

The Existence of Maximal Slicings in Asymptotically Flat Spacetimes

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Abstract. We consider Cauchy data (g, π) on \mathbb{R}^3 that are asymptotically Euclidean and that satisfy the vacuum constraint equations of general relativity. Only those (g, π) are treated that can be joined by a curve of “sufficiently bounded” initial data to the trivial data $(\delta, 0)$. It is shown that in the Cauchy developments of such data, the maximal slicing condition $\text{tr} \pi = 0$ can always be satisfied. The proof uses the recently introduced “weighted Sobolev spaces” of Nirenberg, Walker, and Cantor.

Consider the set \mathcal{C} of spacetimes which are the Cauchy developments of initial data (g, π) on \mathbb{R}^3 which are asymptotically Euclidean and which satisfy the constraint equations [see (3) and (4) below] in the dynamical formulation of general relativity [1]. In 1968, Brill and Deser [2] conjectured that one can maximally slice any such spacetime, i.e. one can find spacelike hypersurfaces on which $\text{tr} \pi = 0$. In a Hamiltonian analysis of general relativity $\text{tr} \pi$ assumes the role of a gauge variable (see for example [12]) and so one would expect that the $\text{tr} \pi = 0$ condition can be met in any such spacetime. Here we prove that the Brill-Deser conjecture is true.

We consider only those (g, π) which can be joined by a curve of “sufficiently bounded” initial data (to be explained later) to flat space $(\delta, 0)$. Thus we are considering the component \mathcal{C}_0 of $(\delta, 0)$ in the set of asymptotically Euclidean solutions of the constraint equations. \mathcal{C}_0 is restricted to those 3-metrics which are derived from Lorentz metrics on \mathbb{R}^4 that are near the “background” Minkowski metric. The set \mathcal{C}_0 is discussed in [7–11].

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In her note [6], Choquet-Brubhat proves a theorem for spacetimes with compact spacelike hypersurfaces which is similar to our step 2 below. She also notes her proof extends to yield the local result for spacetimes with noncompact spacelike hypersurfaces. The authors became aware of [6] after the present work was completed.

We shall prove:

Theorem. Let $(g, \pi) \in \mathcal{G}_0$. Then in the Cauchy development of (g, π) there is a slice on which the trace of the second fundamental form is zero. (Recall that this entails $\text{tr} \pi = 0$).

There is a similar theorem for the component of \mathcal{G} containing any given (g, π) with $\text{tr} \pi = 0$ or in the case of compact hypersurfaces, $\text{tr} \pi / \mu_g = \text{constant}$ (see [6] and [12]). The constant depends on the hypersurface. This theorem is proven similarly to the one in this paper.

The proof requires the use of the weighted Sobolev spaces $M_{p,s}^p$ introduced in [3]. For compact hypersurfaces, the usual Sobolev spaces $W^{s,p}$ will do, as in [9].

Definition. Let $\sigma(x) = (1 + |x|^2)^{1/2}$. For $1 \leq p \leq \infty$, s a nonnegative integer, and $\delta \in \mathbb{R}$, let $M_{p,s}^p(\mathbb{R}^n, \mathbb{R}^q)$ be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^q)$ with respect to the norm

$$\|f\|_{p,s,\delta} = \sum_{|\alpha| \leq s} (\|D^\alpha f\|_{p,\delta+|\alpha|}).$$

The elementary properties of these spaces are discussed in [3, 4].

The important technical result for this paper is

Lemma 1. [5]. Let $n > m$ and $A_\alpha = \sum_{|\alpha| \leq m} \bar{a}_\alpha D^\alpha$ be an elliptic homogeneous operator on \mathbb{R}^n . Suppose we have an elliptic operator $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ on \mathbb{R}^n satisfying for $s \geq m$, $a_\alpha \in C^{-m}$ and

$$\sup |D^\gamma(a_\alpha(x)) \cdot \sigma^{m-|\alpha|+|\gamma|}| < \infty \quad \text{for } |\alpha| < m$$

and

$$\limsup |D^\gamma(a_\alpha(x) - \bar{a}_\alpha) \sigma^{|\alpha|}| < \varepsilon \quad \text{for } |\alpha| = m$$

and $|\gamma| \leq s - m$. Then if $p > n/(n-m)$ and $0 \leq \delta < -m + (p-1)/p$, and ε is sufficiently small, A is an isomorphism between $M_{p,s,\delta}^p$ and $M_{p,s-m,\delta+m}^p$.

