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GENERAL RELATIVITY AS A HAMILTONIAN SYSTEM (*)

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1. Introduction.

Recently the authors have obtained results on the structure of the set of solutions to the constraint equations of general relativity [8]. In the present article we shall explain how these results tie in with the dynamics of general relativity. Specifically, we want to show how to make the space of solutions of the full non-linear vacuum field equations of general relativity into an honest mooth manifold (under certain technical conditions) and to show how this becomes a symplectic manifold when isometric metrics are identified. This makes use of some very general results of Marsden-Weinstein [11]. This symplectic structure is analogous to that obtained in classical field theories; cf. Lichnerowicz [9], Segal [12] Chernoff-Marsden [2] and the articles of P. L. Garcia and I. Segal in this volume. However the present case is complicated by the presence of constraints and the necessity of passing to a quotient space (when isometric metrics are identified). This necessity of passing to a quotient space is already recognized in the formal constructions of Fadeev [5].

We shall also mention how the results on the constraint set can be used to justify linearization of the field equations. In other words, we establish conditions under which a solution of the linearized field equations actually approximates, to first order, an exact solution to the non-linear field equations.

We begin by supplying some necessary background.

2. Dynamics of general relativity.

Let V denote a 4 dimensional manifold and let ${}^{(4)}g$ denote a Lorentz metric on V. We use the superscript ${}^{(4)}$ to avoid confusion with

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3-metrics used below. The Einstein field equations in vacuo are

$$\text{(1)} \qquad \operatorname{Ric}({}^{(4)}g) = 0\,, \quad \text{i.e. } R_{\alpha\beta} = 0$$

where Ric ($= R_{\alpha\beta}$) denotes the Ricci tensor of ${}^{(4)}g$. Arnowitt-Deser-Misner [1] showed how equation (1) may be regarded as a Hamiltonian evolution equation for an evolving 3-metric $g(t)$. We can describe these results, in a special case, as follows; cf. [6]. Let $M \subset V$ be a space-like hypersurface. Assume M is compact or else impose appropriate asymptotic conditions ([7]). If we choose a Gaussian normal coordinate system i.e. an exponential tubular neighborhood about M, then in a neighborhood of M, V becomes $M \times [-\varepsilon, \varepsilon]$ and our metric ${}^{(4)}g$ takes the form

$$\text{(2)} \qquad {}^{(4)}g_{\alpha\beta}\, dx^\alpha dx^\beta = -dt^2 + g_{ij}\, dx^i dx^j$$

where $x^\alpha = (x^i, t)$. Thus we have induced a 3-metric $g(t)$ on each hypersurface $M \times \{t\}$.

Let $\mathcal{M}$ denote the set of all C^∞ riemannian metrics on M. Then $\mathcal{M}$ is an open cone in the linear space S_2 of all C^∞ symmetric 2-tensors on M. Thus the tangent space to $\mathcal{M}$ at $g \in M$ is simply given by

$$T_g \mathcal{M} = S_2\,.$$

For many technical results one needs to consider metrics of Sobolev class H^s, (s sufficiently large). One then uses regularity arguments to recover the results in C^∞; cf. § 3 below.

Define a weak pseudo-riemannian metric on $\mathcal{M}$, the De Witt metric, by

$$\mathfrak{J}_g\colon T_g\mathcal{M} \times T_g\mathcal{M} \to \mathbf{R}\,,$$

$$\mathfrak{J}_g(h_1, h_2) = \int_M (\operatorname{tr} h_1 \cdot \operatorname{tr} h_2 - h_1 h_2)\, d\mu_g\,,$$

where $\operatorname{tr} h = h_i{}^i$, $h_1 k_2 = (h_1)_{ij}(k_2)^{ij}$ and $d\mu_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \ldots \wedge dx^n$ is the volume on M induced by the metric g. Here, weak refers to the fact that $\mathfrak{J}$ is non-degenerate in the weak sense:

$$\mathfrak{J}_g(h_1, h_2) = 0 \quad \text{for all } h_2 \Rightarrow h_1 = 0\,.$$

(Non-degenerate in the strong sense would entail that the induced map of $T_g\mathcal{M}$ to $T_g^*\mathcal{M}$ is bijective, but this is not the case here.)

