this way to your examples you generally assumed a Hilbert space $L^2(R^n)$. In field theory one has a Fock space, which is a Hilbert space, but not of the above form. But I suppose that the particular form $L^2(R^n)$ is not necessary for most of the discussion – is this so ?

Pr **Marsden** – Yes. For example, in the Hamiltonian formulation of fluid mechanics the spaces $W^{s,p} = L^s_p$ are very useful.

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DEFORMATIONS OF NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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RESUME

Dans cet article, nous examinons en quel sens la linéarisation d'un système d'équations aux dérivées partielles non linéaire approche le système complet. Nous appliquons ces idées à l'étude des déformations de l'équation de courbure scalaire et des équations d'Einstein en relativité générale, ainsi qu'à l'étude des ensembles de métriques riemanniennes à courbure scalaire donnée. On montre que ces systèmes sont linéairement stables sous des hypothèses très générales ; nous étudions aussi les cas exceptionnels d'instabilité linéaire.

ABSTRACT

In this article we examine in what sense the linearization of a system of nonlinear partial differential equations approximates the full nonlinear system. These ideas are applied to study the deformations of the scalar curvature equation and Einstein's equations of general relativity, as well as the set of metrics wirth prescribed scalar curvature. We show that these systems are linearization stable under general hypotheses ; in the exceptional cases of instability, we study the isolation of solutions.

0 – INTRODUCTION

Let M be a compact manifold, let X and Y be Banach manifolds of maps over M, such as spaces of tensor fields on M and let

$$\Phi : X \to Y$$

be a non-linear differential operator between X and Y ; we assume Φ itself is a differentiable map. Thus for given $y_0 \in Y$,

$$\Phi(x) = y_0 \qquad (1)$$

as an equation for $x \in X$, is a system of partial differential equations. If $x_0 \in X$ is a solution to (1), we will say that a differentiable curve $x(\lambda)$,

$|\lambda| < \delta$, $\delta > 0$, of solutions to (1), $\Phi(x(\lambda)) = y_0$, with $x(0) = x_0$ is a *finite deformation of* x_0.

To each solution $x_0 \in X$ of (1), we let

$$T\Phi(x_0)\,h = 0\ , \quad h \in T_{x_0}X \tag{2}$$

($T\Phi$ is the tangent, or derivative, of Φ and $T_{x_0}X$ is the tangent space to X at x_0), denote the associated system of *linearized equations about* x_0.

A solution $h \in T_{x_0}X$ of (2) is an *infinitesimal deformation* of x_0. Clearly the tangent $h = x'(0)$ at x_0 of every finite deformation $x(\lambda)$ is an infinitesimal deformation. We now ask the converse question :

When is every infinitesimal deformation of x_0 *actually tangent to a finite deformation ?*

When the answer is affirmative, we say that the equations (1) are *linearization stable* at x_0 ; otherwise the equations are *linearization unstable*. From the implicit function theorem, linearization stability at x_0 will hold if $T\Phi(x_0)$ is surjective and its kernel has a closed complement.

In the following sections, we study the linearization stability and instability of the scalar curvature equation of riemannian geometry (part I) and of the Einstein empty space field equations of general relativity (part II). The relationship between these systems is remarked on in II-1 and II-4.

PART I — DEFORMATIONS OF THE SCALAR CURVATURE EQUATION

I-1. The Submanifold of Riemannian Metrics with Prescribed Scalar Curvature

Let M denote a fixed smooth (C^∞) compact connected oriented n-manifold, $n \geqslant 2$. Let $\mathfrak{M}$ denote the space of smooth riemannian metrics on M, S_2 the space of smooth 2-covariant symmetric tensor fields on M, and $C^\infty = C^\infty(M\,;R)$ the smooth real valued functions on M. For $g \in \mathfrak{M}$, let $R(g) \in C^\infty$ denote the scalar curvature of g, and consider the "scalar curvature map"

$$R(\cdot) : \mathfrak{M} \mapsto C^\infty\ , \quad g \mapsto R(g)$$

as a non-linear second order differential operator. A somewhat remarkable property of $R(\cdot)$ is that, locally, it is almost always a surjection :

I-1.1. **Theorem** (Bourguignon-Fischer-Marsden) — *Let* $g \in \mathfrak{M}$ *and suppose that either*

a) $R(g)$ *is not a constant* $\geqslant 0$, *or*

b) *if* $R(g) = 0$, *then* $\mathrm{Ric}(g) \not\equiv 0$ ($\mathrm{Ric}(g) \in S_2$ *is the Ricci tensor of* g).

Then $R(\cdot)$ *maps any neighborhood of* g *onto a neighborhood of* $R(g)$.

