REDUCTION OF SYMPLECTIC MANIFOLDS WITH SYMMETRY

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We give a unified framework for the construction of symplectic manifolds from systems with symmetries. Several physical and mathematical examples are given; for instance, we obtain Kostant's result on the symplectic structure of the orbits under the coadjoint representation of a Lie group. The framework also allows us to give a simple derivation of Smale's criterion for relative equilibria. We apply our scheme to various systems, including rotationally invariant systems, the rigid body, fluid flow, and general relativity.

§ 1. Introduction

The purpose of this note is to show that when we have a symplectic manifold on which a group acts symplectically, we can reduce this phase space to another symplectic manifold in which, roughly speaking, the symmetries are divided out. This unifies part of Smale's program [18] with certain known facts about Lie groups due to Kostant ([10], theorem 5.3.1). We shall give several physical examples, some of which are known; for example, it is a classical fact that one can reduce a given phase space to a "smaller" symplectic mainifold when one has first integrals in involution. (See, for instance, [22], [4], [16], and references therein.)

When one has a Hamiltonian system on the phase space which is invariant under the group, there is a Hamiltonian system canonically induced on the reduced phase space. Fixed points of this reduced system are called *relative equilibria*; we are able to give a very simple derivation of Smale's criterion for them using the above ideas. We easily obtain Arnold's criterion for stability of relative equilibria for systems on a Lie group; the rigid body is an example. Numerous other physical situations will be presented as well.

We use standard or self-explanatory terminology. For background, see [1], [3], [20] and [21]. Conventions on forms are as in [3].

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§ 2. Construction of the reduced phase space

Let (P, Ω) be a connected symplectic manifold and let G be a Lie group acting on P (from the left) by symplectic diffeomorphisms. [Here P or G may be infinite dimensional and in this case Ω may be only weak; i.e., the map of TP to T^*P induced by Ω may be only injective. This generality is needed only for examples 6 and 7 of § 4 below and will be dealt with at that time.]

For each $\xi \in \mathfrak{G}$, the Lie algebra of G, we let ξ_P denote the corresponding vector field on P; ξ_P is the generator of the action of $\mathbb{R}$ corresponding to the subgroup $\exp(t\xi)$. Since the action is symplectic, the one form $i_{\xi_P}\Omega = \xi_P \lrcorner\, \Omega$ is closed.

By a *moment* for the action we mean a C^∞ map $\psi: P \to \mathfrak{G}^*$ such that if $\hat{\psi}$ denotes the dual map from $\mathfrak{G}$ to the space of smooth functions on P, we have

$$d(\hat{\psi}(\xi)) = \xi_P \lrcorner\, \Omega,$$

i.e., $\langle T_p \psi \cdot v, \xi \rangle = \Omega_p(\xi_P(p), v)$ for $\xi \in \mathfrak{G}$, $v \in T_p P$. In other words, each infinitesimal generator ξ_P has $\hat{\psi}(\xi)$ as a Hamiltonian function. A moment, if it exists, is defined up to an arbitrary additive constant in $\mathfrak{G}^*$. The notion of a moment is due to Souriau [20].

Souriau [20] has shown that the moment ψ is equivariant with respect to a certain affine action (depending on ψ) of G on $\mathfrak{G}^*$, the linear part of which is the dual of the adjoint action. In many cases, for example, if G is semi-simple or if G leaves invariant a one-form θ with $\Omega = d\theta$, we may choose ψ to be equivariant with respect to the co-adjoint action. In this case, if Φ_g: $P \to P$ denotes the action of $g \in G$, equivariance asserts that

$$\psi \circ \Phi_g = (\mathrm{Ad}_{g^{-1}})^* \circ \psi.$$

In the case $\Omega = d\theta$, it is easy to see that one can choose $\psi(p) \cdot \xi = -i_{\xi_P}\theta(p)$.