As is well known, a metric on a manifold induces a symplectic form; one pulls back the canonical symplectic form from the cotangent bundle. Thus $\mathfrak{J}$ induces a symplectic form Ω on $T\mathcal{M}$. As $\mathfrak{J}$ is weakly non-degenerate, the same is true of Ω, so we refer to Ω as a *weak symplectic form.* (Consult [2] for the general theory).

Define a potential $V: \mathcal{M} \to \mathbf{R}$ by $V(g) = 2\int_M R(g)\, d\mu_g$, where $R(g)$ denotes the scalar curvature of g, and consider the Hamiltonian

$$H: T\mathcal{M} \to \mathbf{R}\,, \qquad H(g, k) = \int_M \mathcal{H}(g, k)\, d\mu_g$$

where $\mathcal{H}(g, k) = \frac{1}{2}((\operatorname{tr} k)^2 - k\cdot k) + 2R(g)$.

From H and Ω one can construct in the usual way a Hamiltonian vector field X_H on $T\mathcal{M}$. Actually since Ω is only weakly non-degenerate one must show the existence of X_H, but this can be done; cf. [2, 6, 10]. X_H is a certain non-linear partial differential operator of Hamiltonian type. We write down the explicit formula for X_H in a more general case below.

There arise certain constraint equations. These are:

$$(3) \qquad \begin{cases} \delta\pi = 0\,, \\ \mathcal{H} = 0\,, \end{cases}$$

where $\pi = (\operatorname{tr} k)g - k$ and δ denotes the divergence with respect to the metric g. As will be explained below, these are actually conserved quantities corresponding to invariance under spatial and temporal coordinate transformations respectively.

One of the main results in the dynamics of general relativity is:

THEOREM 1: *Let $g(t)$ and ${}^{(4)}g$ be related by* (2). *Then ${}^{(4)}g$ satisfies* (1) *if and only if $(g(t), k(t) = \dot g(t))$ is an integral curve of X_H described above and the constraint equations* (3) *hold.*

The proof may be found in, for example [6] or [10]. It should be remarked that $\frac{1}{2}k(t)$ is just the second fundamental form of $M \times \{t\} \subset V$.

If one makes a coordinate change in V described by a diffeomorphism $\varphi: V \to V$ then $g(t)$, $k(t)$ are transformed accordingly. They still satisfy Hamiltonian evolution equations but now a «lapse» N and a «shift» X are introduced. Indeed the above theorem is special in that it corresponds to a special coordinate system in V, namely Gaussian normal coordinates. This can be generalized as follows. If

$${}^{(4)}g_{\alpha\beta}\, dx^\alpha dx^\beta = (X\cdot X - N^2)\, dt^2 - 2X_i\, dx^i dt + g_{ij}\, dx^i dx^j$$

where $N: M \times \mathbf{R} \to \mathbf{R}$ is a scalar function and $X: M \times \mathbf{R} \to TM$ is a vector field, then ${}^{(4)}g_{\alpha\beta}$ being Ricci flat is equivalent to the following conditions on g_{ij}:

$$\begin{cases} \dfrac{\partial g}{\partial t} = Nk - L_X g \,, \\[2ex] \dfrac{\partial k}{\partial t} = N\delta_g(k) - 2N \operatorname{Ric} g - L_X k + 2 \operatorname{Hess} N \end{cases}$$

and

$$\begin{cases} \delta\pi = 0 \,, \\ \mathcal{H} = 0 \,, \end{cases}$$

where $S_g(k) = k \times k - \frac{1}{2}(\operatorname{tr} k)k$ (the spray of the De Witt metric); $(k \times k)_{ij} = k_{il}k^l{}_j$, $\operatorname{Hess} N = N_{|i|j}$, and $L_X g = X_{i|j} + X_{j|i}$ is the Lie derivative.

Again the constraints are conserved by these equations and the equations are « Hamiltonian » with respect to the same symplectic structure as before and with energy

$$H(g, k) = \int_M (N \circ \eta_t^{-1}) \mathcal{H}(g, k)\, d\mu_g$$

where η_t is the flow of the vector field X_t. The fact that the symplectic structure is unchanged is important for us below. Details of the above are found in [6]. If the constraints are not imposed, there is an additional term $(N/4)\mathcal{H}$ in the equation for $\partial k/\partial t$.

