regularity theorem ([2, theorem 12.1]), which states in particular that if v_0 is C^∞, so is v_t on the interior of M as long as v_t is defined in H^s, is not immediate in this case. Nevertheless, the result is still true for $\rho_0 \in C^\infty$.

Indeed, an examination of the above proof shows that S^{ρ_0} depends in a C^∞ fashion on ρ_0. The associated exponential map

$$\tilde{E}^{\rho_0}: T\mathcal{D}^s_\mu \to \mathcal{D}^s_\mu$$

therefore also depends smoothly on ρ_0. If $\xi \in \mathcal{D}^s_\mu$ we have

$$(X,Y)_{\rho_0} = (X\circ\xi, Y\circ\xi)_{\rho_0} \circ \xi$$

in place of right invariance. It follows that

$$\tilde{E}^{\rho_0\circ\xi}(X\cdot\xi) = \tilde{E}^{\rho_0}(X)\circ\xi$$

If X and ρ_0 are C^∞, the left side is smooth in ξ, (by the composition lemmas [2, §2]) so the right side is also. Now the argument in theorem 12.1 of [2] applies to yield the result.

Regularity, including at the boundary, can also be proved by the methods of [5].

6. Global Regularity.

It is amusing to note that while global (in time) regular solutions exist for Euler's equations in two dimensions (the Wolibner-Judovich-Kato theorem), this problem remains open for non-homogenous flow in two dimensions, and may well be false. The key fact used in the proof of Wolibner's theorem is conservation of vorticity (Kelvin's circulation theorem). That result is not true for inhomogeneous fluids.

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