In the general case, for $\mu \in \mathfrak{G}^*$, let G_μ denote the isotropy group of μ with respect to the above action of G on $\mathfrak{G}^*$. By equivariance, $\psi^{-1}(\mu)$ is invariant under G_μ so the orbit space $\psi^{-1}(\mu)/G_\mu$ is defined. Note also that, by equivariance, if $p \in \psi^{-1}(\mu)$ and $\Phi_g(p) \in \psi^{-1}(\mu)$, then $g \in G_\mu$. We let

$$P_\mu = \psi^{-1}(\mu)/G_\mu$$

and call P_μ the *reduced phase space.*

§ 3. The symplectic structure on P_μ

It will be useful to have the following notion:

DEFINITION. If f: $M \to N$ is a smooth map, $n \in N$ is called a *weakly regular value* of f if
(a) $f^{-1}(n)$ is a submanifold of M and
(b) for every $m \in f^{-1}(n)$ the inclusion $T_m[f^{-1}(n)] \subset \ker T_m f$ is an equality.

Note that a regular value (wherein $T_m f$ is surjective) is always a weakly regular value, but that there are interesting weakly regular values which are not regular, such as zero angular momentum in the two-body problem.

Let us recall that G acts freely on M when each point $m \in M$ has a trivial isotropy group or, equivalently, each orbit (which is always an immersed submanifold) in M is in one-to-one correspondence with G. The action is proper when the map $(g, m) \mapsto (m, \Phi_g(m))$ is proper (inverse images of compact sets are compact). This holds if, for instance, G is compact, or if $M=G$ and the action is group multiplication. If G acts freely and properly on M, then M/G is a manifold such that the projection $\pi: M \to M/G$ is C^∞ (see Bourbaki [3], p. 63). Coordinates for M/G are provided by a "slice" for the action. Letting $G \cdot m$ denote the orbit through $m \in M$, we may identify $T_{\pi(m)}(M/G) \approx T_m M / T_m(G \cdot m)$. Moreover, $T_m(G \cdot m) = \{\xi_M(m) \mid \xi \in \mathfrak{G}\}$, where ξ_M denotes the infinitesimal generator for the action on M.

THEOREM 1. *Let G be a Lie group acting symplectically on the symplectic manifold (P, Ω). Let ψ be a moment for the action. Let $\mu \in \mathfrak{G}^*$ be a weakly regular value of ψ. Suppose that G_μ acts freely and properly on the manifold $\psi^{-1}(\mu)$. Then if i_μ: $\psi^{-1}(\mu) \to P$ is the inclusion, there is a unique symplectic structure Ω_μ on the reduced phase space P_μ such that $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$, where π_μ is the projection of $\psi^{-1}(\mu)$ onto P_μ.*

To prove this we shall make use of the following:

LEMMA. *For $p \in \psi^{-1}(\mu)$ we have*

(i) $T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_p(\psi^{-1}(\mu))$, *and*

(ii) $T_p(\psi^{-1}(\mu))$ *is the Ω-orthogonal complement of* $T_p(G \cdot p)$.

Proof: (i) Let $\xi \in \mathfrak{G}$, so $\xi_P(p) \in T_p(G \cdot p)$. We must show that $\xi_P(p) \in T_p(\psi^{-1}(\mu))$ iff $\xi \in \mathfrak{G}_\mu$ the Lie algebra of G_μ. Equivariance gives $T_p \psi \cdot \xi_P(p) = \xi_{\mathfrak{G}^*}(\mu)$, so $\xi \in \mathfrak{G}_\mu$ iff $\xi_{\mathfrak{G}^*}(\mu) = 0$ iff $\xi_P(p) \in \ker T_p \psi = T_p(\psi^{-1}(\mu))$.

(ii) For $\xi \in \mathfrak{G}$, $v \in T_p P$ we have $\Omega(\xi_P(p), v) = \langle T_p \psi \cdot v, \xi \rangle$, since ψ is a moment. Thus $v \in \ker T_p \psi$ iff $\Omega(\xi_P(p), v) = 0$ for all $\xi \in \mathfrak{G}$. □

In finite dimensional spaces, it is easy to check that the Ω-orthogonal complement of the Ω-orthogonal complement of a linear subspace $F \subset T_p P$ is again F. Thus by the above lemma, $T_p(\psi^{-1}(\mu))$ and $T_p(G \cdot p)$ are orthogonal complements of each other. We shall use this in the proof of Theorem 1. In the infinite dimensional case, the appropriate assumptions are given in the following lemma. We are grateful to R. Graff and P. Chernoff for pointing out the proof given here.