Sketch of Proof :

First we enlarge $\mathfrak{M}$ to the space $\mathfrak{M}^s$ of riemannian metrics of Sobolev class H^s, $s > n/2 + 1$ (see Ebin [3] for a description of this space), and consider $R(\cdot) : \mathfrak{M}^s \to H^{s-2}$. $\mathfrak{M}^s$ is an open set in the Hilbert space S_2^s, the space of H^s 2-covariant symmetric tensor fields on M, so by the implicit function theorem it suffices for these H^s spaces to show that the derivative

$$\gamma_g = DR(g) : Tg\mathfrak{M}^s \approx S_2^s \to T_{R(g)}H^{s-2} \approx H^{s-2}$$

is surjective for g which satisfies condition (a) or (b). A classical computation given for example in Lichnerowicz [13] gives

$$\gamma_g(h) = \Delta\,\mathrm{tr}\,h + \delta\delta\,h - h\cdot \mathrm{Ric}(g)$$

where $\mathrm{tr}\,h = g^{ij}h_{ij}$ is the trace of h, $\Delta\,\mathrm{tr}\,h = -g^{ab}(\mathrm{tr}\,h)_{|a|b}$ is the Laplacian of g (here a vertical bar denotes covariant differentiation with respect to g), $\delta\delta h = g^{ia}g^{ib}h_{ij|a|b} = h^{ab}_{\ |a|b}$ is the double covariant divergence, and $h.\mathrm{Ric}(g) = h^{ab}R_{ab}$ is the pointwise contraction of h and $\mathrm{Ric}(g)$.

From elliptic theory (see e.g. Berger-Ebin [1]), γ_g is surjective if the L_2 adjoint

$$\gamma_g^* : H^s \mapsto S_2^{s-2}\ , \quad f \mapsto g\Delta f + \mathrm{Hess}\,f - f\,\mathrm{Ric}(g)\ ,$$

(where $\mathrm{Hess}\,f = f_{|i|j}$ is the Hessian of f) is injective and has injective symbol. γ_g^* clearly has injective symbol ; to show γ_g^* is injective let $f \in \ker \gamma_g^*$ so that

$$g\Delta f + \mathrm{Hess}\,f - f\,\mathrm{Ric}(g) = 0 \tag{1.1}$$

Taking the trace and the divergence of 1.1 yields

$$(n-1)\,\Delta f = R(g)f \tag{1.2}$$

and

$$f\,d(R(g)) = 0 \tag{1.3}$$

If $R(g) = 0$, (1.2) implies $f =$ constant so from (1.1) $f\,\mathrm{Ric}(g) = 0$ implies $f = 0$ since $\mathrm{Ric}(g) \not\equiv 0$.

If $R(g) \not\equiv 0, f \not\equiv 0$, and $f^{-1}(0) = \emptyset$, then 1.3 implies $R(g) = C =$ constant. Integrating 1.2 then gives $C\int f\,d\mu_g = 0$ and since $f^{-1}(0) = \emptyset$, $C = R(g) = 0$.

If $R(g) \not\equiv 0, f \not\equiv 0$, and $f^{-1}(0) \neq \emptyset$, and if $x_0 \in f^{-1}(0)$, then $df(x_0) \neq 0$ follows from 1.1 and 1.2. Thus $f^{-1}(0)$ is an $n-1$ dimensional submanifold ans so $d(R(g)) = 0$ on an open dense set and hence everywhere. Thus $R(g) =$ constant and from 1.2, $R(g) =$ constant > 0.

The C^∞ case requires some additional arguments. One needs to show the the image neighborhood of $R(g)$ can be choosen independent of s ; this is possible because one can construct local right inverses for $R(\cdot)$ by maps independant of s. The idea is similiar to one occuring in Ebin [6] and works quite generally when we have L_2 orthogonal splittings for elliptic operators.

Note : We thank J.P. Bourguignon for pointing out that $\gamma_g^* f = 0$, $f \not\equiv 0$ implies $R(g) =$ constant. Previously we had condition (a) replaced with the condition $R(g) \leqslant 0$.

Remark : If $\mathrm{Ric}(g) = 0$, $\ker \gamma_g^* = \{\text{constant functions on } M\}$, and if (M, g) is isometric to S^n with the metric g_0 of a standard sphere of radius r_0 in $\mathbf{R}^{n+1}$, then $\ker \gamma_g^* = \{\text{eigen functions of } \Delta\}$. We conjecture that these are the only cases for which γ_g is not surjective.

For $g_0 \in \mathcal{M}$, let $\rho = R(g_0)$ and let

$$\mathcal{M}_\rho = \{g \in \mathcal{M} : R(g) = \rho\},$$

denote the set of riemannian metrics with prescribed scalar curvature ρ. We consider the following conditions on M, g_0 and ρ :

A
- a) $\dim M = 2$
- b) $\dim M \geqslant 3$ and ρ is not a constant $\geqslant 0$
- c) $\dim M \geqslant 3$, $\rho = 0$ and $\mathrm{Ric}(g_0) \not\equiv 0$ (*Note.* If $\dim M = 3$, $\mathrm{Ric}(g_0) \equiv 0 \Leftrightarrow g_0$ is not flat).

I-1.2. **Theorem** – *Let* $g_0 \in \mathcal{M}$, $\rho = R(g_0)$, *and* $\mathcal{M}_\rho = \{g \in \mathcal{M} : R(g) = \rho\}$ *if* A(*a*) *or* A(*b*) *hold, then* $\mathcal{M}_\rho$ *is a smooth closed submanifold of* $\mathcal{M}$. *If* A(*c*) *holds, then* $\mathcal{M}_\rho$ *is a smooth submanifold in a neighborhood of* g_0.

