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DEFORMATIONS OF THE SCALAR CURVATURE

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§0. Introduction and Statement of Results.

Our main goal is to study the relationship between infinitesimal and actual perturbations of the scalar curvature function with respect to a varying metric and to describe when solutions of the linearized equations can be used to approximate solutions of the nonlinear ones. Many of the results are motivated by corresponding questions in general relativity [17]. This paper supplies details and extensions of the results announced in [19] and described in [20]. The starting point of our proofs is the basic work of Ebin [14] and Berger-Ebin [4]. We begin with a few general notions.

Definition. *Let X and Y be topological vector spaces and $F : X \to Y$ a differentiable mapping. We say F is linearization stable at $x_0 \in X$ iff for every $h \in X$ such that $DF(x_0)\cdot h = 0$ there exists a differentiable curve $x(t) \in X$ with $x(0) = x_0$, $F(x(t)) = F(x_0)$ and $x'(0) = h$.*

The implicit function theorem gives a simple criterion for linearization stability as follows.

Criterion. *Let X, Y be Banach spaces, let $F : X \to Y$ be C^1 and suppose $DF(x_0)$: $X \to Y$ is surjective and its kernel splits, i.e. F is a submersion at x_0. Then F is linearization stable at x_0.*

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If F is a submersion at x_0, we note also that $F^{-1}(F(x_0))$ is a submanifold in a neighborhood of x_0 with tangent space the kernel of $DF(x_0)$.

We shall use the above criterion as our basic method, suitably modified for Frechet spaces, except in a few exceptional cases which are dealt with by a direct analysis.

If linearization stability at x_0 fails, we shall investigate to what extent x_0 is isolated among the solutions of $F(x) = F(x_0)$.

It is obvious that if $x(t)$ is an *actual deformation* of the equation $F(x) = y_0$ (i.e., if $F(x(t)) = y_0$, $x(0) = x_0$), then $DF(x_0)\cdot h = 0$, where $h = x'(0)$, i.e., h is an *infinitesimal deformation.* Linearization stability requires that every infinitesimal deformation be tangent to an actual deformation.

To state our main results, we introduce the following terminology: M is a compact C^∞ n-manifold without boundary (the noncompact case is discussed in §3). $\mathfrak{M}$ denotes the space of C^∞ Riemannian metrics on M, with the C^∞ topology. For $g \in \mathfrak{M}$, $R(g)$ denotes the scalar curvature of g (positive on the standard sphere; conventions as in [30]). C^∞ denotes the C^∞ (real valued) functions on M. We let $\mathrm{Ric}(g)$ be the Ricci tensor of g, Δ be the Laplace–DeRham operator ($= -$Laplace–Beltrami) on scalars and $\mathrm{spec}(\Delta)$ its spectrum, including zero.

Fixing $g_0 \in \mathfrak{M}$, and letting $\rho = R(g_0)$ our first main result states the following.

Theorem A. *The mapping $R : \mathfrak{M} \to C^\infty$, $g \mapsto R(g)$ is linearization stable at g_0 if one of the following holds:*

(i) $n = 2$

(ii) $n \geq 3$ *and* $\rho/(n-1)$ *is not a constant* $\lambda \in \mathrm{spec}(\Delta)$

(iii) $n \geq 3$, $\rho = 0$ *and* $\mathrm{Ric}(g_0)$ *is not identically zero.*

In cases (ii) and (iii), R is a submersion at g_0.

If $n \geq 3$ and either $\rho = 0$ and g_0 is Ricci flat or (M, g_0) is a standard sphere of radius $r_0 = (n(n-1)/\rho)^{\frac{1}{2}}$, then R is not linearization stable at g_0. In fact, if $n \geq 3$ and R is not a submersion at g_0, then R is linearization unstable at g_0.

The proof of theorem A, (ii), (iii) uses the basic criterion for linearization stability of $R : \mathfrak{M}^{s,p} \to W^{s-2,p}$, where $\mathfrak{M}^{s,p}$, $W^{s,p}$ are the corresponding Sobolev spaces of maps of class $W^{s,p} (\equiv H^{s,p} = L_p^{\,s})$, together with a regularity argument. Case (ii) was sharpened over an earlier version by some remarks of J. P. Bourguignon.

Based on work of Obata [35], and a suggestion of J. P. Bourguignon, we make the following, which would cover all cases,

Conjecture. *If $n \geq 3$, $\rho = (n-1)\lambda$ where $\lambda \in \mathrm{spec}(\Delta)$, $\lambda \neq 0$ and if R is linearization unstable at g_0, then (M, g_0) is a standard sphere of radius r_0, $r_0^2 = n(n-1)/\rho$. In other words, we conjecture that if $n > 3$ and $\rho = (n-1)\lambda$ where $\lambda \in \mathrm{spec}(\Delta)$, $\lambda \neq 0$ and (M, g_0) is not a standard sphere, then R is a submersion.*

Closely connected with theorem A is the question of whether or not $\mathfrak{M}_\rho = \{g \in \mathfrak{M} \mid R(g) = \rho\}$ is a submanifold.

THEOREM A′. *Fix $\rho \in C^\infty$ and assume one of the following holds:*
(*i*) $n = 2$
(*ii*) $n = 3$ *and ρ is not a positive constant*
or (*iii*) $n \geq 4$, *ρ is not a positive constant and if $\rho = 0$, there exists a flat metric on M. Then $\mathfrak{M}_\rho \subset \mathfrak{M}$ is a smooth (closed) submanifold.*

The only part which does not come directly from the analysis of theorem A are the cases $\rho = 0$ in (i), (ii), (iii) and $\rho =$ constant > 0 in (i) which depend on Theorem B and are stated in more detail in Theorem B′ below.

At flat metrics, R is not linearization stable. In this situation we show that they are isolated among those metrics with non-negative scalar curvature.

THEOREM B. *Suppose $g_0 \in \mathfrak{M}$ is flat. Then there is a neighborhood U in $\mathfrak{M}$ about g_0 such that $g \in U$ and $R(g) \geq 0$ implies g is flat.*

The proof shows that in fact by coordinate changes and scaling, g can be transformed to g_0. (See remark 4 following Theorem 10).

"Second order versions" of Theorem B (i.e. computations of the Hessian of $g \mapsto \int R(g)\, d\mu(g)$) are done in Brill–Deser [8], Kazdan–Warner [25], Ebin–Bourguignon [unpublished], and Berger [3].

Let $\mathfrak{F}$ denote the flat metrics in $\mathfrak{M}$ and as above, $\mathfrak{M}_0$ consists of metrics g with $R(g) = 0$. Another way of expressing the isolation of solutions of $R(g) = 0$ is as follows

THEOREM B′. *If $n \leq 3$ or if $n \geq 4$ and $\mathfrak{F} \neq \varnothing$, then*

$$\mathfrak{M}_0 = (\mathfrak{M}_0 - \mathfrak{F}) \cup \mathfrak{F}$$

is a disjoint union of closed submanifolds.

In Theorem A′, the case when ρ is not a positive constant is proved by showing R is a submersion on $\mathfrak{M}_\rho^{s,p}$. However this is not the case when $\rho = 0$ and so a direct analysis is needed.

The surjectivity of the derivative $DR(g)$ used in the proof of theorem A is also useful in the proof of some results of Kazdan and Warner; see [26]. For example, using those ideas one has easily that if M admits a metric g with $R(g) = 0$ and $\mathrm{Ric}(g) \neq 0$, then any function ρ on M is the scalar curvature of some metric.

In this paper we shall prove theorems A, A′, B and B′ along with a number of related results and extensions.

We thank J. P. Bourguignon for suggesting substantial improvements in Theorem A, M. Cantor for correcting an error in §3, and Y. Choquet-Bruhat, D. Ebin, J. Guckenheimer, J. Kazdan, R. Palais, F. Warner, A. Weinstein and J. A. Wolf for miscellaneous helpful remarks.

§1. Some Preliminaries.

Except for §3, M will denote a C^∞ compact, connected, oriented n-manifold without boundary, $n \geq 2$. $T_q^p(M)$ denotes the vector bundle of tensors of type (p, q) and $C^\infty(T_q{}^p(M))$ the space of C^∞ cross sections; i.e., C^∞ tensor fields. $W^{s,p}(T_q{}^p(M))$ denotes the space of tensor fields of class $W^{s,p}$; i.e., derivatives up to order s are L_p, $s \geq 0$, $1 < p < \infty$. In case $p = 2$ we write H^s for $W^{s,2}$ as usual. (See e.g. Palais [40, 41] for properties of these spaces).

If we just write $W^{s,p}$ or H^s we mean this class of real valued functions. We let

$\mathfrak{X}^{s,p} = W^{s,p}(T_0{}^1(M))$, the $W^{s,p}$ vector fields,

$S_2{}^{s,p} = W^{s,p}(T_{2,\text{symmetric}}{}^0(M))$, the $W^{s,p}$ symmetric 2-covariant tensors

$\mathfrak{D}^{s,p} =$ the group of $W^{s,p}$ diffeomorphisms of M (see Ebin [14, 15] for a discussion); these are well defined if $s > (n/p) + 1$

$\mathfrak{M}^{s,p} \subset S_2{}^{s,p}$ the open subset of Riemannian metrics of class $W^{s,p}$, defined if $s > n/p$

$S_g{}^{s,p} \subset \mathfrak{M}^{s,p}$ denote a slice at g for the action of $\mathfrak{D}^{s+1,p}$ on $\mathfrak{M}^{s,p}$ (see [14]).

and

$\mu(g)$ denote the volume element of a metric g.

If the ^{s,p} is omitted, C^∞ is understood.

A few other notational conventions follow: g^{-1} will denote the contravariant form of g; i.e. in coordinates g^{ij}, the inverse of g_{ij}. For a tensor t, $g^{-1}t$ then raises an index of t and gt lowers one, in the usual way. If all the indices are lowered, we write $t^\flat$ (totally covariant form) and if they are all raised, we write $t^\sharp$. For $h \in S_2$, tr h means the trace of h (this depends on the metric g, but it will not cause confusion). Also, for example if $h \in S_2$, $X \in \mathfrak{X}$, $h\cdot X$ denotes the pointwise contraction, a one form.

We shall also make heavy use of a splitting lemma. Let E, F be vector bundles over M with a fixed riemannian structure (i.e. inner products on the fibers and a volume fixed on M). Let $D : C^\infty(E) \to C^\infty(F)$ be a kth order differential operator, and $D^* : C^\infty(F) \to C^\infty(E)$ its L_2 adjoint (see Palais [40]).

SPLITTING LEMMA (*cf Berger-Ebin* [4]). *Assume D has injective symbol or that D^* has injective symbol. Then*

$$W^{s,p}(F) = \text{range } D \oplus \ker D^*$$

On the right, D is regarded mapping $W^{s+k,p}(E) \to W^{s,p}(F)$ and $D^* : W^{s,p}(F) \to W^{s-k,p}(E)$. Here $\infty \geq s \geq k$, $1 < p < \infty$. If D has injective symbol, ker D *is finite dimensional and consists of C^∞ elements.*

Remarks.

1. This is proved in [4] in case $p = 2$ and D has injective symbol. It is not difficult to give a direct proof of this using the elliptic estimates and Rellich's theorem (see, e.g. [23], [27] in which "elliptic" means "injective symbol"). The main point to be proved in that the range of D is closed. The case of D^* with injective symbol can be deduced from that for D, for if D^* has injective symbol, Range (D) = Range (DD^*) and as DD^* is elliptic, this is closed.

2. If an operator D maps into a product space $F = F_1 \oplus \cdots \oplus F_l$ with different orders in each factor $D_i : C^\infty(E) \to C^\infty(F_i)$ and if one computes the symbol of each D_i separately and each of these is injective, then the basic

elliptic estimates, and hence the splitting lemma still hold. Similarly, if $E = E_1 \oplus \cdots \oplus E_m$, the symbol may be computed as an $m \times l$ matrix of subsymbols. This is a remark of L. Hörmander [23] and is needed in applications; cf. §5.

3. Applying the splitting theorem to the Laplacian on a Riemannian manifold results at once in the Hodge decomposition on forms.