LEMMA. *Let E be a reflexive Banach space. Let ω be a continuous alternating bilinear form on E. For $M \subset E$ a closed subspace, define $M^\perp = \{y \in E \colon \omega(x, y) = 0$ for all $x \in M\}$. Define B: $E \to E^*$ by*

$$\langle Bx, y \rangle = \omega(x, y).$$

Assume that B is one-to-one with closed range. (This is a condition intermediate between weak and strong nondegeneracy of ω.)

Then for every closed subspace M, we have $M = (M^\perp)^\perp$.

Proof: The hypothesis on B implies that B is a homeomorphism onto its range. So BM is closed in E^*. Suppose that $x \notin M$. Then $Bx \notin BM$. The Hahn–Banach theorem together with reflexivity then yields the existence of a vector $y \in E$ with $\langle Bx, y\rangle \neq 0$ but $\langle Bz, y\rangle = 0$ for all $z \in M$. In other words, $\omega(x, y) \neq 0$ but $y \in M^\perp$. Thus x is *not* in $(M^\perp)^\perp$, and so $(M^\perp)^\perp \subseteq M$. The reverse inclusion is trivial. □

Proof of Theorem 1: For $v \in T_p(\psi^{-1}(\mu))$, let $[v] \in T_{\pi_\mu(p)} P_\mu$ denote the corresponding equivalence class in $T_p \psi^{-1}(\mu)/T_p(G_\mu \cdot p)$, so $[v] = T\pi_\mu \cdot v$. The assertion $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$ becomes

$$\Omega_\mu([v], [w]) = \Omega(v, w), \qquad v, w \in T_p \psi^{-1}(\mu).$$

Thus Ω_μ is unique. Moreover, Ω_μ is well-defined because of the Lemma. Ω_μ is also smooth because quantities on a quotient M/G are smooth when they have smooth pull-backs to M. Thus Ω_μ is a well-defined smooth two-form on P_μ.

To show that Ω_μ is symplectic we first show that Ω_μ is non-degenerate; $\Omega_\mu([v], [w]) = 0$ for all $w \in T_p \psi^{-1}(\mu)$ implies $v \in T_p(G_\mu \cdot p)$ by the Lemma, or $[v] = 0$. It remains to show that Ω_μ is closed. But from $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$ and $d\Omega = 0$, we conclude that $\pi_\mu^*(d\Omega_\mu) = 0$, so $d\Omega_\mu = 0$ since $T\pi_\mu$ is surjective. □

Remarks: Even if $\Omega = d\theta$ and the action leaves θ invariant, Ω_μ need not be exact. For $\mu \neq 0$, θ does not project to a one-form on P_μ because $\theta(\xi_P)(p) = \psi(p)\,\xi \neq 0$.

As a consequence, observe that P_μ is even-dimensional. If ψ is a submersion, then $\dim P_\mu = \dim P - \dim G - \dim G_\mu$.

If μ is a regular value of ψ, the action is always locally free near $\psi^{-1}(\mu)$.

§ 4. Examples

1. If G acts on a manifold M, we obtain a symplectic action on T^*M which preserves the canonical one-form θ on T^*M (see [1], [20]). A moment for this action is given by ψ: $T^*M \to \mathfrak{G}^*$:

$$\langle \psi(\alpha), \xi\rangle = \langle \alpha, \xi_M(m)\rangle, \qquad \alpha \in T_m M.$$

This moment is Ad*-equivariant. This result is due to Souriau [20], but was also obtained by Marsden [11] and Smale [18].

We conclude from Theorem 1 that if G_μ acts freely and properly on $\psi^{-1}(\mu) = \{\alpha \in T^*M \mid \langle \alpha, \xi_M(m)\rangle = \langle \mu, \xi\rangle$ for all $\xi \in \mathfrak{G}\}$, then $\psi^{-1}(\mu)/G_\mu$ is a symplectic manifold. If the $\xi_M(m)$ span a space of dimension $= \dim \mathfrak{G}$ at m, then it is easy to see that each point of $T_m^* M$ is regular.