Sketch of Proof :

If A(*b*) or A(*c*) holds, the theorem follows from surjectivity of γ_g, the implicit function theorem, and a regularity argument similiar to that in I-1.1.

If $\dim M = 2$, we need only consider the case $\rho =$ constant $\geqslant 0$ since otherwise γ_g is surjective. If $\rho =$ constant > 0, (M, g_0) is C^∞ isometric to a standard 2-sphere S^2 in $\mathbf{R}^3$ of radius $r_0 = \left(\frac{2}{\rho}\right)^{1/2}$. Thus

$$\mathcal{M}_\rho = \theta_{g_0} = \mathcal{D}(g_0) = \{g \in \mathcal{M} : g = \varphi^* g_0 \in \mathcal{D}\}$$

where $\mathcal{D} = \mathrm{Diff}(M)$ is the group of smooth diffeomorphism of M, and θ_{g_0} is the orbit of g_0. By [6], $\mathcal{M}_\rho = \theta_{g_0}$ is then a smooth closed submanifold of $\mathcal{M}$.

If $\rho = 0$, $M = T^2$ and $\mathcal{M}_\rho = \mathcal{F}$, the set of smooth flat riemannian metrics on T^2. From [9], $\mathcal{F}$ is a smooth closed submanifold of $\mathcal{M}$.

Remark : If $\dim M \geqslant 3$ and $\rho = 0$, then under the hypothesis that $\mathcal{F} \neq \phi$, $\mathcal{M}_0$ is also a smooth submanifold of $\mathcal{M}$ (see I.3).

J. Kazdan and F. Warner (see [12] and the references there in) study the equation $R(g) = \rho$ by considering metrics which are conformally equivalent to g. Their interest in this equation is to find which functions are the scalar curvature of some metric rather than the structure of the space $\mathcal{M}_\rho$ and so is somewhat complementary to our considerations.

I.2. Linearization Stability of $R(g) = \rho$

As another application of I.1.1, we have the following result concerning the linearization stability of $R(g) = \rho$.

I.2.1. **Theorem** – *Let* $g_0 \in \mathcal{M}$ *and let* $\rho = R(g_0)$. *Assume that one of the conditions of* (A) *hold. Then the equation* $R(g) = \rho$ *is linearization stable about* g_0 ; *i.e. for any* $h \in S_2$ *satisfying the linearized equation.*

$$DR(g_0) \cdot h = \Delta \operatorname{tr} h + \delta\delta h - h \cdot \mathrm{Ric}(g_0) = 0$$

There exists a C^∞ *curve* $g(\lambda) \in \mathcal{M}$ *such that* $g(0) = g_0$, $g'(0) = h$, *and* $R(g(\lambda)) = \rho$.

Sketch of Proof :

Under conditions A(*b*) and A(*c*), $\ker \gamma_{g_0} = T_{g_0}\mathcal{M}_\rho$ (which implies linearization stability) follows from surjectivity of γ_{g_0} and similiarly of $\dim M = 2$ and ρ is not a constant $\geqslant 0$. If $\dim M = 2$ and $\rho =$ constant $\geqslant 0$, then γ_{g_0} is definitely not surjective. However, an analysis of the symmetric 2-tensors on a flat 2-torus and a standard 2-sphere shows that even in these cases $\ker \gamma_{g_0} = T_{g_0}\mathcal{M}_\rho$.

Remark : Linearization stability does not follow from the manifold structure of the level set $\mathcal{M}_\rho$ alone ; one needs more, viz. $\ker \gamma_{g_0} = T_{g_0}\mathcal{M}_\rho$. As a finite dimensional example of the type of pathology that might develop, consider

$$\Phi : \mathbf{R}^2 \mapsto \mathbf{R}, \quad (x, y) \mapsto x(x^2 + y^2)$$

Then $\Phi^{-1}(0) =$ Y-axis, a submanifold, but since the origin is a critical point for Φ,

$$d\Phi(0, 0) \cdot (h_1, h_2) = 0$$

for all $(h_1, h_2) \in \mathbf{R}^2$. However, if $h_1 \neq 0$ the infinitesimal deformation (h_1, h_2) is not tangent to any finite deformation).

Here the difficulty can be based to the fact that

$$T_{(0,0)}(\text{Y-axis}) = \mathbf{R} \neq \ker d\Phi(0\,,0) = \mathbf{R}^2$$

so that there are non-integrable infinitesimal deformations.

By extending the analysis of the map $R(\cdot)$ to second order one can also prove the following linearization instability result ([9]).

1.1.2. **Theorem** – Let $\dim M \geqslant 3$, $g_0 \in \mathfrak{M}$, $R(g_0) = \rho$ and suppose that either

a) $\mathrm{Ric}(g_0) = 0$

b) $(M\,, g_0)$ is isometric to a standared n-sphere in $\mathbf{R}^{n+1}$ of radius $r_0 = \left(\frac{n(n-1)}{\rho}\right)^{1/2}$. Then $R(g) = \rho$ is linearization unstable about g_0.

Thus the question of linearization stability or instability of $R(g) = \rho$ remains open only if $\dim M \geqslant 3$, $\rho =$ constant > 0, and $(M\,, g_0)$ is not isometric to a standard n-sphere of radius $r_0 = \left(\frac{n(n-1)}{p}\right)^{1/2}$.