The main consequence of the splitting lemma we shall need is the following: *if D^* is injective and has injective symbol, then D is surjective.*

We note that an injective operator need not have injective symbol as is seen by considering the example $D\psi(x_1, x_2) = \Delta_1\Delta_1\psi(x_1, x_2) + \Delta_2\psi(x_1, x_2) + \psi(x_1, x_2)$ where $\psi : M_1 \times M_2 \to \mathbf{R}$ is defined on the product of two Riemannian manifolds and Δ_1 and Δ_2 are their respective Laplacians.

Some more notation: for $X \in \mathfrak{X}$, $g \in \mathfrak{M}$, $L_X g$ denotes the Lie derivative (in coordinates $(L_X g)_{ij} = X_{i|j} + X_{j|i}$, where a vertical bar denotes covariant differentiation). For $h \in S_2$, $\delta_g h$ denotes the divergence of h; in coordinates (note the sign), $(\delta_g h)^i = -h^{ij}{}_{|j}$. Thus we have two differential operators (notation following Berger–Ebin [4]); for g fixed in $\mathfrak{M}^{s+1,p}$, $s > n/p$,

$$\alpha_g : \mathfrak{X}^{s+1,p} \to S_2^{s,p}; \qquad X \mapsto L_X g$$

$$\delta_g : S_2^{s,p} \to \mathfrak{X}^{s-1,p}; \qquad h \mapsto \delta_g h.$$

It is easy to check that $\alpha_g^* = 2\delta_g$. Also, the symbol of α_g is injective so we get the splitting of Berger–Ebin:

$$S_2^{s,p} = \overset{\circ}{S}{}_2^{s,p} \oplus \alpha_g(\mathfrak{X}^{s+1})$$

where

$$\overset{\circ}{S}{}_2^{s,p} = \ker \delta_g .$$

We refer to this as the *canonical decomposition.*

Note that g is required to be of class $W^{s+1,p}$ (g is assumed C^∞ in [4] but $W^{s+1,p}$ suffices; g of class $W^{s,p}$ is not sufficient in general since for $X \in \mathfrak{X}^{s+1,p}$, $L_X g$ would be only $W^{s-1,p}$; see however §4 below).

We shall write the canonical splitting as

$$h = \overset{\circ}{h} + L_X g. \tag{1}$$

Notice that X in (1) is unique up to a killng field of g, i.e. an X with $L_X g = 0$. A by product of the splitting theorem is a quick proof of the fact that the killing fields form a finite dimensional Lie algebra.

A couple of simple identities will be useful for later purposes. First of all, one computes that

$$\delta(L_X g) = \Delta X + (d\delta X)^\# - 2\,\mathrm{Ric}(g)\cdot X \tag{2}$$

where $\Delta X = ((d\delta + \delta d)X^\flat)^\#$ is the Laplace–de Rham operator on vector fields. Δ is related to the *rough Laplacian* $\bar{\Delta}X = -g^{ik}X^i{}_{|j|k}$ by the Weitzenböck formula (see e.g. Nelson [35]):

$$\Delta X = \bar{\Delta}X + \mathrm{Ric}(g)\cdot X. \tag{3}$$

Secondly, if Hess ψ is the Hessian of $\psi : M \to R$ (in coordinates Hess $\psi = \psi_{|i|j}$), then applying (2) to $X = \operatorname{grad}\psi$, using $L_X g = 2$ Hess ψ yields the identity

$$\delta \text{ Hess } \psi = (d\Delta\psi)^{\#} - \mathrm{Ric}(g)\cdot d\psi. \tag{4}$$

As in Ebin [14], we let $A : \mathfrak{D}^{s+1,p} \times \mathfrak{M}^{s,p} \to \mathfrak{M}^{s,p}$ denote the action by pull back: $(\eta, g) \mapsto \eta^* g$. For $s > n/p$, it is a continuous (right) action. Let $\mathcal{O}_g{}^{s,p}$ denote the orbit through $g \in \mathfrak{M}^{s,p}$. For $g \in \mathfrak{M}^{s+k,p}$, $\mathcal{O}_g{}^{s,p}$ is a C^k submanifold and $T_g\mathcal{O}_g{}^{s,p} = \alpha_g(\mathfrak{X}^{s+1,p})$; i.e. the $L_X g$ piece of (1) is in the direction of isometric changes and roughly speaking, $\overset{\circ}{h}$ is in the direction of true geometric deformations i.e. in the direction of a slice $S_g{}^{s,p}$.

Thus the canonical splitting may be written as $T_g\mathfrak{M}^{s,p} = T_g\mathcal{O}_g{}^{s,p} \oplus T_g S_g{}^{s,p}$.

These same ideas can be used to prove a number of splittings of importance in geometry and relativity and will be a basic tool used in this paper; see Berger–Ebin [4] and §5 for further details).

§2. Local Surjectivity of R.

Let us begin with the following remark:

LEMMA 1. *Let $s > n/p + 1$. Then R maps $\mathfrak{M}^{s,p}$ into $W^{s-2,p}$ and* Ric *maps $\mathfrak{M}^{s,p}$ into $S_2{}^{s-2,p}$ and are C^∞ mappings.*

Proof. The easiest way to see this is to use the local formulas:

$$\begin{aligned} R_{ij} = &-\tfrac{1}{2}g^{ab}\frac{\partial^2 g_{ij}}{\partial x^a\partial x^b} + \tfrac{1}{2}g^{ab}\frac{\partial^2 g_{ai}}{\partial x^b\partial x^j} + \tfrac{1}{2}g^{ab}\frac{\partial^2 g_{aj}}{\partial x^b\partial x^i} \\ &- \tfrac{1}{2}g^{ab}\frac{\partial^2 g_{ab}}{\partial x^i\,\partial x^j} - g_{ab}g^{kl}\Gamma_{ij}{}^a\Gamma_{kl}{}^b + g^{ab}g_{kl}\Gamma_{ai}{}^k\Gamma_{bj}{}^l \end{aligned}$$

and

$$R = g^{ij}R_{ij} \qquad (\text{i.e.}\quad R = g^{-1}\cdot\mathrm{Ric})$$

together with the multiplicative properties of Sobolev spaces (these are conveniently summarized in Palais [40]). For instance, since $W^{s-1,p}$ is a ring, the last two terms of R_{ij} are rational combinations of g and Dg, so are smooth functions of g. Since $s > n/p + 1$, multiplication $W^{s,p} \times W^{s-2,p} \to W^{s-2,p}$ is continuous bilinear, so the first four terms are smooth functions of g as well. ■

Let Δ_L denote the Lichnerowicz Laplacian (see [30]), acting on symmetric two tensors; in coordinates

$$\Delta_L h = -g^{ab}{}_{ij|a|b} + R_{ia}h^a{}_j + R_{ja}h^a{}_i - 2R_i{}^a{}_j{}^b h_{ab}\,.$$

The classical computations (see, e.g. [30]) of the variations of Ric and R give

LEMMA 2. *In Lemma* 1 *the derivatives of* Ric *and* R *are given by*

$$D\,\mathrm{Ric}(g)\cdot h = \tfrac{1}{2}(\Delta_L h - \alpha_g \delta_g h - \mathrm{Hess}\,\mathrm{tr}\,h) \tag{1}$$

and

$$\begin{aligned} DR(g)\cdot h &= -h\cdot \mathrm{Ric}(g) + g^{-1}(D\,\mathrm{Ric}(g)\cdot h) \\ &= \Delta(\mathrm{tr}\,h) + \delta\delta h - h\cdot \mathrm{Ric}(g). \end{aligned} \tag{2}$$

Note. Recall our conventions: $\Delta f = -g^{ab} f_{|a|b}$.

In connection with these formulas, some simple identities are worth noting: $\mathrm{tr}(\Delta_L h) = \Delta(\mathrm{tr}\,h)$, $\mathrm{tr}\,\mathrm{Hess}(\mathrm{tr}\,h) = -\Delta\,\mathrm{tr}\,h$, and $\mathrm{tr}(\alpha_g \delta_g h) = -2\delta\delta h$.

Following notation in Berger–Ebin [4], we let

$$\gamma_g(h) = \Delta(\mathrm{tr}\,h) + \delta\delta h - h\cdot \mathrm{Ric}(g).$$

It is easy to see that

$$\gamma_g{}^* f = g\Delta f + \mathrm{Hess}\,f - f\,\mathrm{Ric}(g). \tag{3}$$

Now we can prove a $W^{s,p}$ version of theorem A, in cases (ii), (iii).

THEOREM 1. *Let* $g \in \mathfrak{M}^{s,p}$, $\infty > s > n/p + 1$ *and assume that either* (*i*) $R(g) = \rho$ *is not a constant* $\lambda(n-1)$ *where* $\lambda \in \mathrm{spec}(\Delta)$, *or* (*ii*) $\rho = 0$ *and* $\mathrm{Ric}(g) \not\equiv 0$. *Then* γ_g *is surjective, its kernel splits and* $R : \mathfrak{M}^{s,p} \to W^{s-2,p}$ *maps any neighborhood of* g *onto a neighborhood of* ρ.

Proof. By the basic splitting theorems, it suffices to show that $\gamma_g{}^*$ is injective with injective symbol, for then $\gamma_g : S_2{}^{s,p} \to W^{s-2,p}$ will be surjective and its kernel will have a closed complement, namely Range $\gamma_g{}^*$. The local surjectivity of R then follows by the implicit function theorem (e.g. see Lang [28]).

The symbol of $\gamma_g{}^*$ is $\sigma_\xi : s \mapsto (-g\,\|\xi\|^2 + \xi \otimes \xi)s$ which is injective if $\xi \neq 0$ and $n \geq 2$, for its trace is $(1-n)\,\|\xi\|^2 s$.

To show $\gamma_g{}^*$ injective, assume $f \in \ker \gamma_g{}^*$, so that

(a) $\gamma_g{}^* f = g\Delta f + \mathrm{Hess}\,f - f\,\mathrm{Ric}(g) = 0$.

Taking the trace yields

(b) $(n-1)\Delta f = R(g) f$,

and combining (a) and (b) yields

(c) $\mathrm{Hess}\,f = (\mathrm{Ric}(g) - (1/n - 1) g R(g)) f$.

First we show that if $f \not\equiv 0$, then $R(g) = (n-1)\lambda$ where λ is a constant in $\mathrm{spec}(\Delta)$. Indeed, taking the divergence of (a) yields

(d) $-(d\Delta f)^{\#} + \delta\,\mathrm{Hess}\,f + df\cdot \mathrm{Ric}(g) - f\delta(\mathrm{Ric}(g)) = 0$.

From (4)§2 and the contracted Bianchi identity $\delta \mathrm{Ric}(g) = -\frac{1}{2}(d(R(g)))^{\#}$, (d) reduces to

$$\tfrac{1}{2} f d(R(g)) = 0.$$

If f is never zero, $d(R(g)) = 0$, so $R(g) =$ constant, and from (b) is $(1/n - 1)$ times an eigenvalue of the Laplacian, and hence ≥ 0.

On the other hand, assume there is some $x_0 \in M$ with $f(x_0) = 0$. We must have $df(x_0) \neq 0$. To see this, assume $df(x_0) = 0$, let $\gamma(t)$ be a geodesic starting at x, and let $h(t) = f(\gamma(t))$. Hence from (c), $h(t)$ satisfies the linear second order differential equation

$$\begin{aligned} h''(t) &= (\text{Hess } f)_{\gamma(t)} \cdot (\gamma'(t), \gamma'(t)) \\ &= \left\{ \left(\text{Ric}(g) - \frac{1}{n-1} gR(g) \right)_{\gamma(t)} \cdot (\gamma'(t), \gamma'(t)) \right\} h(t) \end{aligned}$$

with $h(0) = f(x_0) = 0$ and $h'(0) = df(\gamma(0)) \cdot \gamma'(0) = 0$. Thus f is zero along $\gamma(t)$ and so (by the Hopf–Rinow theorem) f vanishes on all of M. Thus df cannot vanish on $f^{-1}(0)$ so 0 is a regular value of f, and so $f^{-1}(0)$ is an $n - 1$ dimensional submanifold of M. Hence $d(R(g)) = 0$ on an open dense set and hence everywhere.

If condition (i) holds, we can conclude from the above that $f = 0$.