One refers to N as the *lapse function* and to X as the *shift vector field*. They were first introduced by J. A. Wheeler. Below in § 6, 7 we shall see that these lapse and shift functions are closely connected with the constraint equations and the invariance groups of general relativity.

3. Geometry of the constraint set.

We now summarize some basic results on the constraint set defined by (3).

Let us begin by supposing M is compact; we consider the following conditions on a point $(g, k) \in T\mathcal{M}$.

(4)
- (i) any vector field X satisfying $L_X g = 0$ and $L_X \pi = 0$ is zero (L_X denotes the Lie derivative),
- (ii) if $k \neq 0$ then g is not flat,
- (iii) $\operatorname{tr}(k)$ is constant.

Let $\mathcal{C}_\delta$ denote the set of g, k such that $\delta\pi = 0$ and $\mathcal{C}_{\mathcal{H}}$ those for which $\mathcal{H}$ is identically zero, and let $\mathcal{C} = \mathcal{C}_\delta \cap \mathcal{C}_{\mathcal{H}}$, the constraint set.

THEOREM 2: (i) *If* $(g, k) \in T\mathcal{M}$ *satisfies* (i) *of* (4), *then* $\mathcal{C}_\delta$ *is a smooth submanifold of* $T\mathcal{M}$ *in a neighborhood of* (g, k).

(ii) *If* (g, k) *satisfies* (ii) *of* (4), *then* $\mathcal{C}_{\mathcal{H}}$ *is a smooth submanifold near* (g, k).

(iii) *If* (g, k) *satisfies* (i), (ii) *and* (iii) *then* $\mathcal{C}$ *is a smooth submanifold near* (g, k).

Furthermore, the tangent spaces of $\mathcal{C}_\delta$, $\mathcal{C}_{\mathcal{H}}$, $\mathcal{C}$ *are obtained by linearizing the equations* $\delta\pi = 0$ *and* $\mathcal{H} = 0$. *For example regarding* δ *as a map of* $T\mathcal{M}$ *to vector fields,*

$$T_{(g,k)}\mathcal{C}_\delta = \{(h, \omega) | D\delta \cdot (g, k) \cdot (h, \omega) = 0\}$$

where D *denotes the Fréchet derivative.*

The details of proof may be found in [8], but we can easily explain the method here. Namely we first replace the space $\mathcal{M}$ by the corresponding Sobolev space $\mathcal{M}^s$, so that we have a Hilbert manifold. Let $\mathfrak{X}^s$ denote the H^s vector fields on M. Then $\delta: T\mathcal{M}^s \to \mathfrak{X}^{s-1}$ is a smooth map for s large enough $(s > n/2 + 1)$. It is a certain nonlinear differential operator. We then show that the derivative $D\delta(g, k)$: $S_2^s \times S_2^s \to \mathfrak{X}^{s-1}$ is surjective, where (g, k) satisfies (i) of (4). Here S_2^s denotes the symmetric two tensors of class H^s. To do this, one uses elliptic theory; namely one shows that the symbol of $D\delta(g, k)$ is injective and that the adjoint of $D\delta(g, k)$ has trivial kernel. Once this is done, that $\delta^{-1}(0)$ is a submanifold then follows from the implicit function theorem; i.e. δ is a submersion at and hence in a neighborhood of (g, k).

One proceeds with $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}$ in a similar manner. Finally a regularity argument enables one to pass from H^s to C^∞.

The following is an important but immediate deduction from theorem 1.

COROLLARY: *Let* $(g, k) \in \mathcal{C}$ *let* (4) *hold and let* (h, w) *satisfy the linearized constraint equations*:

$$D\delta\,(g, k) \cdot (h, w) = 0$$

$$D\mathcal{H}(g, k) \cdot (h, w) = 0\,.$$

Then there is a curve $(g(\lambda), k(\lambda))$ *of exact solutions to the constraint equa-*

tions tangent to (h, w):

$$g(0) = g\ ,\qquad k(0) = k\ ,$$

$$\left.\frac{dg}{d\lambda}\right|_{\lambda=0} = h\ ,\qquad \left.\frac{dk}{d\lambda}\right|_{\lambda=0} = w\ .$$

If the conditions of (4) do not hold then the point (g, k) may be genuinely singular; i.e. solutions of the linearized equations need not correspond to a curve of exact solutions. This is discussed in [8].