1.3. Isolated solution of $R(g) = 0$ and the manifold of metrics with zero scalar curvature

When $R(g_0) = 0$ we have seen from I.1.1. that γ_{g_0} is surjective iff $\mathrm{Ric}(g_0) \not\equiv 0$, and in $\dim M \geqslant 3$, the lack of surjectivity of γ_{g_0} leads to linearization instability of

$$R(g) = 0 \tag{3.1}$$

at g_0. Moreover, if $g_0 = g_F \in \mathcal{F}$ is a flat solution of 3.1, then we assert that there are no non-flat solutions which are near g_F. Thus the flat metrics are an isolated set of solutions to 3.1. This result is somewhat surprising in view of the fact that the scalar curvature is a relatively weak mesure of the curvature.

I.3.1. **Theorem** – *Let $g_F \in \mathcal{F}$. Then there exists a neighborhood $U_{g_F} \subset \mathfrak{M}$ of g_F such that if $g \in U_{g_F}$ and $R(g) \geqslant 0$, then g is also in $\mathcal{F}$.*

Sketch of Proof ([9]) :

Let $d\mu_{g_F}$ denote the volume element of g_F, and define

$$\Phi : \mathfrak{M} \mapsto \mathbf{R}\,, \quad g \mapsto \int R(g)\, d\mu_{g_F}\,.$$

Then the critical points of Φ are those metrics $g \in \mathcal{F}$ such that $d\mu_g = c\, d\mu_{g_F}$ for some constant $c > 0$. At the critical point g_F the second derivative of Φ is given by

$$d^2\Phi(g_F)\cdot(h\,,h) = -\frac{1}{2}\int(\nabla h)^2 d\mu_{g_F} - \frac{1}{2}\int(d\,\mathrm{tr}\,h)^2 d\mu_{g_F} + \int(\delta h)^2 d\mu_{g_F}$$

Let S be a slice at g_F (see [6] for the definition of a slice), and let $\Phi_S = \Phi\,|\,S$. Since $T_{g_F}S = \{h \in S_2 : \delta h = 0\}$,

$$d^2\Phi_S(g_F)\cdot(h\,,h) = -\frac{1}{2}\int(\nabla h)^2 d\mu_{g_F} - \frac{1}{2}\int(d\,\mathrm{tr}\,h^2)\,d\mu_{g_F}$$

Thus $d^2\Phi_S(g_F)\cdot(h\,,h) \leqslant 0$ for all $h \in T_{g_F}S$, and $d^2\Phi_S(g_F)(h\,,h) = 0$ implies $\nabla h = 0$.

It follows that there exists a neighborhood $V \subset S$ of g_F such that $\Phi_S \leqslant 0$ on V and if $g \in V$ and $\Phi_S(g) = 0$, then g is flat.

Now let $U = \mathcal{D}(V) = \{\varphi_g^* \in \mathfrak{M} : \varphi \in \mathcal{D}, g \in V\}$. By Ebin's Slice Theorem [6], U fills out a neighborhood of g_F. Thus if $g \in U$, $R(g) \geqslant 0$, there exists a $\varphi \in \mathcal{D}$ such that $\varphi_g^* \in V$.

Thus $\Phi_S(\varphi_g^*) = \int R(\varphi_g^*)\, d\mu_{g_F} = \int R(g) \circ \varphi\, d\mu_{g_F} \geqslant 0$ and since $\Phi_S \leqslant 0$, on V, $\Phi_S(\varphi_g^*) = 0$ implies φ_g^* is flat. Thus g is also flat.

Using I.3.1, we can now study the set $\mathfrak{M}_0 = \{g \in \mathfrak{M} : R(g) = 0\}$. Let $\mathcal{E}_0 = \{g \in \mathfrak{M} : \mathrm{Ric}(g) = 0\}$, the set of Ricci flat metrics. It is not known if there exist any non-flat Ricci flat metrics on a compact M. However, if $\mathcal{F} \neq \emptyset$, then $\mathcal{E}_0 = \mathcal{F}$ [11].

I.3.2. **Theorem** – *Assume that $\mathcal{F} \neq \emptyset$. Then*

$$\mathfrak{M}_0 = (\mathfrak{M}_0 - \mathcal{F}) \,\dot\cup\, \mathcal{F}$$

is the disjoint union of smooth closed submanifolds of $\mathfrak{M}$; hence $\mathfrak{M}_0$ is itself a smooth closed submanifold.

Proof : From I.1.2, $\mathfrak{M}_0 - \mathcal{E}_0$ is a smooth submanifold of $\mathfrak{M}$. If $\mathcal{F} \neq \emptyset$, $\mathcal{E}_0 = \mathcal{F}$, which is also a smooth closed submanifold of $\mathfrak{M}$. Thus

$$\mathfrak{M}_0 = (\mathfrak{M}_0 - \mathcal{F}) \cup \mathcal{F}$$

is the union of smooth submanifolds of $\mathfrak{M}$, $\mathcal{F}$ closed. From I.3.1, $\mathfrak{M}_0 - \mathcal{F}$ is also closed.