Assume (ii) holds. Then from (b), f = constant and from (a), $f \text{ Ric}(g) = 0$, so $f = 0$. ∎

Note. We thank J. P. Bourguignon for pointing out that $\gamma_g^* f = 0$, $f \neq 0$ implies $R(g)$ = constant. Previously we had assumed $\rho \leq 0$, $\rho \neq 0$.

Remark. If $p > n$, then we can choose $s = 2$ in Theorem 1 and so R hits a whole neighborhood of ρ in L_p.

We now prove that theorem 1 remains valid if $s = \infty$, by a regularity argument, as in [14].

THEOREM 2. *Let $g \in \mathfrak{M}$ and assume (i) or (ii) holds in Theorem 1. Then $R : \mathfrak{M} \to C^\infty$ maps any neighborhood of g onto a neighborhood of $\rho = R(g) \in C^\infty$.*

Proof. From theorem 1 we know that if $k \in$ Range γ_g^* in $S_2^{s,p}$ and λ is small, that $R(g + \lambda k)$ covers a neighborhood of g in $W^{s-2,p}$. What we need to show here is that if $R(g + \lambda k)$ is C^∞, then k is C^∞. However, $R(g + \lambda k) \in C^\infty$ implies $\gamma_g \cdot k \in C^\infty$, by differentiating in λ. If $k = \gamma_g^* f$, then $\gamma_g \gamma_g^* f \in C^\infty$. But from §1, $\gamma_g \gamma_g^*$ is elliptic. Hence f and so k is C^∞. ∎

Following is a list of miscellaneous remarks concerning these results:

1. If $n = 2$, then (ii) of Theorem 1 cannot hold. If $n = 3$, $\text{Ric}(g) \neq 0$ is equivalent to g being not flat. For $n \geq 4$ it is unknown if $\text{Ric}(g) = 0$ implies g is flat, although this implication does hold if M admits some flat metric (Fischer–Wolf [22]).

2. If $\text{Ric}(g) = 0$, then $\ker \gamma_g^*$ consists of constant functions, so γ_g is not surjective in this case. See however §3.

3. If $M = S^n \subset \mathbf{R}^{n+1}$ is the standard sphere of radius r_0 with g_0 the standard metric, $\text{Ric}(g_0) = (n - 1/r_0^2)g$ and $R(g_0) = n(n-1)/r_0^2$. Thus $f \in \ker \gamma_{g_0}^*$ if and only if

$$\text{Hess } f = -\frac{f}{r_0^2} g_0$$

from (c) in the proof of theorem 1. On the standard sphere this is actually equivalent to $\Delta f = (n/r_0^2)f$ (see [5]); i.e. f is an eigenfunction for the first (non-zero) eigenvalue of Δ. Thus γ_{g_0} is not surjective in this case and misses the space $\ker \gamma_{g_0}{}^* = \{f \mid \Delta f = (n/r_0^2)f\}$.

A theorem of Obata [37-38] states that if (M, g) admits a non-trivial solution of

$$\text{Hess}\, f = -c^2 fg \qquad c > 0,$$

then the manifold is isometric to a standard sphere.

We conjecture that if $\text{Hess}\, f = (\text{Ric}(g) - gR(g)/(n-1))f$ *admits a non-constant solution* f *then the space is Ricci flat or is isometric to a standard sphere.*

This is essentially the conjecture of §0. If true, then R would be linearization stable at g_0 iff (M, g_0) is not Ricci flat or a standard sphere, $n \geq 3$ (see Theorems 7, 8, 9). In support of the conjecture, we offer these remarks: (a) If (M, g) is an Einstein space, $\text{Ric}(g) = \lambda g$, $\lambda > 0$, then by Obata's theorem, the conjecture is true.

(b) If $\text{Ric}(g) \neq 0$ is parallel then the conjecture is true by an examination of the proof of Obata's theorem.

(c) If $n = 2$ the conjecture is true.

4. Using local surjectivity of R together with an approximation lemma, Kazdan and Warner [26] have been able to recover some of their results on what functions are realizable as scalar curvatures.

Example. If there exists $g \in \mathfrak{M}$ *such that* $R(g) = 0$, $\text{Ric}(g) \neq 0$, *then* $R : \mathfrak{M} \to C^\infty$ *is surjective.*

Proof. By theorem 1, R is locally surjective. Hence constants of either sign, but near zero, are scalar curvatures. Global surjectivity now follows from the Kazdan–Warner result [25], Theorem 4.3. ∎

5. From Theorems 1, 2 and the implicit function theorem, we have established (ii), and (iii) of Theorem A.

In Theorem 3 we establish part of Theorem A′. Here

$$\mathcal{E}_0^{s,p} = \{g \in \mathfrak{M}^{s,p} \mid \text{Ric}(g) = 0\} \quad \text{and} \quad \infty \geq s > \frac{n}{p} + 1.$$

THEOREM 3. (i) *Let* $\rho \in W^{s-2,p}$ *and assume* ρ *is not a constant* ≥ 0. *Then* $\mathfrak{M}_\rho^{s,p} \subset \mathfrak{M}^{s,p}$ *is a* C^∞ *closed submanifold.*

(ii) $\mathfrak{M}_0^{s,p} - \mathcal{E}_0^{s,p} \subset \mathfrak{M}^{s,p}$ *is a* C^∞ *submanifold.*

In either case, the tangent space at g *is equal to* $\ker DR(g)$.

Proof. Immediate from the fact that R is a submersion on the given spaces, from Theorem 1, with Theorem 2 being used if $s = \infty$. ∎

The case $n = 2$ and the case $\rho = 0$ for (ii) and (iii) of Theorem A′ will be given in §6 and 7.

§3. The Noncompact Case.

We now consider the above results when M is replaced by $\mathbf{R}^n$. Presumably similar results hold for general noncompact manifolds satisfying suitable curvature and completeness restrictions, such as those in Cantor [9].

The results depend on choosing the correct function spaces. We briefly note these here. See Nirenberg–Walker [34] and Cantor [10] for details

Let $|\ |_p$ denote the L_p norm on $\mathbf{R}^n$, $\sigma(x) = \sqrt{1 + |x|^2}$, $\delta \in \mathbf{R}$ and $s \geq 0$ an integer. Let

$$||f||_{s,p,\delta} = \sum_{|\alpha| \leq s} |\sigma^{\delta+|\alpha|} D^\alpha f|_p$$

where D^α is $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ corresponding to a multi-index α.

Let $M_{s,\delta}^p(\mathbf{R}^n, \mathbf{R}^m)$ be the completion of $C_0^\infty(\mathbf{R}^n, \mathbf{R}^m)$ (C^∞ functions with compact support) in the norm $||\cdot||_{s,p,\delta}$.

One can establish various Sobolev type inequalities for these spaces analogous to the usual ones. The spaces are primarily designed so the following holds:

Lemma 1. *For $p > n/(n-2)$, $1/p + 1/p' = 1$, $-(n/p) < \delta < -2 + n/p'$, the Laplacian is an isomorphism (onto)*

$$\Delta : M_{s,\delta}^p \to M_{s-2,\delta+2}^p.$$

Proof. See Cantor [10], Nirenberg–Walker [34]. ∎

Similar results hold for other elliptic operators. Letting $\mathfrak{M}_{s,\delta}^p$ denote the metrics of class $M_{s,\delta}^p$, we have:

Theorem 4. *Assume $n \geq 3$, $s > n/p + 1$, $\delta \geq 0$ and the inequalities of lemma 1 hold. (It is possible to choose $\delta \geq 0$ if $p > n/(n-2)$). Then*

$$R : \mathfrak{M}_{s,\delta}^p \to M_{s-2,\delta+2}^p$$

covers a neighborhood of zero. The equation $R(g) = 0$ is linearization stable about the Euclidean metric in these spaces.

Proof. That R is a C^∞ map and has derivative γ_{g_0} is proved as before. Here g_0 is the standard metric on $\mathbf{R}^n$.

We claim the following splitting holds:

$$M_{s-2,\delta+2}^p = \text{Range } \gamma_{g_0} \oplus \ker \gamma_{g_0}^*$$

(cf. §1 and a corresponding splitting in Cantor [10]).

To prove this, let $f \in M_{s-2,\delta+2}^p$ and let $k = \Delta^{-1} f \in M_{s,\delta}^p$. Then Hess $k \in M_{s-2,\delta+2}^p$ since Hess k consists just of second derivatives of k. Let $h = (g_0 f + \text{Hess } k)/(n-1) \in M_{s-2,\delta+2}^p$ and $h_1 = \Delta^{-1} h \in M_{s,\delta}^p$ and $f_1 = f - \gamma_{g_0} h_1 \in M_{s-2,\delta+2}^p$. One checks $\gamma_{g_0}^* f_1 = 0$, so we have our splitting. (General considerations of functional analysis show that the splitting is topological as well as algebraic, as in §1; cf. [4]).

Thus, as in Theorem 1, what we need to do is check that $\gamma_{g_0}^*$ is injective. But $\gamma_{g_0}^* f = g_0 \Delta f + \text{Hess } f = 0$ implies $\Delta f = 0$, so as Δ is an isomorphism, $f = 0$. ∎

We thank M. Cantor for correcting an error in an earlier version of this proof.

Remarks and Examples. 1. Notice the difference with Theorem 1 (ii). In the noncompact case with the proper asymptotic conditions Δ is an isomorphism; thus γ_{g_0} is surjective even if $\mathrm{Ric}(g_0) = 0$. In the compact case Δ is not an isomorphism and γ_{g_0} is not surjective if $\mathrm{Ric}(g_0) = 0$.

2. It follows that any function near zero is a scalar curvature. This is related to the non-compact results of Kazdan and Warner [24]. However the metrics constructed by the methods here, being in a C^1 neighborhood of the standard metric, are necessarily complete. On $\mathbf{R}^2$ a characterization of when a function is the scalar curvature of a metric can be given. See [24].

3. The following corollary partially answers a question raised by N. O'Murchadha and J. W. York [39]: *On* $\mathbf{R}^3$ *there exists a* C^∞ *riemannian metric* g *which is* $\delta_{ij} + O(1/r)$ *at* ∞, and has scalar curvature $\rho \geq 0$, ρ *is* C^∞ *of compact support and* $\rho \not\equiv 0$ (*precisely, in the function spaces above, choose* $n = 3$, $p > 3$ and $\delta < 2p - 3/p$; *then* $M_{s,\delta}{}^p$ *includes functions* $O(1/r)$ *at* ∞; *i.e. no higher order asymptotic behaviour, like* $O(1/r^2)$ *can be expected*).

4. For compact manifolds, the flat metrics are isolated amongst the metrics g with $R(g) \geq 0$ (see §8). The above shows that this is not true in the noncompact case. It is true that the flat metrics are isolated amongst Ricci flat metrics on $\mathbf{R}^3$.

5. It would be interesting to prove the slice theorem and the results of §4 for the noncompact case and the case of a manifold with boundary.

§4. The Space of Flat Riemannian Metrics.

We return again to the case of M compact, and study the space $\mathfrak{F}^{s,p}$ of flat metrics of class $W^{s,p}$. For $g \in \mathfrak{M}^{s,p}$, ler $\Gamma(g)$ denote its Levi-Cevita connection. Let $\mathfrak{K}^{s,p}$ denote the set of flat riemannian connections of class $W^{s,p}$. Thus, if $s > n/p+1$, $g \in \mathfrak{F}^{s,p}$ implies $\Gamma(g) \in \mathfrak{K}^{s-1,p}$.

The following regularity theorem will be basic:

THEOREM 5. *Let* $s > n/p + 1$ *and* $\Gamma \in \mathfrak{K}^{s-1,p}$. *Then there exists* $\phi \in \mathfrak{D}^{s+1,p}$ *such that* $\phi^*\Gamma \in \mathfrak{K}$; *i.e.* $\phi^*\Gamma$, *the pull back of* Γ *by* ϕ *is a* C^∞ *flat riemannian connection. Similarly, if* $g \in \mathfrak{F}^{s,p}$, *there exists a* $\phi \in \mathfrak{D}^{s+1,p}$ *such that* $\phi^*g \in \mathfrak{F}$.