2. If we specialize Example 1, taking $M = G$ with G acting on itself by left multiplication, then the moment ψ: $T^*G \to \mathfrak{G}^*$ is given by

$$\psi(\alpha) = (TR_g)^* \cdot \alpha \in T_e^* G = \mathfrak{G}^*, \qquad \alpha \in T_g G,$$

where R_g denotes right translation (see [2], [13], [9]). Thus each $\mu \in \mathfrak{G}^*$ is regular and

$\psi^{-1}(\mu)$ is the graph of the right invariant one-form ω_μ whose value at e is μ. Now $G_\mu = \{g \in G \mid L_g^* \, \omega_\mu = \omega_\mu\}$, so the action of G_μ on $\psi^{-1}(\mu)$ is left translation on the base point. Thus $\psi^{-1}(\mu)/G_\mu \approx G/G_\mu \approx G \cdot \mu \subset \mathfrak{G}^*$. Thus the reduced phase space is just the orbit of μ in $\mathfrak{G}^*$. That this is a symplectic manifold then follows from Theorem 1. The rather special construction in this case is due to Kirillov–Kostant; see [10]. If one traces through the definitions, one finds for $\beta \in G \cdot \mu$, $\gamma_1 = (\text{ad } u_1)^* \beta$ and $\gamma_2 = (\text{ad } u_2)^* \beta$, that

$$\Omega_\mu(\gamma_1, \gamma_2) = \beta([u_2, u_1]) .$$

When viewed directly, the symplectic structure on $G \cdot \mu \subset \mathfrak{G}^*$ seems rather special. However, it becomes natural when viewed in the context of reduced phase spaces. Moreover, the proof becomes more transparent. This example is studied further in §§ 6, 7 below.

3. If G acts on M and leaves a given closed two-form F on M invariant, then we get a symplectic action on T^*M with the symplectic form $\Omega_F = \Omega + \pi^* F$, where Ω is the canonical form and π: $T^*M \to M$ the projection. Such a situation arises when one has a particle moving in the "electromagnetic field" F (cf. [19]). Now suppose that $F = dA$ is exact and A is invariant. Then the moment is given by

$$\langle \psi(\alpha), \xi \rangle = \langle \alpha - A, \xi_M(m) \rangle$$

(this corresponds to the classical prescription of replacing p by $p - (e/c) A$ in an electromagnetic potential A). The verification is the same as in Example 1. Thus again, if μ is a weakly regular value and G_μ acts freely and properly on $\psi^{-1}(\mu)$, we can form the reduced phase space P_μ.

4. Let $G = SO(3)$ and P a symplectic manifold. Here $\mathfrak{G} \approx R^3$ and the adjoint action is the usual one. For $\mu \in R^3$, $\mu \neq 0$, $G_\mu = S^1$ corresponding to rotations about the axis μ. Since G is semi-simple, a symplectic action of G on P has an Ad*-equivariant moment ψ by Souriau [20]. One refers to ψ as "angular momentum" in this case. The reduction of P to $\psi^{-1}(\mu)/S^1$ is a generalization of the procedure called "elimination of the nodes" (cf. Smale [18] and Whittaker [22], p. 344).

5. Suppose we have the situation of Theorem 1, and in addition G is Abelian. Ad*-equivariance means that the generating functions $\hat{\psi}(\xi)$ are all in involution on P. Furthermore, $G_\mu = G$ for each $\mu \in \mathfrak{G}^*$. If the action is free and μ is a regular value, we can form $P_\mu = \psi^{-1}(\mu)/G$. In this case $\dim P_\mu = \dim P - 2 \dim G$. The reduction to P_μ represents the classical reduction of a Hamiltonian system by integrals in involution.

As a special case, let X_H be a Hamiltonian vector field on P, so that the flow of X_H yields an action of R on P. The moment is just H itself so we get a symplectic structure on $H^{-1}(e)/R$ which is just the space of orbits on each energy surface (we assume that e is a regular value of H).