Remark : If $\dim M = 2$, $\mathfrak{M}_0 = \mathcal{F}$, and if $\dim M = 3$, $\mathcal{E}_0 = \mathcal{F}$ so that in these dimensions we can drop the hypothesis that $\mathcal{F} \neq \emptyset$. Note also that we are allowing the possibility that $\mathfrak{M}_0 - \mathcal{F}$ is empty.

Although $\mathfrak{M}_0$ is a submanifold (under the hypothesis that $\mathcal{F} \neq \emptyset$), in $\dim M \geqslant 3$, $R(g) = 0$ is not linearization stable at a flat solution g_F. This "difficulty" is a consequence of the fact that $\ker \gamma_{g_D}$ is larger than $T_{g_F}\mathfrak{M}_0 = T_{g_F}\mathcal{F}$ since the components of $\mathfrak{M}_0$ have different "dimensionalities".

PART II — LINEARIZATION STABILITY OF THE EINSTEIN EMPTY SPACE FIELD EQUATIONS

II.1. The Einstein equations as a Hamiltonian system

We now consider the Einstein empty space field equations of general relativity as a system of non-linear evolution equations. The problem of linearization stability for an evolution equation is interesting only when there are some non-linear constraints on the initial data of the form $\Phi(x) = y_0$, as in the introduction. Then linearization stability of Φ implies that the evolution equation is also linearization stable (see II.2).

Other approches to the problem of linearization stability of the Einstein equations are given by Brill and Deser [2], Choquet-Bruhat and Deser [3] and O'Murchadha and York [14].

Let ${}^{(4)}g$ be a smooth Lorentz metric of signature $-{+}{+}{+}$ on a 4-manifold V. The Einstein empty space field equations are that the Ricci tensor of ${}^{(4)}g$ vanish :

$$\mathrm{Ric}\,({}^{(4)}g) = 0$$

These equations can be converted to a Hamiltonian system as follows : Let M denote a spacelike hypersurface of $(V\,,{}^{(4)}g)$ and let g be the induced riemannian metric on M and k the second fundamental form of the embedded hypersurface M. Assume that M is compact, and let $\mathcal{M}$ be the space of smooth riemannian metrics on M as before, $T\mathcal{M} \approx \mathcal{M} \times S_2$ its tangent bundle, and for $g \in \mathcal{M}$, $T_g^*\mathcal{M} = S^2 \otimes \mu_g$, the space of 2-contravariant symmetric tensor densities (we are using μ_g and $d\mu_g$ interchangeable for the volume element induced by g). Let $T^*\mathcal{M} = \bigcup_{g \in \mathcal{M}} T_g^*\mathcal{M}$ denote the "cotangent bundle" of $\mathcal{M}$.

Here we are taking the dual in the L_2 inner product but use only the closed subspace of such elements continuous in the C^∞ topology, so the dual of S_2 is $S^2 \otimes \mu_g$.

We introduce the De Witt "weak" metric on $\mathcal{M}$ (see [7]) by

$$\mathcal{G}_g(k\,,k) = \int (k \cdot k - \mathrm{tr}\,k)^2)\, d\mu_g$$

for $k \in T_g\mathcal{M} \approx S_2$. The "momentum" conjugate to the "velocity" k is then given by :

$$\mathcal{G}_g^b : T_g\mathcal{M} \mapsto T_g{}^*\mathcal{M}\,, \qquad k \mapsto \mathcal{G}_g^b(k) = (k - (\mathrm{tr}\,k)\,g)\,\mu_g = \pi$$

with inverse

$$\mathcal{G}_g^\# : T^*g\,\mathcal{M} \mapsto Tg\mathcal{M}\,, \qquad \pi \mapsto \pi' - \frac{1}{2}(\mathrm{tr}\,\pi')\,g$$

where π' is the "tensor part" of π ; $\pi = \pi' \otimes \mu_g$.

We define a hamiltonian on $T^*\mathcal{M}$ as follows :

$$H : T^*\mathcal{M} \to \mathbf{R}\,, (g\,,\pi) \mapsto \frac{1}{2}\,\mathcal{G}_g(\mathcal{G}_g^\#(\pi)\,,\mathcal{G}_g^\#(\pi)) - \int R(g)\, d\mu_g$$

$$= \int (\pi' \cdot \pi' - \frac{1}{2}(\mathrm{tr}\,\pi')^2 - R(g))\, d\mu_g$$

$$= \int \mathcal{H}(g\,,\pi)\, d\mu_g$$

Where $\mathcal{H}(g\,,\pi) = \left(\pi' \cdot \pi' - \frac{1}{2}(\mathrm{tr}\,\pi')^2 - R(g)\right) d\mu_g$ is the hamiltonian density.