We thank Alan Weinstein for pointing out the following proof. We first prove a local version:

LEMMA. *Let* $\Gamma \in \mathfrak{K}^{s-1,p}$. *Then the coordinate change to normal coordinates is of class* $W^{s+1,p}$. (*One could use* C^k *spaces here as well.*)

Proof. Let $\Gamma_{jk}{}^i$ be the Christoffel symbols of Γ in a coordinate system x^j, and let $\bar{x}^i(x^j)$ be the coordinate change to normal coordinates so that $\bar{\Gamma}_{jk}{}^i = 0$. Thus from the transformation rules for the Christoffel symbols,

$$\frac{\partial^2 \bar{x}^i}{\partial x^j\, \partial x^k} = \frac{\partial \bar{x}^i}{\partial x^a}\, \Gamma_{jk}^{a}\,, \tag{1}$$

and the Christoffel symbols $\Gamma_{jk}{}^a$ are of class $W^{s-1,p}$.

We know by construction of normal coordinates from the exponential map that $\bar{x}^i(x^j)$ has the same differentiability as the Christoffel symbols (see e.g. Lang [28, p. 96]); thus $\bar{x}^i(x^j)$ is of class $W^{s-1,p}$ and $(\partial \bar{x}^i/\partial x^a)$ is of class W^{s-2}. Thus from multiplication properties of Sobolev spaces, the right hand side of (1) is also of class $W^{s-2,p}$. Thus $\bar{x}^i(x^k)$ is of class $W^{s,p}$, and from (1) again is actually of class $W^{s+1,p}$. ∎

Now we prove the first part of Theorem 5. Using the exponential map we get a new differentiable structure on M in which the connection Γ is smooth. Call this manifold M_1. The identity is a map of class $W^{s+1,p}$ (by the lemma) so can be $W^{s+1,p}$ approximated by a C^∞ diffeomorphism $f : M \to M_1$. Pulling back Γ on M_1 by f gives a C^∞ flat connection on M which is $W^{s,p}$ close and $W^{s+1,p}$ diffeomorphic to the original connection. ∎

Recall from Wolf [45], Theorem 3.3.1 that all C^∞ flat riemannian connections on M are affinely equivalent. This, together with Theorem 5 gives:

COROLLARY 1. *Let $s > n/p + 1$ and $\Gamma_1, \Gamma_2 \in \mathcal{K}^{s-1,p}$. Then Γ_1, Γ_2 are affinely equivalent; i.e. there is $\phi \in \mathfrak{D}^{s+1,p}$ with $\phi^*\Gamma_1 = \Gamma_2$.*

We recall that in order for the canonical splitting to hold at g in $\mathfrak{M}^{s,p}$, g had to be of class $W^{s+1,p}$ (see §0). However if g is flat we can use theorem 5 to transport the splitting and the slice (see Ebin [14]) of a C^∞ metric to deduce the following:

COROLLARY 2. *Let $g \in \mathfrak{F}^{s,p}$, $s > n/p + 1$. Then there is a slice S at g. Also, the canonical decomposition $S_2^{s,p} = \overset{\circ}{S}_2{}^{s,p} \oplus \alpha_g(\mathfrak{X}^{s+1,p})$ is valid at g.*

For $h \in S_2$, $g \in \mathfrak{M}$ let $g^{-1} \circ h$ denote h regarded as an endomorphism of T_xM at each $x \in M$; in coordinates $g^{-1} \circ h$ is $h^i{}_j$. Set, for $g \in \mathfrak{M}^{s,p}$

$$E_g : S_2^{s,p} \to \mathfrak{M}^{s,p}; \qquad h \mapsto g \exp(g^{-1} \circ h).$$

For $\Gamma \in \mathcal{K}^{s-1,p}$, set

$$\mathfrak{F}_\Gamma{}^{s,p} = \{g \in \mathfrak{F}^{s,p} \mid \Gamma(g) = \Gamma\}$$

and

$$S_{2,\parallel}{}^{s,p} = \{h \in S_2^{s,p} \mid \nabla h = 0\}$$

where ∇ is the covariant derivative of Γ.

THEOREM 6. *For $s > n/p + 1$, $\mathfrak{F}^{s,p} = \mathfrak{D}^{s+1,p}(\mathfrak{F}_\Gamma{}^{s,p})$ and $\mathfrak{F}_\Gamma{}^{s,p}$ are closed submanifolds of $\mathfrak{M}^{s,p}$. Moreover, for $g \in \mathfrak{F}_\Gamma{}^{s,p}$,*

$$T_g\mathfrak{F}_\Gamma{}^{s,p} = S_{2,\parallel}{}^{s,p}$$

and

$$T_g\mathfrak{F}^{s,p} = S_{2,\parallel}{}^{s,p} \oplus \alpha_g(\mathfrak{X}^{s+1,p}).$$

Proof. That $\mathfrak{F}^{s,p} = \mathfrak{D}^{s+1,p}(\mathfrak{F}_\Gamma{}^{s,p})$ follows from Corollary 1 of Theorem 5.

From Ebin [14], $E_g : S_2^{s,p} \to \mathfrak{M}^{s,p}$ is a C^∞ diffeomorphism. But for $g \in \mathfrak{F}_\Gamma^{s,p}$,

$$E_g(S_{2,\|}^{s,p}) = \mathfrak{F}_\Gamma^{s,p}$$

($\subset$ follows using coordinates in which g and h are constant. To prove $\supset$, let $g_1 \in \mathfrak{F}_\Gamma^{s,p}$, $g_1 = g \exp(h)$. In a coordinate system in which g is constant, the Christoffel symbols of $g \exp(h)$ must be zero. Hence $\exp(h)$ and thus h are constant.) Thus, as $S_{2,\|}^{s,p}$ is a closed linear subspace, $\mathfrak{F}_\Gamma^{s,p}$ is a smooth manifold, with tangent space as claimed.

Now we need to show that the orbit $\mathfrak{D}^{s+1,p}(\mathfrak{F}_\Gamma^{s,p})$ is a smooth manifold. We follow the ideas in Ebin [14] for this result. First we need:

LEMMA 1. *Let g_1, $g_2 \in \mathfrak{F}_\Gamma^{s,p}$. Then the isometry groups of g_1, g_2 have the same Lie algebra and hence the same component of the identity.*

Proof. The Lie algebra of the isometry group of g, $\{X \in \mathfrak{X}^{s+1,p} \mid L_X g = 0\}$, is the Lie algebra of the holonomy group, which depends only on the connection of g. ∎

Denote this Lie algebra by $\mathcal{I}_\Gamma^{s+1,p}$.

We also use the following version of the implicit function theorem which is proved in the usual way (as in Lang [28]). We thank J. Guckenheimer for discussions on this point.

LEMMA 2. *Let $f : N \to P$ be a C^r map of Banach manifolds such that* ker *$Tf \subset TN$ is a C^r subbundle of TN (with a local closed complement), and at each $x \in N$, Range $T_x N$ is closed with a closed complement.*

Then $f(N) \subset P$ is a C^r (locally) immersed submanifold. If f is an open map onto its image, then $f(N)$ is a submanifold.

Now we apply lemma 2 to the map

$$\Psi : \mathfrak{D}^{s+1,p} \times \mathfrak{F}_\Gamma^{s,p} \to \mathfrak{M}^{s,p}; \qquad (\eta, g) \mapsto \eta^* g.$$

Then as in [14], Ψ is a C^∞ map because of the regularity theorem 5, and

$$T_{(\eta,g)}\Psi(X, h_\|) = \eta^*(h_\| + L_{X_\eta} g)$$

where $X_\eta = X \circ \eta^{-1}$, a vector field on M.

The range of $T_{(\eta,g)}\Psi$ is closed by the splitting theorems and, since the space of $h_\|$ is finite dimensional, it is complemented.

Now if $h_\| + L_X g = 0$, $h_\| = 0$ and $L_X g = 0$ since the summands are L_2 orthogonal. Thus the kernel of $T_{(\eta,g)}\Psi$ is $\{(X, h_\|) \mid X_\eta \in \mathcal{I}_\Gamma^{s+1,p}, h_\| = 0\}$. Thus the kernel is exactly the Lie algebra $\mathcal{I}_\Gamma^{s+1,p}$ made right invariant on $\mathfrak{D}^{s+1,p}$ cross the zero bundle on $\mathfrak{F}_\Gamma^{s,p}$. Again, by theorem 5, $\mathcal{I}_\Gamma^{s+1,p}$ consists of C^∞ objects when acted on by a suitable diffeomorphism, and it is finite dimensional. This is enough to guarantee we have a smooth subbundle of $T\mathfrak{D}^{s+1,p}$ (see [16], appendix A for some similar arguments which yield the result stated here).

Thus by lemma 2, the image of Ψ, namely $\mathfrak{F}^{s,p}$ is an immersed submanifold.

However one sees, as in Ebin [14], that the map Ψ is open onto its image so that the image is actually a closed submanifold. ∎

Consider the obvious projection $\pi : \mathfrak{F}^{s,p} \to \mathfrak{K}^{s-1,p}$. It has fiber $\mathfrak{F}_\Gamma{}^{s,p}$ over $\Gamma \in \mathfrak{K}^{s-1,p}$ and the above theorem then shows that $\mathfrak{F}^{s,p}$ is a homogeneous fiber bundle over $\mathfrak{K}^{s-1,p}$ with finite dimensional fiber $\mathfrak{F}_\Gamma{}^{s,p}$.

This bundle is not trivial however because an affine diffeomorphism need not be an isometry (i.e. in lemma 1, we need not have equality of the full isometry groups).

We conclude with a few brief remarks. Let $\widetilde{\mathfrak{M}}_\rho{}^{s,p}$ denote the orbit of $\mathfrak{M}_\rho{}^{s,p}$ under $\mathfrak{D}^{s+1,p}$. Then $\widetilde{\mathfrak{M}}_\rho{}^{s,p}$ may not be a submanifold of $\mathfrak{M}^{s,p}$ (compare Theorem 6). Indeed, the formal tangent space at $g \in \mathfrak{D}_\rho{}^{s,p}$ is $T_g\widetilde{\mathfrak{M}}_\rho{}^{s,p} = T_g\mathfrak{M}_\rho{}^{s,p} + \alpha_g(\mathfrak{X}^{s+1})$ and the second summand is not defined if g is only $W^{s,p}$.

We conjecture that $\widetilde{\mathfrak{M}}_\rho$ is a submanifold of $\mathfrak{M}$. This is because the orbit map $\Phi : \mathfrak{D} \times \mathfrak{M}_\rho \to \mathfrak{M}$ has derivatives whose ranges and kernels are locally isomorphic; i.e. $T_g\mathfrak{M}_\rho + \alpha_g(\mathfrak{X}) \cong T_{\bar{g}}\mathfrak{M}_\rho + \alpha_{\bar{g}}(\mathfrak{X})$ which follows from the fact that $T_g\mathfrak{M}_\rho \cap \alpha_g(\mathfrak{X}) \cong \mathfrak{X}_\rho/\mathfrak{I}_g$ where $\mathfrak{X}_\rho = \{X \in \mathfrak{X} \mid L_X\rho = 0\}$ and where $\mathfrak{I}_g$ is the Lie algebra of Killing fields of g. The problems of differentiability mentioned above should disappear as $s \to \infty$.

§5. Splittings of S_2.

We shall now discuss some splittings of S_2 related to $\mathfrak{M}_\rho$ and other submanifolds of $\mathfrak{M}$ which are discussed in Barbance [1], Deser [13], Berger–Ebin [4], York [46–47], and Moncrief [32].

We begin with splittings based on the operator γ_g using the basic splitting lemma of §1.

The following formulas are useful:

$$\begin{aligned} DR(g)\cdot L_Xg &= L_X(R(g)) \\ D\,\mathrm{Ric}(g)\cdot L_Xg &= L_X(\mathrm{Ric}(g)) \end{aligned} \tag{1}$$

where X is a vector field and L_X is the Lie derivative. These formulas follow at once from $\eta^*(R(g)) = R(\eta^*g)$ and $\eta^*(\mathrm{Ric}(g)) = \mathrm{Ric}(\eta^*g)$.

Notice that any infinitesimal deformation of $R(g) = \rho$ of the form $h = L_Xg$ can be immediately integrated to $\phi_t{}^*g$, where ϕ_t is the flow of X, as follows from (1).