6. Let $\mathscr{D}$ denote the group of C^∞-diffeomorphisms of a finite dimensional Riemannian manifold M. Suppose that M is compact, or restrict to diffeomorphisms which are "asymptotic to the identity" ([7]). Now $T_e \mathscr{D} = \mathscr{X}(M) =$ the vector fields on M ([2], [6])

and we put on $\mathscr{D}$ the L_2 metric which is obtained from $\mathscr{X}(M)$ by right invariance. Thus $\mathscr{D}$ acting on $T\mathscr{D}$ on the right is a symplectic action. As in Example 2, we conclude that for each $X \in \mathscr{X}(M)$, the set $\{\eta^* X \mid \eta \in \mathscr{D}\} \subset \mathscr{X}(M)$ is a weak symplectic manifold. The symplectic structure is

$$\Omega_X(\eta^* L_{Y_1} X, \eta^* L_{Y_2} X) = \int_M \langle X, [Y_2, Y_1] \rangle \, dx .$$

One may similarly restrict to volume preserving diffeomorphisms and divergence free vector fields. This symplectic manifold is important in fluid mechanics. See Arnold [2] and Ebin–Marsden [6].

Here the manifolds are Fréchet. Properly, one should use Sobolev spaces, as in [6]. One can show that the orbit of X above is a smooth submanifold by using techniques of Ebin [5].

7. Let M and $\mathscr{D}$ be as in Example 6. Let $\mathscr{M}$ denote the space of all Riemannian metrics on M. Define the DeWitt metric on $\mathscr{M}$ by

$$\mathscr{G}_g(h, k) = \int_M [\langle h, k \rangle - (\operatorname{tr} h)(\operatorname{tr} k)] \, d\mu_g ,$$

where $h, k \in T_g \mathscr{M}$ = the symmetric 2-tensors on M, $\langle h, k \rangle$ is the inner product of h, k using the metric g, tr denotes the trace, and μ_g is the volume element associated with g. $\mathscr{G}_g$ is a weak metric and gives a (weak) symplectic structure on $T\mathscr{M}$.

The space $T\mathscr{M}$ is a basic (weak) symplectic manifold used in general relativity. We shall now describe its reduced phase space in the presence of the symmetry group $\mathscr{D}$. (See [7], [12] for the connections of these ideas with general relativity.) $\mathscr{D}$ acts symplectically on $T\mathscr{M}$ by pull-back. The moment for this action is not difficult to compute (see [7], [12]). It is:

$$\psi(g, k) \cdot X = 2 \int_M \langle X, \delta\pi \rangle \, d\mu_g ,$$

where $\pi = k - \frac{1}{2}(\operatorname{tr} k) g$ and δ is the divergence taken with respect to g. Of particular interest is the case $\psi^{-1}(0) = \{(g, k) \in T\mathscr{M} \mid \delta\pi = 0\}$ (referred to as the divergence constraint in general relativity).

The isotropy group is all of $\mathscr{D}$, so the reduced phase space is $\psi^{-1}(0)/\mathscr{D}$. If we work near a metric with no isometries (asymptotically the identity if M is not compact), then $\psi^{-1}(0)/\mathscr{D}$ is a manifold by the Ebin–Palais Slice Theorem [5], using [8]. We conclude by Theorem 1 that $\psi^{-1}(0)/\mathscr{D}$ is a (weak) symplectic manifold. This is the basic space one uses for a dynamical formulation of general relativity. It is related to the "superspace" $\mathscr{M}/\mathscr{D}$ in that all "geometrically equivalent" objects have been identified.

By making a further reduction by dividing out by the "relativistic time translation group" (see [7] or [12]) and passing to the surface of zero energy, one obtains a symplectic structure on the space of solutions of the four dimensional Einstein field equations with solutions leading to the same spacetime identified. Details of this will be given elsewhere.