Remark. The smoothness condition of the a_α may be relaxed by taking completions in the appropriate Banach space of linear operators. This fact is used implicitly below.

We shall apply Lemma 1 where A is the Laplacian with respect to some asymptotically flat metric on \mathbb{R}^3 . Thus $n=3$ and $m=2$. We assume p and δ are as in the theorem and $s > n/p + 2$. The (g, π) we shall consider will be of the form $g = \delta + h$ with $h \in M_{p,s,\delta}^p$ and $\pi \in M_{p,s-1,\delta+1}^p$ (see [8]). All norms are taken with respect to the flat background metric. Note that for $g \in \mathcal{G}_0$ these norms are equivalent to those induced by g . Note we may take $\text{tr} \pi \in M_{p,s-1,\delta+2}^p$. The topology on the space of initial data is given by the $M_{p,s,\delta}^p$ norms.

Maximal Slicings

The required slicing will be chosen so that the shift vector is a density:

$$\partial g / \partial \lambda = 2N(\pi - \frac{1}{2}(\text{tr} \pi)g)$$

$$\partial \pi / \partial \lambda = -N(\text{Ric}(g) - \frac{1}{2}(\text{tr} \pi)g)$$

$$\mathcal{K}(g, \pi) = (\pi \cdot \pi - \frac{1}{2}(\text{tr} \pi)^2)g$$

$$\delta_p \pi = 0$$

and using $p = \text{tr} \pi / \mu_g = 2$

$$\partial p / \partial \lambda = 2(k \cdot k - p^2)N$$

Step 1. If $p=0$ for some which the dynamics is d.

Proof. Writing $N = 1 + \lambda$

$$\partial p / \partial \lambda = 2k \cdot k + 2(k \cdot k - p^2)N$$

Thus the equation $\partial p / \partial \lambda = 0$ for each λ . Thus for this will be maintained. \square

In what follows we will choose $p=0$ at $\lambda=1$ by choosing

Step 2. (Local Argument) neighborhood V of (g_0, π_0) such that $p=0$ at $\lambda=1$. (E reached by the dynamics).

Proof. Let $F = \mathcal{G}_0 \times \mathcal{M}_{p,\pi}^p$ open set) by

$$F((g, \pi), N) = (\text{the function } N \text{ such that } (g, \pi, N) \in F)$$

Then using smoothness of a smooth mapping. The differential δN is

$$D_N F((g_0, \pi_0), 0) \cdot \delta N = 0$$

where $k_0(\lambda)$ is the evolution of $g_0(\lambda)$.

Since we are only concerned with the local argument from Lemma 1 the smoothness of N is not needed. Thus by the implicit function theorem for $N(g, \pi)$ near 0 and (g, π)

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The required slicing will be determined by a lapse function $N(\lambda, x) = (-g^{00})^{-1/2}$. Letting the shift vector $g_{0i} = X_i(\lambda, x) = 0$, the Einstein Equations read (here π is a density):

$$\partial g / \partial \lambda = 2N(\pi - \frac{1}{2}(\text{tr } \pi)g)(1/\mu_g) = -2Nk \quad (1)$$

$$\begin{aligned} \partial \pi / \partial \lambda = & -N(\text{Ric}(g) - \frac{1}{2}R(g)g)\mu_g + \frac{1}{2}N(\pi \cdot \pi - \frac{1}{2}(\text{tr } \pi)^2)/\mu_g \\ & - 2N(\pi \times \pi - \frac{1}{2}(\text{tr } \pi)\pi)/\mu_g + (\text{Hess } N - g\nabla^2 N)\mu_g \end{aligned} \quad (2)$$

$$\mathcal{H}(g, \pi) = (\pi \cdot \pi - \frac{1}{2}(\text{tr } \pi)^2)/\mu_g - R(g)\mu_g = 0 \quad (3)$$

$$\delta_g \pi = 0 \quad (4)$$

and using $p = \text{tr } \pi/\mu_g = 2 \text{tr } k$, we find from the above equations that

$$\partial p / \partial \lambda = 2(k \cdot k - \nabla^2 N)N. \quad (5)$$

Step 1. If $p=0$ for some λ , we may choose an N such that p is zero for all λ (for which the dynamics is defined).