Since $\gamma_g{}^*$ has injective symbol, (Theorem 1, **§2**), $S_2{}^{s,p}$ splits as

$$S_2{}^{s,p} = \ker \gamma_g \oplus \gamma_g{}^*(W^{s+2,p}), \qquad g \in \mathfrak{M}^{s+2,p}$$

as in Berger–Ebin [4]. Explicitly, we write this as

$$h = \bar{h} + (g\Delta f + \mathrm{Hess}\, f - f\,\mathrm{Ric}(g)) \tag{2}$$

where

$$\Delta(\mathrm{tr}\,\bar{h}) + \delta\delta\bar{h} - \bar{h}\cdot\mathrm{Ric}(g) = 0$$

The portion $\tilde{h}$ is tangent to the submanifold $\mathfrak{M}_\rho$ (if conditions of Theorem 2 hold), and the remaining piece is L_2 orthogonal to it; i.e. $T_g\mathfrak{M} = T_g\mathfrak{M}_\rho \oplus \text{Range } (\gamma_g^*)$.

We can further split $\tilde{h}$ into two parts; $\tilde{h} = \overset{\circ}{\tilde{h}} + L_X g$ according to the canonical decomposition, with $L_X g$ being "inessential" in that it is tangent to the orbit through g. If $R(g) = \rho = \text{constant}$, then $\gamma_g(L_X g) = 0$ by (1), i.e. Range $\alpha_g \subset \ker \gamma_g$, so in this case we have the finer splitting of Berger–Ebin [4]:

$$
\begin{aligned}
(3) \qquad S_2 &= (\ker \gamma_g \cap \ker \delta_g) \oplus (\ker \gamma_g \cap \alpha_g(\mathfrak{X})) \oplus \text{Range } \gamma_g^* \\
&= (\ker \gamma_g \cap \ker \delta_g) \oplus \alpha_g(\mathfrak{X}) \oplus \text{Range } \gamma_g^*.
\end{aligned}
$$

explicitly written

$$
\begin{gathered}
h = \overset{\circ}{\tilde{h}} + L_X g + g\Delta f + \text{Hess } f - f \text{ Ric}(g), \\
\text{where } \delta\overset{\circ}{\tilde{h}} = 0 \text{ and } \Delta \text{ tr } \overset{\circ}{\tilde{h}} - \overset{\circ}{\tilde{h}} \cdot \text{Ric}(g) = 0
\end{gathered}
$$

This finer splitting may be written $T_g\mathfrak{M}_\rho = T_g(\mathfrak{M}_\rho \cap S_g) \oplus T_g\, \mathcal{O}_g$. It is not hard to show that $\mathfrak{M}_\rho \cap S_g$ is a manifold under the conditions of Theorem 2 by restricting R to the slice S_g.

Lemma. *If g is Einstein, $\text{Ric}(g) = \lambda g$, $\lambda \neq 0$ then $\text{tr}(\overset{\circ}{\tilde{h}}) = 0$ while if $\lambda = 0$, $\text{tr } \overset{\circ}{\tilde{h}}$ is constant.*

Proof. From (2) $\Delta(\text{tr } \overset{\circ}{\tilde{h}}) = \overset{\circ}{h} \cdot \text{Ric}(g) = \lambda(\text{tr } \overset{\circ}{\tilde{h}})$, so if $\lambda < 0$, $\text{tr } \overset{\circ}{\tilde{h}} = 0$. If $\lambda > 0$ then it is known (Lichnerowicz [30]) that the first eigenvalue of Δ is $\lambda_1 \geq \lambda n/(n-1) > \lambda$, so again $\text{tr } \overset{\circ}{\tilde{h}} = 0$. If $\lambda = 0$, $\text{tr } \overset{\circ}{\tilde{h}} = \text{constant}$. ■

Thus if g is Einstein we get the splittings:

(4) (a) $(\lambda \neq 0)$ $h = h^{TT} + L_X g + g\Delta f + \text{Hess } f - f \cdot \text{Ric } g$
(where h^{TT} satisfies $\delta h^{TT} = 0$, $\text{tr}(h^{TT}) = 0$)

(b) $(\lambda = 0)$ $h = \left(h^{TT} + \frac{c}{n} g\right) + L_X g + (g\, \Delta f + \text{Hess } f)$,

$$c = \int \text{tr } h \, d\mu(g)/\text{volume } (M).$$

These splittings are useful in general relativity because the TT part (= transverse-traceless) is believed to describe the space of true gravitational degrees of freedom ([13], [21], [46]).

By utilizing the conformal group in place of $\mathfrak{D}$, York [47] has introduced another decomposition which is very useful in general relativity, since it has no curvature restrictions. This decomposition uses the conformal Lie derivative in place of the Lie derivative in the canonical splitting:

$$\tau_g : \mathfrak{X} \to S_2, \qquad X \mapsto L_X g + \frac{2}{n}(\delta X) g$$

with adjoint $\tau_g^*(h) = 2\delta h + (2/n) \text{ grad } (\text{tr } h)$. τ_g has injective symbol

$$\sigma_{\xi_x}(\tau_g)\cdot X = X_i\xi_j + X_j\xi_i - \frac{2}{n}(X^a\xi_a)g_{ij} \; ;$$

setting this equal to zero and contracting with $\xi_i X_j$ gives

$$\left(1 - \frac{2}{n}\right) ||X\cdot\xi||^2 + ||\xi||^2\, ||X||^2 = 0 \quad \text{so} \quad X = 0$$

since $\xi \neq 0$ and this is the sum of two non-negative terms, and so we have the splitting

$$S_2 = \ker \tau_g{}^* \oplus \tau_g(\mathfrak{X});$$

thus each h can be decomposed as

$$h = \tilde{h} + L_X g + \frac{2}{n}(\delta X)g$$

with $\tilde{h}$ satisfying $\delta\tilde{h} + 1/n \text{ grad } (\text{tr } \tilde{h}) = \delta(\tilde{h} - 1/n \; (\text{tr } \tilde{h})g) = 0$. Setting $h^{TT} = \tilde{h} - 1/n \; (\text{tr } \tilde{h})g$ and noting that $\text{tr } h = \text{tr } \tilde{h}$ gives York's splitting

$$S_2 = S_2{}^{TT} \oplus \tau_g(\mathfrak{X}) \oplus (C^\infty \otimes \{g\}),$$

written as

$$h = h^{TT} + L_X g + \frac{2}{n}(\delta X)g + \frac{1}{n}(\text{tr } h)g. \tag{5}$$

The part h^{TT} is orthogonal to the conformal orbit $\mathcal{C}_g$ of g in $\mathfrak{M}$.

The tangent space to $\mathcal{C}_g$ splits orthogonally to a piece $L_X g + (2/n)(\delta X)g$ tangent to $\mathcal{C}_g \cap \mathfrak{M}_\mu$, where $\mathfrak{M}_\mu$ is the space of metrics with a fixed volume μ ([14]) and a piece $(1/n)(\text{tr } h)g$ tangent to Pg, the metrics pointwise conformal to g. The underlying geometry involved in these splittings is considered in more detail in [21].

Let $T^*\mathfrak{M}$ denote the bundle over $\mathfrak{M}$ whose fiber at $g \in \mathfrak{M}$ is $S_2 \otimes \mu_g$; i.e. tensor densities. We shall write elements of $T^*\mathfrak{M}$ as (g, π). Moncrief [32] has introduced a splitting of the tangent space to $T^*\mathfrak{M}$ which is of basic importance in relativity. It is based on the mapping $\Phi : T^*\mathfrak{M} \to C_d{}^\infty \times \mathfrak{X}_d$, d standing for 'densities',

$$\Phi(g, \pi) = (\mathcal{H}(g, \pi), \delta\pi)$$

where $\mathcal{H}(g, \pi) = \{\tfrac{1}{2}(\text{tr } \pi)^2 - \pi\cdot\pi + R(g)\}\mu_g$; see [17, 20]. The set $\mathcal{C} = \Phi^{-1}(0)$ is the manifold of constraints of general relativity. We have the splitting

$$T_{(g,\pi)}(T^*\mathfrak{M}) = \ker D\Phi(g, \pi) \oplus \text{Range } D\Phi(g, \pi)^* \tag{6}$$

since $D\Phi(g, \pi)^*$ always has injective symbol ([17]). Let

$$J = \begin{pmatrix} 0 & I \\ -1 & 0 \end{pmatrix},$$

the usual symplectic matrix. One can show by a direct calculation that

$$\text{Range } J \circ D\Phi(g, \pi)^* \subset \ker D\Phi(g, \pi)$$

so we get a finer splitting:

$$(7) \quad \ker D\Phi(g, \pi) = \ker D\Phi(g, \pi) \cap \ker(D\Phi(g, \pi) \circ J) \oplus \text{Range } (J \circ D\Phi(g, \pi)^*).$$

The first summand can be used as a model for the space of 'true gravitational degrees of freedom' and is a generalized TT part.

If $\pi = \beta g$ and g is Einstein, this reduces to the Barbance–Deser–Berger–Ebin decomposition above; note that if $\pi = 0$, $\mathcal{H}$ reduces to $R(g)\mu_g$. The geometry of these ideas will be treated in greater detail elsewhere.

§6. Linearization Stability of $R(g) = \rho$.

We begin by establishing results which will fill in some missing cases in Theorems A and A′, namely A(i) and A′(i).

LEMMA 1. *Let $g_0 \in \mathfrak{M}^{s,p}$, $h_\| \in S_{2,\|}{}^{s,p}$ and set*

$$g(\lambda) = g_0 \exp(\lambda g_0{}^{-1} \circ h_\|).$$

Then $g'(0) = h_\|$ and $\text{Ric}(g(\lambda)) = \text{Ric}(g_0)$.

Proof. A straightforward calculation as in Theorem 6: $\Gamma(g(\lambda)) = \Gamma(g_0)$ and Ric depends only on the connection. ■

In general $R(g(\lambda)) \neq R(g_0)$. However if $\text{Ric}(g_0) = 0$, then $h_\| + L_X g$ is an infinitesimal deformation of $R(g) = 0$ and if ϕ_λ is the flow of X, $\phi_\lambda{}^*g(\lambda)$ is an actual deformation of $R(g) = 0$ tangent to $h_\| + L_X g$.

LEMMA 2. *Let $M = S^2$. Let $g \in \mathfrak{M}$, $h \in S_2$. Then there exists $f \in C^\infty(M, \mathbf{R})$ and $Y \in \mathfrak{X}$ such that*

$$(1) \qquad h = fg + L_Y g.$$

If $\delta h = 0$ and $d(\text{tr } h) = 0$, then $h = (c/2)g$, $c = \text{tr } h = \text{const.}$

Proof. Any two riemannian metrics on S^2 are conformally equivalent (Wolf [45], 2.5.17). Let $h = g'(0)$ where $g(\lambda) \in \mathfrak{M}$, $g(0) = g$. Thus there exist $\phi_\lambda \in \mathfrak{D}$, $\phi_0 =$ identity with $g(\lambda) = \phi_\lambda{}^*(P_\lambda g)$, $P_\lambda : M \to \mathbf{R}$, $P_\lambda > 0$, $P_0 = 1$. Hence $g'(0) = h = fg + L_Y g$ where $f = P_0'$, $Y = (d\phi_\lambda/d\lambda) \mid \lambda = 0$. Thus the h^{TT} part in York's decomposition (5) of §5, is zero. ■

LEMMA 3. *Let $M = T^2$ with g a flat metric. Let $h \in S_2$. Then*

$$h = h_\| + fg + L_Y g, \qquad h_\| \in S_{2,\|}.$$

If $\delta h = 0$ and d tr $h = 0$, then $h = h_\| + (c/2)g$, c a constant.

Proof. On T^2, every metric is pointwise conformal to a flat metric. Thus the conformal orbit of g is everything except possibly $\mathfrak{F}$. But $S_{2,\|} = T_g\mathfrak{F}_\Gamma$ is orthogonal to $fg + L_Y g$, so the h^{TT} piece must lie in it, again using York's decomposition. The result follows. ∎

One can prove lemma 3 directly in coordinates by showing $\delta h = 0$, $\operatorname{tr} h = 0$ implies $\nabla h = 0$; i.e. on a flat two torus, transverse traceless (^{TT}) is the same as covariant constant.