If M is non-compact, say $M = \mathbf{R}^3$, then as was mentioned above, asymptotic conditions are imposed. For example in (i) of (4) only X's vanishing at infinity are allowed and (ii) may be dropped. Thus g the usual metric on $\mathbf{R}^3$ and $k = 0$ is allowed, and $\mathcal{C}$ is a smooth manifold near this point. The corollary then reduces to a theorem of Y. Choquet and S. Deser (see [8] for this and related references).

4. Reduced phase spaces.

We now describe one additional piece of background material that we shall need. This is concerned with a method for the construction of phase spaces when symmetry groups are present, and is taken from Marsden-Weinstein [11].

Basically, the result is a non-commutative generalization of the classical fact that if one has k first integrals in involution, then one can reduce the symplectic manifold to another one in which $2k$ variables have been eliminated.

Let P, Ω be a weak symplectic manifold and let G be a Lie group which acts on P by symplectic diffeomorphisms (= symplectomorphisms = canonical transformations). If we let $\Phi_g\colon P \to P$ denote the action on P corresponding to $g \in G$, then we are assuming $\Phi_g^* \Omega = \Omega$.

Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ its dual (as a vector space). For $\xi \in \mathfrak{g}$, let ξ_P denote the infinitesimal generator on P; thus ξ_P is a vector field on P defined by

$$\xi_P(p) = \frac{d}{dt}\Phi_{\exp t\xi}(p)\Big|_{t=0}\ .$$

Following terminology of J. M. Souriau, we suppose $\psi\colon P \to \mathfrak{g}^*$ is a moment for this action. This means that for each $\xi \in \mathfrak{g}$, ξ_P is a Hamiltonian vector field with energy given by $p \mapsto \psi(p)\cdot\xi$.

For example, it is well known that if Φ_g is the canonical lift of an action Ψ_g on M to $P = T^* M$, then

$$\psi: P \to \mathfrak{g}^*, \qquad \psi(\alpha_x)\cdot\xi = \langle \alpha_x, \xi_M(x)\rangle$$

where $\alpha_x \in T_x^* M$.

This particular moment just constructed is Ad^* equivariant, and we suppose this is true in general:

$$\psi \circ \Phi_g = (\mathrm{Ad}_{g^{-1}})^* \circ \psi ,$$

where $(\mathrm{Ad}_{g^{-1}})^*$ is the coadjoint action of G on $\mathfrak{g}^*$.

In concrete cases, ψ corresponds to a conserved quantity such as angular momentum. It is easy to see that ψ is conserved by any Hamiltonian vector field X_H for which $H \circ \Phi_g = H$.

Now let $\mu \in \mathfrak{g}^*$ be a regular value of ψ so that $\psi^{-1}(\mu)$ is a submanifold of P. Set

$$G_\mu = \{g \in G \,|\, \mathrm{Ad}^*_{g^{-1}}\mu = \mu\}$$

the isotropy subgroup of G for μ. From Ad* equivariance it is easy to see that G_μ acts on $\psi^{-1}(\mu)$. Thus we can form the space of orbits

$$P_\mu = \psi^{-1}(\mu)/G_\mu$$

called the *reduced phase space.* Assume P_μ is a manifold; this will hold if, for example the action is free and proper and $T_{[p]}P_\mu = T_p\psi^{-1}(\mu)/T_p(G_\mu\cdot p) = \ker T_p\psi/T_p(G_\mu\cdot p)$ where $[p] = G_\mu\cdot p$ is the orbit of p.

THEOREM 3: *P_μ inherits, in a natural way, a weak symplectic structure from P, Ω.*

For example if P is finite dimensional, it follows that P_μ is even dimensional, which is not a priori obvious. In [11] we show how this result unifies many constructions, such as the symplectic structure on the orbit of a point in $\mathfrak{g}^*$ under the coadjoint action (due to Kostant and Souriau).