Using the canonical (weak) symplectic structure on $T^*\mathcal{M}$, (see the article of Chernoff-Marsden in these proceedings) Hamilton's equations are then the evolution equations :

$$(E)\quad \begin{cases} \dfrac{\partial g}{\partial t} = \pi' - \dfrac{1}{2}(\mathrm{tr}\,\pi')\,g \\[2ex] \dfrac{\partial \pi}{\pi t} = (\pi' \times \pi')\,\mu_g - (\mathrm{tr}\,\pi')\,\pi - 2\,\mathrm{Ric} + \dfrac{1}{2}\,R(g)\,g \end{cases}$$

(where $\pi' \times \pi' = (\pi')^{ik}(\pi')_k^j$ is the "cross-product" of symmetric tensors) with initial conditions $g(0) = g_0$, $\pi(0) = \pi_0$. Remarkably, these evolution equations correspond to six of the equations of $\mathrm{Ric}\,({}^{(4)}g) = 0$ in Gaussian normal coordinates originating on M. The remaining four equations are the conditions that every spacelike hypersurface in a Ricci-flat Lorentz manifold must satisfy, viz.

$$(C)\quad \begin{cases} \mathcal{H}(g\,,\pi) = \left(\pi' \cdot \pi' - \dfrac{1}{2}(\mathrm{tr}\,\pi')^2 - R(g)\right)\mu_g = 0 \\[2ex] \delta(g\,,\pi) = \delta_g\,\pi = 0 \end{cases}$$

These equations are thus constraints on the initial date (g_0, π_0). We shall refer to $\mathcal{H}(g\,,\pi) = 0$ as the hamiltonian constraint and $\delta_g\,\pi = 0$ as the divergence constraint.

The relationship between the equations (E), (C) and $\mathrm{Ric}\,({}^{(4)}g) = 0$ can be summarized as follows :

If M is a spacelike hypersurface of a Ricci-flat spacetime ${}^{(4)}g$, then the induced $(g\,,\pi)$ satisfy (C) and if $\{M_t\}$ is slicing of V by geodesically parallel hypersurfaces (Gaussian coordinates), then the induced $(g_t\,,\pi_t)$ on M_t satisfy (E). Conversely, by means of the existence theory for the evolution equations (E) [5], every solution $(g\,,\pi)$ which satisfies the constraint equations (C) generater a Ricci-flat spacetime in a tubular neighborhood of M.

The presence of $R(g)$ in the Hamiltonian constraint is the reason parts I and II of this paper are closely related. In fact the same methods of I can be used to analyze the linearization stability of the equations (C).

II.2. The main idea for the proof of linearization stability

Those (g, π) which satisfy the constraint equations may be regarded as a certain subset $\mathcal{C}$ of $T^*\mathcal{M}$. We will, according to our general method, show that in a neighborhood of points $(g, \pi) \in \mathcal{C}$ that satisfy certain conditions, $\mathcal{C}$ is a smooth submanifold of $T^*\mathcal{M}$ with tangent space $T_{(g,\pi)}\mathcal{C} = (\ker D\mathcal{H}(g,\pi), \ker D\delta(g,\pi))$. This we do in II.3.

Thus if $(h, \omega) \in T_{(g,\pi)}(T^*\mathcal{M}) \approx S_2 \times (S^2 \otimes \mu_g)$ is a solution to the linearized constraint equations

$$D\mathcal{H}(g, \pi)\cdot(h, \omega) = 0 \qquad (2.1)$$

$$D\delta(g, \pi)\cdot(h, \omega) = 0 \qquad (2.2)$$

$(h, \omega) \in (\ker D\mathcal{H}(g, \pi), \ker D\delta(g, \pi)) = T_{(g,\pi)}\mathcal{C}$ and so is tangent to $\mathcal{C}$. Thus there exists a curve $(g(\lambda), \pi(\lambda)) \in \mathcal{C}$ with $(g(0), \pi(0) = (g, \pi)$ and $(g'(0), \pi'(0) = (h, \omega)$.

Now suppose ${}^{(4)}g$ is a solution to the empty space equations

$$\mathrm{Ric}({}^{(4)}g) = 0.$$

We let :

$$D\,\mathrm{Ric}({}^{(4)}g) : S_2(V) \to S_2(V)$$

denote the derivative of the map $\mathrm{Ric}(\cdot)$: {Lorentz metrics on V} $\to S_2(V)$ at ${}^{(4)}g$; the *linearized Einstein equations* are then

$$D\,\mathrm{Ric}({}^{(4)}g)\cdot{}^{(4)}h = 0 \qquad (2.3)$$

for ${}^{(4)}h \in S_2(V)$. A solution ${}^{(4)}h$ of 2.3 is then an *infinitesimal* deformation of ${}^{(4)}g$.

Now, if ${}^{(4)}g$ is Ricci-flat and M is a spacelike hypersurface, then ${}^{(4)}g$ induces a solution (g, π) to the constraint equations on M, and if M_t is a Gaussian normal slicing, ${}^{(4)}g$ induces a solution $(g(t), \pi(t))$ to the evolution equations (E).

Now suppose ${}^{(4)}h$ is an infinitesimal deformation of ${}^{(4)}g$. Then ${}^{(4)}g$ induces a solution (h, ω) to the linearized constraint equations 2.1, 2.2 about (g, π) on M, and also a solution $(h(t), \omega(t))$ to the linearized evolution equations about the solution $(g(t), \pi(t))$.

If (g, π) satisfies certain conditions, then there exists a curve $(g(\lambda), \pi(\lambda)) \in \mathcal{C}$ tangent to (h, ω) at (g, π). From the existence theory of the evolution equations, each of these solutions $(g(\lambda), \pi(\lambda))$ to the constraint equations generates a flow $(g(\lambda, t), \pi(\lambda, t))$ to the evolution equations which pieces together to form a Ricci-flat Lorentz metric ${}^{(4)}g(\lambda)$ in a tubular neighborhood V′ of M. This curve ${}^{(4)}g(\lambda)$ of Lorentz metrics, after possible adjustment by a curve $\psi(\lambda) : V' \to V'$ of diffeomorphisms of V′, is tangent to ${}^{(4)}h$.