THEOREM 7. *Let $n = 2$. Then $R(g) = \rho$ is linearization stable about any $g \in \mathcal{M}$. Also, $\mathcal{M}_\rho$ is a submanifold with tangent space* $\ker \gamma_g$ *at $g \in \mathcal{M}_\rho$.*

Proof. Theorems 1, 2, 3 cover the case in which ρ is not a constant ≥ 0. So we are left with two cases:

Case 1. $\rho =$ constant > 0. Then for any metric g on M with $R(g) = \rho$, (M, g) is isometric to a standard 2-sphere of radius $r_0 = (2/\rho)^{\frac{1}{2}}$. Thus we can take $M = S^2 \subset \mathbf{R}^3$ and

$$\mathcal{M}_\rho^{s,p} = \mathcal{D}^{s+1,p}(g_0)$$

where g_0 is the standard metric. This set is a C^∞ submanifold of $\mathcal{M}^{s,p}$ as it is an orbit of a C^∞ metric (Ebin [14]), with

$$T_{\phi^* g_0}\mathcal{M}_\rho^{s,p} = \phi^* \alpha_{g_0}(\mathfrak{X}^{s+1,p}).$$

We assert that $\ker \gamma_{g_0} = \alpha_{g_0}(\mathfrak{X}^{s+1,p})$ which will finish the proof in this case. Now since ρ is constant, $\ker \gamma_{g_0} \supseteq \alpha_{g_0}(\mathfrak{X}^{s+1,p})$ (formula (1) of §5). Let $\gamma_{g_0} \overset{\circ}{h} = 0$. Hence by the lemma in §5, $\operatorname{tr} \overset{\circ}{h} = 0$. From lemma 2 it follows that $\overset{\circ}{h} = 0$, so $h = L_X g$. This completes case 1.

Note. This infinitesimal deformation $h = L_X g$ can be explicitly integrated up to $g(\lambda) = \phi_\lambda^* g_0$, $\phi =$ flow of X.

Case 2. $\rho = 0$. In this case any element of $\mathcal{M}_0^{s,p}$ is isometric to a flat metric on the two torus T^2. Hence $\mathcal{M}_0^{s,p} = \mathcal{F}^{s,p}$ which, by Theorem 6 is a submanifold with tangent space $S_{2,\|}^{s,p} \oplus \alpha_g(\mathfrak{X}^{s+1,p})$ at g. This obviously lies in $\ker \gamma_g$. Conversely, let $\gamma_g h = 0$. Then $\gamma_g h = \gamma_g \overset{\circ}{h} = \Delta(\operatorname{tr} \overset{\circ}{h}) = 0$, so $\operatorname{tr} \overset{\circ}{h} =$ constant, so from Lemma 3, $\overset{\circ}{h}$ is parallel. Hence $h \in S_{2,\|}^{s,p} \oplus \alpha_g(\mathfrak{X}^{s+1,p})$ by the canonical decomposition. ∎

Note. In case 2, every infinitesimal deformation can be explicitly integrated up using lemma 1.

In Theorem 7 it should be noted that in order for an equation $F(x) = 0$ to be linearization stable, it is not enough that $F^{-1}(0)$ be a submanifold; its tangent space must also be $\ker DF(x)$. In the exceptional cases of Theorem 7, R is not a submersion, so our basic criterion does not apply.

We have now established all parts of Theorems A, A′ except the last sentence of A and the case $\rho = 0$ in (ii) and (iii) of A′. We shall do this in the next sections.

§7. Linearization Instability.

In §2 we saw that on a standard sphere or on a Ricci-flat space, γ_g is not surjective. We now establish that linearization instability holds if $n \geq 3$ on S^n and on flat spaces. We do this by showing there exist infinitesimal deformations which do not satisfy necessary second order conditions, and hence cannot be tangent to actual deformations. We begin with the following

LEMMA 1. *Let $s > n/p + 1$, $g \in \mathfrak{M}^{s,p}$. Assume $\gamma_g = DR(g)$ is not surjective. Let $f \in \ker \gamma_g^*$, $f \neq 0$. Let $g(\lambda) \in \mathfrak{M}^{s,p}$ be a C^2 curve with $R(g(\lambda)) = \rho$ for all λ, $g(0) = g$ and $h = g'(0)$. Then*

$$DR(g)\cdot h = 0 \tag{1}$$

and

$$\int f\, D^2R(g)\cdot(h, h)\, d\mu(g) = 0. \tag{2}$$

where $D^2R(g) : S_2^{s,p} \times S_2^{s,p} \to W^{s-2,p}$ is the second derivative of R at g.

Proof. (1) is obvious from the chain rule. To prove (2) we differentiate $R(g(\lambda)) = \rho$ twice and evaluate at $\lambda = 0$:

$$0 = \frac{d^2}{d\lambda^2} R(g(\lambda))|_{\lambda=0} = D^2R(g)\cdot(h, h) + DR(g)\cdot g''(0). \tag{3}$$

Multiplying by f and integrating over M gives

$$\begin{aligned} \int f\, D^2R(g)\cdot(h, h)\, d\mu(g) &= -\int f\gamma_g\cdot g''(0)\, d\mu(g) \\ &= -\int \gamma_g^*(f)\cdot g''(0)\, d\mu(g) \\ &= 0. \end{aligned}$$

If $\mathrm{Ric}(g_0) = 0$ then $\ker \gamma_{g_0}^* =$ constant functions, so (2) reads

$$\int D^2R(g_0)\cdot(h, h)\, d\mu(g) = 0. \tag{4}$$

If (M, g_0) is a standard n-sphere than $\ker \gamma_{g_0}^*$ consists of eigenfunctions of Δ_{g_0} with eigenvalue n/r_0^2, so (2) reads:

$$\int f\, D^2R(g_0)\cdot(h, h)\, d\mu(g) = 0 \tag{5}$$

for the $n + 1$ dimensional space of first order spherical eigenfunctions on S^n (See Berger et. al. [5] for elementary properties of the eigenfunctions on S^n).

In case $\dim M = 2$, then from §6 we know that (1) will imply (2) automatically even though $\ker \gamma_g^* \neq \{0\}$. But if $\dim M \geq 3$ then (1) need not imply (2) and in this case we must have linearization instability. To prove this, we shall need the following:

LEMMA 2. *For $g \in \mathfrak{M}$ and $h \in S_2$,*

$$(6)\qquad \begin{aligned} D^2R(g)\cdot(h, h) = -\tfrac{1}{2}(\nabla h)^2 &+ 2\,\mathrm{Ric}\,(g)\cdot(h \times h) \\ &- \tfrac{1}{2}(d\,\mathrm{tr}\,h)^2 + h^{ii}|_k h_i{}^k|_i + 2h\cdot\mathrm{Hess}\,(\mathrm{tr}\,h) \\ &- 2\,\delta h\cdot d(\mathrm{tr}\,h) - \Delta(h\cdot h) - 2\,\delta\delta(h \times h). \end{aligned}$$

where $(h \times k)_{ij} = h_{ij}k_j{}^l$ and $\Delta f = -f_{|i|j}g^{ij}$ is the rough Laplacian.

Proof. This is a straightforward, though tedious calculation starting from $DR(g)\cdot h = \Delta(\mathrm{tr}\,h) + \delta\delta h - h\cdot\mathrm{Ric}\,(g)$ and using the methods of [10]. ■

Some straightforward consequences of (6) follow:

COROLLARY 1. *For $g \in \mathfrak{M}$, $h \in S_2$,*

$$(7)\qquad \begin{aligned} \int D^2R(g)(h, h)\,d\mu(g) = -\frac{1}{2}\int (\nabla h)^2\,d\mu(g) \\ + \int \mathrm{Ric}\,(g)\cdot(h \times h)\,d\mu(g) + \int R^{iaib}h_{ij}h_{ab}\,d\mu(g) \\ - \frac{1}{2}\int (d\,\mathrm{tr}\,h)^2\,d\mu(g) + \int (\delta h)^2\,d\mu(g). \end{aligned}$$

If g is flat, this becomes

$$(7')\qquad \begin{aligned} &\int D^2R(g)(h, h)\,d\mu(g) \\ &= -\frac{1}{2}\int (\nabla h)^2\,d\mu(g) - \frac{1}{2}\int (d\,\mathrm{tr}\,h)^2\,d\mu(g) + \int (\delta h)^2\,d\mu(g). \end{aligned}$$

2. *For $g \in \mathfrak{M}$, $h \in S_2$ and $f \in \ker \gamma_g{}^*$,*

$$(8)\qquad \begin{aligned} \int f\,D^2R(g)(h, h)\,d\mu(g) = \int f\{-\tfrac{1}{2}(\nabla h)^2 - \tfrac{1}{2}(d\,\mathrm{tr}\,h)^2 + (\delta h)^2\}\,d\mu(g) \\ + \int f\{R^{iaib}h_{ij}h_{ab} + 2h\cdot\mathrm{Hess}\,(\mathrm{tr}\,h) - 2\,\delta h\cdot d(\mathrm{tr}\,h)\}\,d\mu(g) \\ - 2\int (df \otimes \delta h)\cdot h\,d\mu(g). \end{aligned}$$

If $\delta h = 0$, $\mathrm{tr}\,h$ = constant, this becomes

$$(8')\qquad \int f\,D^2R(g)\cdot(h, h)\,d\mu(g) = -\frac{1}{2}\int f(\nabla h)^2\,d\mu(g) + \int fR^{iaib}h_{ij}h_{ab}\,d\mu(g).$$

(From §5, note that if g is Einstein, $\gamma_g(h) = 0$ and $\delta h = 0$ implies $\mathrm{tr}\,h$ = constant).

3. *If, in addition, g has constant sectional curvature, and* $trh = 0$,

$$\int f\, D^2R(g)(h, h)\, d\mu(g) = -\frac{\rho}{n(n-1)} \int h \cdot hf\, d\mu(g) - \frac{1}{2} \int f(\nabla h)^2\, d\mu(g). \tag{9}$$

For a standard sphere of radius r_0, $\rho = n(n-1)/r_0^2$.

These formulas follow from lemma 2 and the formula

$$R_{aib}{}^{i} = \frac{\rho}{n(n-1)} (g_{ab}\delta_i{}^i - g_{ai}\delta_b{}^i)$$

for a space of constant curvature.

If (M, g) is Einstein, and $\delta h = 0$, tr $h = 0$ then h is an infinitesimal deformation of R. In the flat case, if h comes from an actual deformation, it must also satisfy

$$\int \nabla h \cdot \nabla h\, d\mu(g) = 0$$

by (7′), i.e. $\nabla h = 0$.

Now we use this to prove instability in the flat case:

THEOREM 8. *Suppose* (M, g) *is flat,* $\dim M \geq 3$. *Then* $R(g) = 0$ *is linearization unstable at g.*

Note. In the noncompact case on $\mathbf{R}^n$, $R(g) = 0$ is linearization stable at the Euclidean metric, as we showed in §3.

Proof. By the Bieberbach theorem [45], p. 105, M admits a riemannian covering by a flat torus T^n. On T^n construct an h with $\delta h = 0$, tr $h = 0$ by setting

$$h_{ij} = \left(\begin{array}{cc|c} 0 & f(x_3, \cdots, x_n) & 0 \\ f(x_3, \cdots, x_n) & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

where the upper left block is 2×2 and the lower right is $(n-2) \times (n-2)$. Here $f : T^{n-2} \to \mathbf{R}$ is any smooth function. Projecting h to M yields h on M with the same properties $\delta h = 0$, tr $h = 0$. However h need not be covariant constant, so by lemmas 1 and 3, $R(g) = 0$ is linearization unstable. ■

Note. Such an example is impossible on T^2 as we showed earlier: $\delta h = 0$, tr h = constant implies $\nabla h = 0$.

THEOREM 9. *The equation* $R(g) = n(n-1)/r_0^2$ *is linearization unstable at the standard metric* g_0 *on a standard sphere* S^n *of radius* $r_0 > 0$ *in* $\mathbf{R}^{n+1}$ *if* $n \geq 3$.

Proof. Let $\overset{\circ}{h}$ be an infinitesimal deformation which is divergence free. By the lemma in §5, this means precisely that $\delta \overset{\circ}{h} = 0$ and tr $\overset{\circ}{h} = 0$; i.e. $\overset{\circ}{h} = h^{TT}$.