In what follows we shall use theorem 3 to give a method of constructing a symplectic structure on the space $T\mathcal{M}$ when the symmetries corresponding to $\mathcal{D}$ and $\mathcal{K}$ have been divided out; we shall then connect this up with the set of solutions to Einsteins equations with isometric metrics identified.

5. The space of solutions to Einsteins equations.

Let V be a fixed four manifold and let $\mathcal{E}_0$ denote the set of Lorentz metrics ${}^{(4)}g$ on V which are Ricci flat. Identify ${}^{(4)}g_1$ and ${}^{(4)}g_2$ if there is a diffeomorphism $\varphi: V \to V$ such that $\varphi^{*(4)}g_1 = {}^{(4)}g_2$. Let $\mathcal{E}_1$ denote the resulting quotient space. This construction can be transcribed in terms of intitial data as follows. Pick a hypersurface $M \subset V$. Then diffeomorphisms of V leaving M invariant induce an equivalence relation on $\mathcal{C} \subset T\mathcal{M}$. Let $\mathcal{E}$ denote the space of equivalence classes of these (g, k). One refers to $\mathcal{E}$ as the space of « true gravitational degrees of freedom ». The space $\mathcal{E}$ corresponds to identifying those (g, k) which are related by a new choice of lapse and shift function; i.e. by a coordinate change on V. The spaces $\mathcal{E}$, $\mathcal{E}_1$ inherit from $T\mathcal{M}$ a symplectic form Ω. That this form is independent of M follows from the fact that the Hamiltonian evolution equations preserve the symplectic form.

It is to be noted that Ω is a well-defined weak symplectic form on $\mathcal{E}$, but on $\mathcal{E}_0$ degeneracies and ambiguity arise.

The above approach has several difficulties. In particular that Ω is a well defined weak symplectic form on $\mathcal{E}$ is a little awkward, but not impossible to show. More serious, it is not at all clear that $\mathcal{E}_0$ or $\mathcal{E}$ are in any reasonable sense smooth manifolds. In fact, in general, they are probably highly singular with the formal tangent space being spurious. (However a recent conjecture of D. Ebin suggests that while $\mathcal{C}$ has singularities, $\mathcal{E}$ may not.)

If however, we combine the ideas from § 2-4 we can obtain a more satisfactory solution. This is done in the next sections.

6. The dynamical group of general relativity.

Let us again fix M and consider the dynamics on $T\mathcal{M}$ as explained above. There is a basic invariance group for this dynamics, namely $\mathcal{D} \times \mathcal{T}$ which we now wish to explain. Here $\mathcal{D}$ denotes the group of all diffeomorphisms of M and $\mathcal{T}$ denotes the additive group of real valued functions on M. Now $\mathcal{D}$ acts on $\mathcal{T}$ by $\tau \mapsto \tau \circ \eta^{-1}$ for $\eta \in \mathcal{D}$. On $\mathcal{D} \times \mathcal{T}$ we put the semi-direct product structure, so group multiplication is

$$(\eta, \tau) \cdot (\varphi, \varrho) = (\eta \circ \varphi, \tau \circ \varphi^{-1} + \varrho) .$$

The tangent space to $\mathcal{D}$ at the identity, e, is the space of vector fields $X \in \mathfrak{X}$ (these correspond to shift functions), while that of $\mathcal{T}$ at 0 is the space $\mathcal{T}$ itself ($N \in \mathcal{T}$ corresponding to lapse functions).

The group $\mathcal{D}\times\mathcal{J}$ acts on $T\mathcal{M}$ as follows: for $\eta\in\mathcal{D}$, map (g, k) to $(\eta_* g, \eta_* k)$. If η_t is the flow of X and $g(t)$, $k(t)$ is the solution with $N=1$, $X=0$, then $(\eta_* g, \eta_* k)$ is the solution with $N=1$ and X as the shift.

Let $\mathcal{J}$ act on g, k as follows. Fix N and let $\tau = tN$. Let τ map g, k to $g(t)$, $k(t)$ the solution of the evolution equations with lapse N and shift 0.