II.3. The constraint submanifolds

Let $\mathcal{C}_{\mathcal{H}} = \mathcal{H}^{-1}(0) = \{(g, \pi) \in T^*\mathcal{M} : \mathcal{H}(g, \pi) = 0\}$, and

$$\mathcal{C}_\delta = \delta^{-1}(0) = \{(g, \pi) \in T^*\mathcal{M} : \delta_g\pi = 0\},$$

the solution sets to the Hamiltonian and divergence constraints respectively.

The following theorems are proved by methods similar to those in I.1.2. an I.2.1.

II.3.1. **Theorem** : *Let* $(g_0, \pi_0) \in \mathcal{M}$. *Then the equation* $\mathcal{H}(g, \pi) = 0$ *is linearization stable at* (g_0, π_0) *iff the following condition holds*

$$C_{\mathcal{H}} : (g_0, \pi_0) \notin \mathcal{F} \times \{0\}$$

If condition $C_{\mathcal{H}}$ *is satisfied, then* $\mathcal{C}_{\mathcal{H}}$ *is a smooth submanifold in a neighborhood of* (g_0, π_0).

Remark : Since dim M = 3, $\mathcal{F} = \mathcal{E}_0$ without the assumption of $\mathcal{F} \neq \emptyset$. Note also that the kinetic terms in $\mathcal{H}(g, \pi)$ involving π help us in the sense that $\mathcal{H}(g, \pi) = 0$ is linearization stable at a (g_F, π), $g_F \in \mathcal{F}$, $\pi \neq 0$; whereas the equation $R(g) = 0$ is not linearization stable at g_F.

II.3.2. **Theorem** : *Let* $(g_0, \pi_0) \in \mathcal{C}_\delta$. *Then the equation* $\delta(g, \pi) = 0$ *is linearization stable at* (g_0, π_0) *iff the following condition is satisfied*

C_δ : *if* $L_X g_0 = 0$ *and* $L_X \pi_0 = 0$, *then* $X = 0$; *here* L_X *denotes Lie differentiation with respect to the vector field* X. *If condition* C_δ *is satisfied, then* $\mathcal{C}_\delta$ *is a smooth submanifold in a neighborhood of* (g_0, π_0).

As in I.3, $\mathcal{F} \times \{0\}$ is also a submanifold of $T^*\mathcal{M}$.

Thus

$$\mathcal{C}_{\mathcal{H}} = (\mathcal{C}_{\mathcal{H}} - (\mathcal{F} \times \{0\})) \cup (\mathcal{F} \times \{0\})$$

is the disjoint union of submanifolds. However, because of the kinetic terms in $\mathcal{H}(g, \pi)$, $\mathcal{F} \times \{0\}$ is not an isolated set of solutions to the hamiltonian constraint. In fact, we have :

II.3.3 : **Theorem** : *Let* $(g_F, 0) \in \mathcal{F} \times \{0\}$. *Then in every neighborhood*

$$U_{(g_F, 0)} \subset T^*$$

of $(g_F, 0)$, *there exists a* $(g, \pi) \in U_{(g_F, 0)}$ *such that* $\mathcal{H}(g, \pi) = 0$, $\delta_g \pi = 0$, *but* $(g, \pi) \notin \mathcal{F} \times \{0\}$.

As a consequence of I.3.3., $\mathcal{C}_{\mathcal{H}} - (\mathcal{F} \times \{0\})$ is not closed so that $\mathcal{C}_{\mathcal{H}}$ itself need not be a submanifold. Moreover, since the divergence constraint does not have any isolated solutions, we can conclude that : *There are no isolated solutions to the empty space constraint equations of general relativity.*

In order to insure that $\mathcal{C} = \mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ is a submanifold at those (g, π) which satisfy $C_{\mathcal{H}}$ and C_δ, we need the additional assumption that $\mathrm{tr}\, \pi' =$ constant.

II.3.4. **Theorem** : *Let* $(g_0, \pi_0) \in \mathcal{C}$ *satisfy conditions* $C_{\mathcal{H}}$, C_δ, *and* tr: $\pi'_0 =$ *constant. Then* $\mathcal{C}$ *is a smooth submanifold in a neighborhood of* (g_0, π_0) *with tangent space* $T(g_0, \pi_0)\mathcal{C} = (\ker D\mathcal{H}(g_0, \pi_0), \ker D\delta(g_0, \pi_0))$ *thus the equations* $\mathcal{H}(g, \pi) = 0$, $\delta(g, \pi) = 0$ *are simultaneously linearization stable at* (g_0, π_0).

The above is proved by showing that the intersection $\mathcal{C}_{\mathcal{H}} \cap \mathcal{C}_\delta$ is transversal in a neighborhood of those (g, π) that satisfy the above conditions. We do not know if the $\mathrm{tr}\, \pi' =$ constant condition is necessary ; indeed it would be an important result if this condition could be dropped.