Now suppose in addition that $\overset{\circ}{h}$ comes from an actual deformation. Then by (9) we find that $\overset{\circ}{h}$ must satisfy

$$\frac{1}{r_0^2}\int_{S^n} f(\overset{\circ}{h})^2\, d\mu(g_0) - \frac{1}{2}\int_{S^n} f(\nabla \overset{\circ}{h})^2\, d\mu(g) \tag{9'}$$

for all eigenfunctions f on S^n with eigenvalue n/r_0^2 ; i.e.: for linear functions f on $\mathbf{R}^{n+1}$ restricted to S^n.

We claim that there exists an $\overset{\circ}{h}$ satisfying $\delta\overset{\circ}{h} = 0$, tr $\overset{\circ}{h} = 0$, but not (9′) on S^n, $n \geq 3$.

Let h be a symmetric two tensor on $\mathbf{R}^{n+1}$. If tr $h = 0$, $\delta h = 0$ and if the rows of h are, as vectors in $\mathbf{R}^{n+1}$, tangent to S^n, then the pull back of h to S^n is an $\overset{\circ}{h}$ such that tr $\overset{\circ}{h} = 0$, $\delta\overset{\circ}{h} = 0$ on S^n. This is easily checked using the fact that $\delta h = 0$ is equivalent to being orthogonal to Lie derivatives.

On $\mathbf{R}^{n+1}$, $n \geq 3$ an example of such an h is given by the following matrix:

$$h_{ij}(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) = \left[\begin{array}{cccc|c} 0 & x_4x_3 & -2x_2x_4 & x_2x_3 & 0 \\ x_4x_3 & 0 & x_1x_4 & -2x_1x_3 & 0 \\ -2x_2x_4 & x_1x_4 & 0 & x_1x_2 & 0 \\ x_2x_3 & -2x_1x_3 & x_1x_2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where the last column of zeros is $(n - 3) \times (n + 1)$. This h has its rows and columns tangent to S^n, each row and column is divergence free and h is symmetric.

Thus the pull back of h to S^n defines a symmetric two tensor $\overset{\circ}{h}$ with $\delta\overset{\circ}{h} = 0$, tr $\overset{\circ}{h} = 0$. However using known properties of the standard sphere, (see [5]) it is not hard to see that the first term in (9′) vanishes, but the second does not. Hence $\overset{\circ}{h}$ cannot come from an actual deformation and the result follows. ∎

Recently, using elliptic theory and results of Hörmander [23], Bourguignon-Ebin and Marsden (not yet published) have shown that the set of TT tensors h (i.e. satisfying $\delta h = 0$ and tr $h = 0$) is infinite dimensional, as are those with support in a given open set $\Omega \subset M$, if dim $M \geq 3$. In fact, the dimensions are still correct to make the same conclusion for h satisfying $\delta h = 0$, tr $h = 0$ and $h \cdot \mathrm{Ric}\,(g) = 0$ (since these equations define a non-elliptic operator from a bundle with six dimensional fiber to one with dimension five for dim $M = 3$).

However, from Rellich's theorem, on a suitable Ω where $f > 0$, (8′) cannot be zero on an infinite dimensional space of h.

Hence we can conclude that, more generally than Theorems 8 and 9, $R(g)$ *is linearization unstable at* g_0 *if and only if* $DR(g_0)$ *is not surjective, in dimensions* ≥ 3.

Of course as we have proved, in dimension 2, $R(g)$ is linearization stable and $DR(g_0)$ is not surjective.

Open Case. $n \geq 3$, $R(g_0) = \rho$ where ρ = constant $= \lambda(n - 1) > 0$ where

$\lambda \in \operatorname{spec}(\Delta)$ and Ric (g_0) is not parallel (in particular (M, g_0) is not a standard sphere).

Our conjecture of §1 can be formulated by saying that this case should be linearization stable.

§8. Isolated Solutions of $R(g) = 0$.

The aim of this section is to show that if g_F is flat and if g is near g_F and $R(g) \geq 0$, then g is also flat, i.e. to prove Theorem B. This is somewhat surprising in that $R(g)$ is a relatively weak measure of the curvature. "Near" here means near in a $W^{s,p}$ topology, $s > n/p + 1$. For instance, in three dimensions near in $W^{2,p}$, $p > 3$ suffices.

We begin by setting up some notations and lemmas.

Fix a smooth volume $d\mu$ on M and define

$$\Psi: \mathfrak{M}^{s,p} \to \mathbf{R}, \qquad g \mapsto \int R(g)\, d\mu.$$

(Compare with [7] where $\int R(g)\, d\mu(g)$ is used; here μ is fixed, independent of g.)

Lemma 1. *$g \in \mathfrak{M}^{s,p}$ is a critical point of Ψ iff* $\operatorname{Ric}(g) = 0$ *and* $d\mu = c\, d\mu(g)$ *for c a constant > 0. At a critical point, the second derivative of Ψ is:*

$$d^2\Psi(g)\cdot(h, h) = -\frac{1}{2}\int h\cdot \Delta_L h\, d\mu - \frac{1}{2}\int (d \operatorname{tr} h)^2\, d\mu + \int (\delta h)^2\, d\mu. \tag{1}$$

Proof. For $g \in \mathfrak{M}^{s,p}$, let $\rho \in W^{s,p}(M; \mathbf{R})$, $\rho > 0$ be such that $d\mu = \rho\, d\mu(g)$. Then

$$d\Psi(g)\cdot h = \int DR(g)\cdot h\, d\mu = \int (\Delta \operatorname{tr} h + \delta\delta h - h\cdot \operatorname{Ric}(g))\rho\, d\mu(g)$$
$$= \int (g\, \Delta\rho + \operatorname{Hess} \rho - \rho \operatorname{Ric}(g))\cdot h\, d\mu(g).$$

This vanishes for all $h \in S_2^{s,p}$ iff

$$\gamma_g^*(\rho) = g\Delta\rho + \operatorname{Hess}\rho - \rho \operatorname{Ric}(g) = 0. \tag{2}$$

Contracting (2) gives

$$(n-1)\Delta\rho = R(g)\rho, \qquad \rho > 0. \tag{3}$$

From the proof of Theorem 1, §2 this implies $R(g) =$ constant ≥ 0. Integrating (3) over M gives

$$0 = (n-1)\int \Delta\rho\, d\mu(g) = R(g)\int \rho\, d\mu(g).$$

If $R(g) = \text{constant} > 0$, then $\int \rho \, d\mu(g) = 0$, contradicting $\rho > 0$. Thus $R(g) = 0$, so from (3), $\rho = \text{constant} = c > 0$, so that $d\mu = c\, d\mu(g)$. Since $\rho = \text{constant} > 0$, from (2), $\operatorname{Ric}(g) = 0$.

Now suppose g is a critical metric of Ψ, so $\operatorname{Ric}(g) = 0$ and $d\mu = c\, d\mu(g)$. Thus

$$d^2\Psi(g)\cdot(h, h) = \int D^2R(g)\cdot(h, h)\, d\mu$$

$$= c \int D^2R(g)\cdot(h, h)\, d\mu(g).$$

The theorem now follows from Lemma 2 of §7. ■

Now let

$$\hat{\mathfrak{N}}_\mu^{s,p} = \{g \in \mathfrak{M}^{s,p} \mid d\mu(g) = c\, d\mu \text{ for a constant } c > 0\} \subset \mathfrak{M}^{s,p}$$

and let

$$\mathfrak{D}_\mu^{s,p} = \{\eta \in \mathfrak{D}^{s,p} \mid \eta^*\mu = \mu\} \subset \mathfrak{D}^{s,p}$$

the volume preserving diffeomorphisms. By known results [14, 16] these are smooth manifolds, with

$$T_g\hat{\mathfrak{N}}_\mu^{s,p} = \{h \in S_2^{s,p} \mid \operatorname{tr} h = \text{constant}\}$$

and

$$T_\phi\mathfrak{D}_\mu^{s,p} = \{X \mid X \text{ is a divergence free vector field covering } \phi\}.$$

Recall that we set $\mathcal{E}_0^{s,p} = \{g \in \mathfrak{M}^{s,p} \mid \operatorname{Ric}(g) = 0\}$, and $\mathfrak{F}^{s,p}$ is the flat metrics (see §4).

We can rephrase the first part of lemma 1 by saying that the critical set of Ψ is $\mathcal{E}_0^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p}$. Remember that if $\mathfrak{F} \neq \emptyset$, $\mathcal{E}_0^{s,p} = \mathfrak{F}^{s,p}$.

LEMMA 2. *Let g_1, $g_2 \in \mathfrak{M}^{s,p}$ and assume $\Gamma(g_1) = \Gamma(g_2)$ i.e. g_1 and g_2 have the same connection. Then $\mu(g_1) = c\mu(g_2)$ for a constant $c > 0$ ($\mu(g)$ is the volume element of g).*

Proof. Let $\mu(g_1) = \rho\mu(g_2)$ for ρ a function > 0. However the covariant derivatives of $\mu(g_1)$ and $\mu(g_2)$ are zero in the common connection, so $\nabla\rho = 0$ and thus ρ is constant. ■

LEMMA 3. *Let $g_F \in \mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p}$ and let Γ be the connection of g_F, and μ the volume element of g_F. Then*

$$\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p} = \mathfrak{D}_\mu^{s+1,p}(\mathfrak{F}_\Gamma^{s,p})$$

and this set is a smooth submanifold of $\mathfrak{M}^{s,p}$.

Proof. To show $\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p} = \mathfrak{D}_\mu^{s+1,p}(\mathfrak{F}_\Gamma^{s,p})$, let $g \in \mathfrak{D}_\mu^{s+1,p}(\mathfrak{F}_\Gamma^{s,p})$ and $\bar{g} = \phi^*g \in \mathfrak{F}_\Gamma^{s,p}$. By lemma 2, $\mu(\bar{g}) = c\mu$ and so $\mu(g) = c\mu$ as well. Thus $g \in \hat{\mathfrak{N}}_\mu^{s,p}$, and clearly g is flat. Conversely, let $g \in \mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p}$ so $\mu(g) = \bar{c}\mu$.

Now $\phi^*g \in \mathfrak{F}_\Gamma^{s,p}$ for some $\phi \in \mathfrak{D}^{s+1,p}$ by Theorem 6. However $\mu(\phi^*g) = c\mu$ again by lemma 2. But $\mu(\phi^*g) = \phi^*\mu(g) = \bar{c}\phi^*\mu$. By integrating over M, we see that $c = \bar{c}$, so ϕ is μ-volume preserving.

That the $\mathfrak{D}_\mu^{s+1,p}$ orbit of $\mathfrak{F}_\Gamma^{s,p}$ is a smooth manifold is proved in exactly the same way as we did in Theorem 6. ∎

The tangent space is

$$\begin{aligned}T_g(\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p}) &= \{h \in S_2^{s,p} \mid h = h_\parallel + L_X g \\ &\qquad \text{where } h_\parallel \text{ is parallel and tr } h = \text{constant}\} \\ &= \{h \in S_2^{s,p} \mid h = h_\parallel + L_X g \\ &\qquad \text{where } h_\parallel \text{ is parallel and } \delta X = 0\}.\end{aligned}$$

The two expressions correspond to the set equality in lemma 3 (see §4).

If $\mathfrak{F} \neq \emptyset$ it follows that $\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p}$ is exactly the critical set of Ψ.

LEMMA 4. *In lemma 3, let S be a slice at g_F. (See corollary 2 of Theorem 5). Then $\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu^{s,p} \cap S = N$ is a submanifold of S in a neighborhood of g_F. (If S is constructed as in* [14], *$N = \mathfrak{F}_\Gamma^{s,p}$).*

Proof. From Theorem 6 we see that both N and $F_\Gamma^{s,p}$ are obtained by exponentiating $S_2^{s,p}$, so $N = F_\Gamma^{s,p}$ which is a smooth manifold. ∎

LEMMA 5. *Let P be a Banach manifold and $f : P \to \mathbf{R}$ a C^2 function. Suppose that $Q \subset P$ is a submanifold, $f = 0$ and $df = 0$ on Q and that there is a smooth normal bundle neighborhood U of Q such that if E_x is the normal complement to T_xQ in T_xP then $d^2f(x)$ is weakly negative definite on E_x* (i.e. $d^2f(x)(v, v) \leq 0$ *with equality only if $v = 0$). Let $\langle\, , \rangle_x$ be a weak Riemannian structure* [14, 16] *with a smooth connection and assume that f has a smooth $\langle\, , \rangle_x$-gradient, $Y(x)$. Assume $DY(x)$ maps E_x to E_x and is an isomorphism for $x \in Q$. Then there is a neighborhood U of Q such that $y \in U$, $f(y) \geq 0$ implies $y \in Q$.*

Proof. These hypotheses are sufficient for the Morse lemma to hold in the normal variables as is seen by a proof analogous to the Moser-Weinstein-Palais-Tromba treatment of the Morse lemma (see [28], [42], [43]). The result therefore follows. ∎

To second order, Brill [7] and Brill–Deser [8] show that the flat solutions of $R(g) = 0$ are isolated, and Kazdan–Warner [25, §5] show that near a flat metric, to second order there are no metrics with positive scalar curvature. The following extends these results to a full neighborhood of the flat metrics.