THEOREM 4: *The above defines a symplectic action of $\mathcal{D}\times\mathcal{J}$, with the semi-direct product structure, on $T\mathcal{M}$. Furthermore, this action has a moment given by*

$$\psi\colon T\mathcal{M}\to(\mathcal{X}\times\mathcal{J})^*\,,$$
$$\psi(g, k)\cdot(X, N) = 2\int_M X\cdot\delta\pi\, d\mu_g + \int_M N\mathcal{H}\, d\mu_g\,.$$

Instead of giving the details of the above, we shall confine ourselves to a few remarks pertinent to the proof.

(*a*) One needs to use the semi-direct product structure, for if $g(t)$, $k(t)$ is a solution with given N and shift zero and if X is given with flow η_t, then the solution with lapse $N\circ\eta_t^{-1}$ and shift X is $\eta_* g(t)$, $\eta_* k(t)$.

(*b*) The action is symplectic in $\mathcal{J}$ because of the Hamiltonian character of the equations and it is symplectic in $\mathcal{D}$ by properties of pull back.

(*c*) The action may actually be defined only near 0 in $\mathcal{J}$ because initial data may be propagated only a finite amount, but this does not affect the argument.

(*d*) That we have an action is actually not trivial and requires some computation. In any case, that ψ is the moment is computed from standard formulas (see [6], [10]). The infinitesimal statement that we have an action is expressed by the following commutation relations: setting $P(X) = 2\int_M X\cdot\delta\pi d\mu$, $T(N) = \int_M N\mathcal{H} d\mu$, we have

$$\left|\begin{aligned} &\{P(X), P(Y)\} = P([X, Y])\,,\\ &\{T(N), T(Q)\} = 0\,,\\ &\{T(N), P(X)\} = T(X(N))\,.\end{aligned}\right.$$

Here the first expression is clear from general properties of momentum functions (cf. [2]). The Poisson brackets are, of course, computed in our symplectic structure on $T\mathcal{M}$. The second commutation relation reflects the fact that $\mathfrak{J}$ is abelian and the last corresponds to the semi-direct product structure on $\mathfrak{D}\times\mathfrak{J}$ (cf. Fadeev [5]).

7. Construction of $\mathcal{E}$ as a quotient manifold of $\mathcal{C}$.

THEOREM 5: *Let ${}^{(4)}g$ on V have a space like hypersurface M on which ${}^{(3)}g$, ${}^{(3)}k$ satisfy the conditions* (4). *Then in a sufficiently small neighborhood of M and restricting to metrics sufficiently close to g, k we have*

$$\mathcal{E}\simeq\mathcal{C}/(\mathfrak{D}\times\mathfrak{J})$$

and $\mathcal{C}/(\mathfrak{D}\times\mathfrak{J})$ has the structure of a smooth manifold. Moreover, $\mathcal{E}$ has a smooth weakly nondegenerate symplectic form naturally inherited from that on $T\mathcal{M}$ and which coincides with that in § 5.

The important feature to note here is that it is essential to pass to the quotient in order for the symplectic form to be non degenerate. It may be degenerate on $\mathcal{E}_0$ or $\mathcal{C}$ alone.

Granting our previous work, the proof of theorem 5 is not difficult. Indeed, § 2 shows that $0\in(\mathfrak{X}\times\mathfrak{J})^*$ is a proper value of ψ defined in theorem 4 and correspondingly $\mathcal{C}=\psi^{-1}(0)$ is a manifold; since the isotropy of 0 is the whole group $\mathfrak{D}\times\mathfrak{J}$, we conclude from § 3 that $\mathcal{C}/(\mathfrak{D}\times\mathfrak{J})$ is indeed a symplectic manifold if it is a manifold. To see the latter one can show the action is free since g, k have no infinitesimal isometries (cf. Ebin [4]) for $\mathfrak{D}$ and from the dynamics for $\mathfrak{J}$. Thus $\mathcal{C}/(\mathfrak{D}\times\mathfrak{J})$ is a smooth manifold.