II.4. Integration infinitesimal deformations of the Einstein equations

As explained in II.2, we can use theorem II.3.4 to prove that the Einstein equations are linearization stable about a solution ${}^{(4)}g$ which satisfies certain mild conditions.

II.4.1. **Theorem** : *Let* $({}^{(4)}g, V)$ *be a smooth Lorentz Ricci-flat manifold, and let* ${}^{(4)}h \in S_2(V)$ *be an infinitesimal deformation of* ${}^{(4)}g$,

$$D\,\mathrm{Ric}\,({}^{(4)}g) \cdot {}^{(4)}h = 0.$$

Assume that V *has a compact connected oriented spacelike hypersurface* M *with induced riemannian metric g, second fundamental form k, and momentum* $\pi = (k - (\mathrm{tr}\, k)\, g)\, \mu_g$ *which satisfy the conditions*

a) $(g, \pi) \notin \mathcal{F} \times \{0\}$

b) *If* $L_X g = 0$ *and* $L_X \pi = 0$, *then* $X = 0$

c) $\mathrm{tr}\, \pi' =$ *constant.*

Then there exists a tubular neighborhood $V' \subset V$ *of* M *and a smooth curve* ${}^{(4)}g(\lambda)$ *of Ricci-flat Lorentz metrics on* V' *such that* ${}^{(4)}g(0) = {}^{(4)}g \restriction V'$ *and* ${}^{(4)}g'(0) = {}^{(4)}h \restriction V'$.

Thus under certain weak conditions on the background solution ${}^{(4)}g$ and in a tubular neighborhood of M, a solution of the linearized Einstein equations actually approximates to first order a curve of exact solutions to the non-linear equations. Because these conditions are so weak, presumably any spacetime which has compact spacelike hypersurfaces has a hypersurface M satisfying these conditions and thus is linearization stable in a tubular neighborhood. Moreover, by using standard arguments and by considering the maximal development of the Cauchy data of the curve of spacetime ${}^{(4)}g(\lambda)$ [4], there will be a maximal common development (which approximates the maximal development of ${}^{(4)}g(0)$) for which the spacetime is linearization stable.

Finally we emphasise that the similarity of the linearization stability question of Einstein's equations for a Lorentz metric and of the scalar curvature equation of a riemannian metric is due to the fact that the question for the Einstein equations reduces to the question for the constraint equations ; these in turn are to a large extent dominated by the behavior of the scalar curvature terms for the induced riemannian metric g. Thus because of the dynamical aspect of Lorentz manifolds, the study of Ricci-flat Lorentz manifolds are in some ways more manageable than the study of Ricci-flat riemannian manifolds. Indeed, we have not been able to establish whether or not $\mathrm{Ric}\,(g) = 0$ is linearization stable about a nonflat solution (providing such a metric exists).

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DISCUSSION

Pr **Tarski** – In this lecture and on various previous occasions, the degeneracy of the system of Einstein equations was emphasized. Now, in classical mechanics, there are systems which are be described in terms of parameters which yield degenerate systems, and alternately in terms of parameters which yield nondegenerate systems. Would there be a corresponding (nondegenerate) description of the Einstein system ? Would it have some advantages ?

Pr **Fischer** – There is presumably a corresponding nondegenerate description of the Einstein equations and it would be of great advantage to have such a description at hand. This aspect of the Einstein equations is under very active investigation. Perhaps the most promising approach is that due to A. Lichnerowicz, *J. Math. pures et appl.* **23** (1944) 37-63, Y. Choquet-Bruhat, *Commun. Math. Phys.* **21** (1971) 211-218, and N. O'Murchadha and J.W. York, *J. Math. Phys.* **14** (1973) 1551-1557.

The main idea is to consider the space of (g, π) with $\delta\pi = 0$, $\mathrm{tr}\,\pi = 0$ and (g, π) equivalent to (g, π) if they are conformally equivalent (these includes a coordinate transforma-tion) as the dynamical space of "true degrees of gravitational freedom".

In a related approach using the methods of J. Marsden and A. Weinstein, Reports on *Math. Phys.* **5** (1974) we have been able to show (a publication in preparation) thatthe above quotient space can be obtained by factoring $C_S \cap C_H$ by a suitable symmetry group. The resulting space is a nondegenerate symplectic manifold, and the program now is to construct a non-degenerate hamiltonian system on this space for which the coordinate degeneracies of the Einstein equations have been factored out.

Pr. **Kostant** – Do your results on scalar curvature overlap with results of Kazdan and Warner on scalar curvature ?

Pr **Fischer** – Our results are complementary rather than overlapping. We do not consider the question of whether or not these is a given metric with a prescribed scalar curvature ρ but rather the structure of the space $\mathfrak{M}_\rho = \{g \in \mathfrak{M} : R(g) = \rho\}$ which may possibly be empty. However, Karzdan and Warner have pointed out to us that by utilizing an approximation lemma in the $W^{s,p}$ spaces, local surjectivity of $R(\cdot)$ can be used to prove many of their results concerning what functions can be realized as scalar curvatures in dimensions $\geqslant 2$; see J. Kazdan and F. Warner, A Direct Approach to the Determination of Gaussian and Scalar Curvature Functions, *Inv. Math.* **28** (1975) 227-230.