THEOREM 10. *Let $g_F \in \mathfrak{F}^{s,p}$, $s > n/p + 1$. Then there exists a neighborhood $U \supset \mathfrak{M}^{s,p}$ of g_F such that if $g \in U$ and $R(g) \geq 0$ then g is also in $\mathfrak{F}^{s,p}$.*

Proof. Let $d\mu$ denote the volume element of g_F, let Γ denote its Levi–Civita connection, and let $\Psi : \mathfrak{M}^{s,p} \to \mathbf{R}$, $\Psi(g) = \int R(g)\, d\mu$, as above. Also, let S be a slice at g_F, and Ψ_S denote the restriction of Ψ to S.

The manifold $N \subset S$ (lemma 4) consists of critical points of Ψ_S and we have formula (1) for $d^2\Psi_S$. Since $\delta h = 0$ for h tangent to S at g_F, we get

$$d^2\Psi_S(g_F)\cdot(h, h) = -\frac{1}{2}\int (\nabla h)^2\, d\mu - \frac{1}{2}\int (d\operatorname{tr} h)^2\, d\mu.$$

Now $d^2\Psi_S(g_F)\cdot(h, h) = 0$ iff $\nabla h = 0$; i.e. h is tangent to N as well as S. Thus $d^2\Psi_S(g_F)$ is (weakly) negative definite on a complementary bundle to N at g_F and hence also in a neighborhood W of g_F in S.

From Lemma 5, it follows that N is a (locally) isolated set of zeros of Ψ in S. In fact, there is a neighborhood V of $N \cap W$ in S such that if $g \in V$, $\Psi(g) \geq 0$ then $g \in N$. Here we choose $\langle h, h\rangle_g = \int (h\cdot h + \nabla h\cdot\nabla h)d\mu(g)$ and as in [14] this has a smooth connection. Also, from the proof of Lemma 1, $Y(g) = P_g(1 + \Delta)^{-1}\gamma_g{}^*\rho$, a smooth vector field fulfilling Lemma 5 on S as one computes, where P_g is the orthogonal projection to the slice.

Now let $U = \mathfrak{D}^{s+1,p}(V)$ be the saturation of V. By the slice theorem [14, 15] this is a neighborhood of g in $\mathfrak{M}^{s,p}$.

By Lemma 3, the $\mathfrak{D}_\mu{}^{s+1,p}$ saturation of N is $\mathfrak{F}^{s,p} \cap \hat{\mathfrak{N}}_\mu{}^{s,p}$ and by Moser's theorem [33, 14, 16] the $\mathfrak{D}^{s+1,p}$ saturation of this is $\mathfrak{F}^{s,p}$. Thus the $\mathfrak{D}^{s+1,p}$ saturation of N is $\mathfrak{F}^{s,p}$.

Let $g \in U$ and suppose $R(g) \geq 0$. Then there is $\phi \in \mathfrak{D}^{s+1,p}$ (near the identity) such that $\phi^*g \in V$. But $\Psi(\phi^*g) = \int R(\phi^*g)\, d\mu = \int R(g) \circ \phi J(\phi)\, d\mu \geq 0$ (J is the μ-Jacobian of ϕ). Hence $g \in N$, so $g \in \mathfrak{F}^{s,p}$; i.e. g is flat. ∎

Remarks. 1. For dim $M = 2$, we need not restrict to a neighborhood U. Indeed, from the Gauss–Bonnet Theorem, if $\mathfrak{F}^{s,p} \neq \varnothing$, the Euler–Poincaré characteristic $\chi_M = 0$, so that for any metric g, $\int R(g)\, d\mu(g) = 0$. Thus $R(g) \geq 0$ implies $R(g) = 0$ implies g is flat.

2. Usually one considers an integrated scalar curvature $\Phi(g) = \int R(g)\, d\mu(g)$ with volume element $d\mu(g)$ induced from g rather than $\Psi(g) = \int R(g)\, d\mu$ with fixed volume element $d\mu$, as e.g. in [3]. One then has at a critical flat metric

$$(4) \qquad d^2\Phi(g_F)\cdot(\overset{\circ}{h}, \overset{\circ}{h}) = -\frac{1}{2}\int (\nabla \overset{\circ}{h})^2\, d\mu(g_F) + \frac{1}{2}\int (d\operatorname{tr}\overset{\circ}{h})^2\, d\mu(g_F)$$

rather than

$$(5) \qquad d^2\Psi(g_F)\cdot(\overset{\circ}{h}, \overset{\circ}{h}) = -\frac{1}{2}\int (\nabla \overset{\circ}{h})^2\, d\mu(g_F) - \frac{1}{2}\int (d\operatorname{tr}\overset{\circ}{h})^2\, d\mu(g_F).$$

There is now an important sign change in the second term due to the contribution of the volume element. Because of this sign change, a flat metric g_F is a saddle point of the integrated scalar curvature, even when restricted to a slice [14] whereas $\int D^2R(g_F)\cdot(\overset{\circ}{h}, \overset{\circ}{h})\, d\mu(g) \leq 0$ on a slice. Thus the behavior of the integrated scalar curvature is somewhat different from the pointwise scalar curvature $R(g)$ at g_F.

Thus the indefinite sign in (4) is a property of the volume element and not the pointwise scalar curvature.

3. Brill [7] uses (4), together with the condition $\operatorname{tr}\overset{\circ}{h} = $ constant implied by the first order equations $DR(g)\cdot h = \Delta\operatorname{tr}\overset{\circ}{h} = 0$ to deduce $\nabla\overset{\circ}{h} = 0$ and hence his second order result.

To extend this result to a full neighborhood of the flat metrics, one may *not* make use of the first order condition $\operatorname{tr} \overset{\circ}{h} =$ constant. The indefinite sign in (4) in fact is a nontrivial difficulty. We introduced the map Ψ with fixed volume element $d\mu$ since this difficulty is not present in (5).

4. If g is near g_F and if $R(g) \geq 0$, the proof actually shows that $g = \eta^*(g_F + h_\parallel)$ for some η near the identity and some parallel tensor $h_\parallel$. In particular, any other flat metric near g_F must be obtained this way by a coordinate change η and a "scale change" $h_\parallel$, which leaves the connection fixed. Thus the *local* structure of the space of flat metrics $\mathfrak{F}$ is exactly what one might expect, from $\mathfrak{F} = \mathfrak{D}(\mathfrak{F}_\Gamma)$.

As a consequence of Theorem 10 we have the following structure theorem for $\mathfrak{M}_0^{s,p}$:

THEOREM 11. *Let $s > n/p + 1$, $\dim M \geq 3$, and if $\dim M \geq 4$, assume $\mathfrak{F}^{s,p} \neq \emptyset$. Then*

$$\mathfrak{M}_0^{s,p} = (\mathfrak{M}_0^{s,p} - \mathfrak{F}^{s,p}) \cup \mathfrak{F}^{s,p}$$

is the disjoint union of two C^∞ closed submanifolds, and hence $\mathfrak{M}_0^{s,p}$ is itself a C^∞ closed submanifold of $\mathfrak{M}^{s,p}$; similarly, $\mathfrak{M}_0 = (\mathfrak{M}_0 - \mathfrak{F}) \cup \mathfrak{F}$ is a union of C^∞ closed submanifolds of $\mathfrak{M}$.

Proof. If $\dim M = 3$, $\operatorname{Ric}(g) = 0$ implies g is flat. If $\dim M \geq 4$ and $\mathfrak{F}^{s,p} \neq \emptyset$, then from [22], $\mathfrak{F}^{s,p} = \mathcal{E}_0^{s,p}$. Thus in either case

$$\mathfrak{M}_0^{s,p} = (\mathfrak{M}_0^{s,p} - \mathcal{E}_0^{s,p}) \cup \mathcal{E}_0^{s,p} = (\mathfrak{M}_0^{s,p} - \mathfrak{F}^{s,p}) \cup \mathfrak{F}^{s,p}$$

which from Theorems 3 and 6 is the disjoint union of C^∞ submanifolds.

Obviously $\mathfrak{F}^{s,p}$ is closed. On the other hand, $\mathfrak{M}_0^{s,p} - \mathfrak{F}^{s,p}$ is closed in $\mathfrak{M}^{s,p}$ since $\mathfrak{M}_0^{s,p}$ is closed in $\mathfrak{M}^{s,p}$ and theorem 10 implies that $\mathfrak{F}^{s,p}$ is open in $\mathfrak{M}_0^{s,p}$. ∎

This result completes the proof of Theorem A′.

Remarks. 1. If $\dim M = 2$, $\mathfrak{M}_0^{s,p} = \mathfrak{F}^{s,p}$, also a C^∞ closed submanifold (§4).

2. We are allowing the possibility that $\mathfrak{M}_0^{s,p} - \mathfrak{F}^{s,p}$ is empty, and if $\dim M = 3$, we are also allowing the possibility that $\mathfrak{F}^{s,p}$ is empty.

3. There are various conditions on M, $\dim M \geq 4$, that imply a Ricci-flat metric is flat; see [22]. If we adopt any of these condtions, then we can drop the $\mathfrak{F}^{s,p} \neq \emptyset$ assumption.

4. If $\dim M \geq 4$, $\mathfrak{F}^{s,p} = \emptyset$, then $\mathfrak{M}_0^{s,p}$ is still the union $\mathfrak{M}_0^{s,p} = (\mathfrak{M}_0^{s,p} - \mathcal{E}_0^{s,p}) \cup \mathcal{E}_0^{s,p}$, with $\mathfrak{M}_0^{s,p} - \mathcal{E}_0^{s,p}$ a submanifold (Theorem 3). But we do not know if $\mathcal{E}_0^{s,p}$, if non-empty, is a submanifold.

5. Although $\mathfrak{M}_0^{s,p}$ is a manifold (under the hypothesis $\mathfrak{F}^{s,p} \neq \emptyset$), the equation $R(g) = 0$ is not linearizationstable at a flat solution, as we have seen in Theorem 8. The difficulty can be traced to the fact that $\mathfrak{M}_0^{s,p}$ is a union of closed manifolds of different "dimensionalities," $\mathfrak{F}^{s,p}$ being finite-dimensional modulo the orbit directions as was explained in §4.

6. The following comments, related to the above isolation results and calculations of Y. Muto [34], and which show Theorem 10 is sharp, were communicated by P. Ehrlich.

H. Karcher has observed that suitably deforming a flat product metric g_0 on $M = T^2 \times S^1$ on the T^2 factor produces a smooth curve $g(t)$ of metrics through g_0, non-flat for $t \neq 0$ but with $\int_M R(g(t))\mu(g(t)) = 0$ for all t. In this example the integral is forced to vanish by the two-dimensional Gauss–Bonnet Theorem. However P. Ehrlich and B. Smyth have observed (unpublished) the following result in $\mathfrak{M}/\mathfrak{D}$ which seems to show that the example of Karcher is not entirely an accident of the two dimensional Gauss–Bonnet Theorem. Let $[g_0]$ be any Ricci flat Riemannian structure in $\mathfrak{M}/\mathfrak{D}$ ($[g_0]$ denotes its equivalence class). Then given any open neighborhood U of $[g_0]$ in $\mathfrak{M}/\mathfrak{D}$ there is a Riemannian structure $[g] \in U$ with $\int_M R(g)\mu(g) = 0$ but $\mathrm{Ric}(g) \neq 0$.

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