Finally, one must show that $\mathcal{E}$ and $\mathcal{C}/(\mathfrak{D}\times\mathfrak{J})$ are identifiable. This is easily shown by fixing M and mapping $[{}^{(4)}g]\in\mathcal{E}_1$ ([ ] stands for its equivalence class) to $[(g, k)]$, where g, k is the induced metric and second fundamental form on M. If one now traces through the definition carefully, it is seen that this is a bijection from $\mathcal{E}$ to $\mathcal{C}/\mathfrak{D}\times\mathfrak{J}$. This then completes theorem 5. This construction seems to be analogous to that given by De Witt [3].

Fadeev [5] uses his formal construction to discuss quantization of general relativity (see also the article of Segal in this volume). There remains the task of using the precise manifold structure on $\mathcal{E}$ obtained above to justify his calculations. Unfortunately the fact that $\mathcal{E}$ is only a «local manifold»; i.e. for metrics near a given one and in a neighborhood of M, and might have severe singularities in general, may hamper this program.

8. Linearization stability.

A common practice in general relativity is the following perturbation procedure. One starts with a given spacetime V and ${}^{(4)}g$ on V satisfying the field equations:

$$\mathrm{Ric}\,{}^{(4)}g = 0 .$$

One then linearizes about ${}^{(4)}g$ and seeks to solve the linearized equations:

$$D\,\mathrm{Ric}({}^{(4)}g)\cdot{}^{(4)}h = 0$$

for a tensor ${}^{(4)}h$. Then ${}^{(4)}g + \lambda\,{}^{(4)}h$ for λ small is supposed to be a first order approximation to an exact solution. Written out, the linearized equations are:

$$\Box_L{}^{(4)}h - \alpha({}^{(4)}g)\cdot\delta({}^{(4)}h - \tfrac{1}{2}\,\mathrm{tr}\,({}^{(4)}h)\,g) = 0$$

where $\Box_L$ is the Lichnerowicz d'Alembertian computed from ${}^{(4)}g$ (see Lichnerowicz [9]) and where $\alpha(g)\cdot X = L_X g$.

The results of § 3 can be used to justify the above procedure in many cases (this result is one of the motivations for the results of § 2 and also was for Y. Choquet and S. Deser who obtained a special case as we have mentioned).

THEOREM 6: *Let ${}^{(4)}g$ be a Lorentz metric on V satisfying $R_{\alpha\beta} = 0$ and let ${}^{(4)}h$ be a solution of the linearized equations. Let M be a space-like hypersurface with induced metric g and second fundamental form k. Assume (for M compact) that g, k satisfy (4) of § 3. Then there exists a smooth curve ${}^{(4)}g(\lambda)$ of exact solutions of $R_{\alpha\beta} = 0$ such that*

$${}^{(4)}g(0) = {}^{(4)}g$$

and

$$\frac{d}{d\lambda}\,{}^{(4)}g(\lambda)\big|_{\lambda=0} = {}^{(4)}h .$$

Here ${}^{(4)}g(\lambda)$ are defined in some neighborhood of M in V.

The proof is simple. Namely we choose, say Gaussian normal coordinates around M for ${}^{(4)}g$; i.e. work with lapse 1 and shift $= 0$. The solution ${}^{(4)}h$ induces on M solutions to the linearized constraint equations (h, w) since solutions ${}^{(4)}g$ to $R_{\alpha\beta}$ always induce solutions (g, k) to the constraint equations. But from § 3, $\mathcal{C}$ is a manifold near (g, k), so (h, w) is a tangent vector to $\mathcal{C}$ at (g, k). This (h, w) is tangent to a curve $g(\lambda)$, $k(\lambda)$ in $\mathcal{C}$. But any point in $\mathcal{C}$ defines an exact solution of $R_{\alpha\beta}=0$ near M by the existence theory for the Cauchy problem in general relativity (cf. [7]). This defines the ${}^{(4)}g(\lambda)$ we wanted.

As an example we conclude that Minkowski space is linearization stable; i.e. satisfies the conclusions of theorem 6.

Interestingly the analogous results for the Riemannian (as opposed to Lorentz) case are rather different. Indeed solutions of $\mathrm{Ric}=0$ may often be isolated; for example the flat metric in $\mathbf{R}^n$ is geometrically isolated. The Lorentz case is more tractible because the problem can be reduced to a consideration of the constraint equations. The latter involves just the *scalar* curvature, and this is a much more flexible object to work with.

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