Proof. Writing $N = 1 + \tilde{N}$ (so that \tilde{N} is close to 0 when N is close to 1), we find

$$\partial p / \partial \lambda = 2k \cdot k + 2(k \cdot k - \nabla^2 \tilde{N})\tilde{N}.$$

Thus the equation $\partial p / \partial \lambda = 0$ may be solved using Lemma 1 for $\tilde{N}(\lambda) \in M_{1,\delta}^{p+1,\delta}$ for each λ . Thus for this choice of $N = 1 + \tilde{N}$ in the dynamics the condition $p=0$ will be maintained. \square

In what follows we show that whatever p equals at $\lambda=0$, we may achieve $p=0$ at $\lambda=1$ by choosing a suitable N . Throughout, we shall take $\partial N / \partial \lambda = 0$.

Step 2. (Local Argument). Let $(g_0, \pi_0) \in \mathcal{C}_0$ and suppose $\text{tr } \pi_0 = 0$. Then there is a neighborhood V of (g_0, π_0) such that if $(g, \pi) \in V$ then there is an $N \in M_{1,\delta}^{p+1,\delta}$ such that $p=0$ at $\lambda=1$. (By a suitable choice of scale, we may assume $\lambda=1$ will be reached by the dynamics.)

Proof. Let $F = \mathcal{C}_0 \times M_{1,\delta}^{p+1,\delta}(\mathbb{R}^3, \mathbb{R}) \rightarrow M_{1,\delta}^{p+1,\delta+2}(\mathbb{R}^3, \mathbb{R})$ be defined (on a suitable open set) by

$$F((g, \pi), N) = \{\text{the function } p \text{ at } \lambda=1 \text{ determined by Equations (1), (2), (5)}\}.$$

Then using smoothness properties of the evolution equations (see [8]), F is a smooth mapping. The derivative with respect to N at $((g_0, \pi_0), 0)$ in the direction δN is

$$D_N F((g_0, \pi_0), 0) \cdot \delta N = \left(\int_0^1 (k_0(\lambda) \cdot k_0(\lambda) - \nabla_1^2) d\lambda \right) \delta N \quad (6)$$

where $k_0(\lambda)$ is the evolution of k_0 for the given (g_0, π_0) and ∇_1^2 is the Laplacian for $g_0(\lambda)$.

Since we are only considering functions that are independent of λ , it follows easily from Lemma 1 that the operator (6) is an isomorphism (see also [9]). Thus by the implicit function theorem we can uniquely solve $F((g, \pi), N) = 0$ for $N(g, \pi)$ near 0 and (g, π) near (g_0, π_0) . This proves step 2. \square

Step 3 (Globalization). Let (g_0, π_0) be joined to (g, π) by a continuous curve $(g(\alpha), \pi(\alpha))$ in \mathcal{C}_0 , $\alpha \in [0, 1]$. Let J be the set of α for which the resulting space time has a maximal slice. Then $0 \in J$ and step 2 shows that J is open. We can always work in a neighborhood of the curve $(g(\alpha), \pi(\alpha))$ so that the evolution times used in step 2, can be chosen to be uniform along the curve.

To show J is closed, let $\alpha_m \in J$ and $\alpha_m \rightarrow \alpha$. Let N_m be the unique lapse functions given by step 2. In order to demonstrate that V_m^2 remains uniformly elliptic and the slices "uniformly spacelike", we may take a sequence of coordinate transformations f_m on the slices S_m chosen so as to keep the eigenvalues of g_m (relative to the flat background metric) bounded away from zero. Since $k_0(m)$ remains uniformly bounded and V_m^2 remains uniformly elliptic for α_m , $m \rightarrow \infty$, the N_m will converge to a function N . This N is the required zero of F .

Thus $J = [0, 1]$ and our proof is complete. \square

Note Added in Proof. The hypotheses of Lemma 1 should include that a_0 is non-positive. In our application, $a_0 = -\int_0^\infty k_0(\lambda) \cdot k_0(\lambda) d\lambda \leq 0$.

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