§ 5. Hamiltonian systems on the reduced phase space

THEOREM 2. *Let the conditions of Theorem 1 hold. Let K be another group acting symplectically on P with a moment φ. Let the actions of K and G commute and φ be invariant under G. Then*

(i) *K leaves ψ invariant,*

(ii) *the induced action of K on P_μ is symplectic and has a moment which is naturally induced from the moment φ.*

Proof: (i) is a standard fact. To prove (ii), let Ψ_k denote the action of $k \in K$ on P. By (i), $\psi^{-1}(\mu)$ is invariant under this action, and since the action commutes with that of G, we get a well-defined action on P_μ. Also, if $\tilde{\Psi}_k$ is the induced action on P_μ,

$$\pi_\mu^* \tilde{\Psi}_k^* \Omega_\mu = \Psi_k^* \pi_\mu^* \Omega_\mu = \Psi_k^* i_\mu^* \Omega = i_\mu^* \Psi_k^* \Omega = i_\mu^* \Omega .$$

Hence, $\tilde{\Psi}_k^* \Omega_\mu = \Omega_\mu$. Similarly, from the definition of the moment we see that the induced moment is a moment for the induced action: namely, the induced moment $\tilde{\varphi}$ satisfies $\tilde{\varphi} \circ \pi_\mu = \varphi$, thus for $[v] = T\pi_\mu \cdot v \in TP_\mu$, $\xi \in \mathfrak{G}_K$, we have

$$\langle T\tilde{\varphi} \cdot [v], \xi \rangle = \langle T\varphi \cdot v, \xi \rangle = \Omega(\xi_P, v) = \Omega_\mu(\xi_{P\mu}, [v])$$

since, as easily seen, the generators ξ_P, ξ_{P_μ} on $\psi^{-1}(\mu)$ and P_μ are related by the projection π_μ. □

For example, if we consider Example 2 of § 4 and let $G = K$ acting on T^*G by lifting the *right* action, we can apply Theorem 2 to conclude that the natural action of G on the orbit $G \cdot \mu \subset \mathfrak{G}^*$ is a symplectic action. The induced moment is easily seen to be just the identity map: $\tilde{\varphi}(\mathrm{Ad}_g^* \mu) = \mathrm{Ad}_g^* \mu \in \mathfrak{G}^*$.

The fact that G acts symplectically on the orbit $G \cdot \mu$, so that $G \cdot \mu$ is a "homogeneous Hamiltonian G-space", is a known and useful result. See Kostant [10] and Souriau [20], p. 116.

Taking $K = R$ in Theorem 2, we are led to:

COROLLARY 3. *Let the conditions of Theorem 1 hold and let X_H be a Hamiltonian vector field on P with H invariant under the action of G. Then the flow of X_H induces a Hamiltonian flow on P_μ whose energy $\tilde{H}$ is that induced from H; i.e., $\tilde{H} \circ \pi_\mu = H \circ i_\mu$.*

For example, if $\langle\, , \rangle$ is a left invariant metric on a group G, the Hamiltonian $H(v) = \frac{1}{2}\langle v, v \rangle$, which yields geodesics on G, induces a Hamiltonian system on the orbits in $\mathfrak{G}^* \approx \mathfrak{G}$. Note that the original Hamiltonian system on P is completely determined by the induced systems on the reduced spaces P_μ.

Similarly, in each of the other examples of § 4, if we start with a given Hamiltonian system on P, invariant under G, then we can, with no essential loss of information, pass to the Hamiltonian system on the reduced phase space.

§ 6. Relative equilibria and relative periodic points

DEFINITION. In the situation of Corollary 3 above, a point $p \in P$ such that $\pi_\mu(p) \in P_\mu$ is a critical point [resp. periodic point] for the induced Hamiltonian system on P_μ is called a *relative equilibrium* [resp. *relative periodic point*] of the original system.

Poincaré [14] considered relative periodic points in the n-body problem on an equal footing with ordinary periodic points. Indeed, in general, the only "true" dynamics is that taking place in the reduced phase space P_μ.

The following shows that our definition coincides with the standard ones (Smale [18], Robbin [15]).

THEOREM 4. (i) *$p \in P$ is a relative equilibrium iff there is a one-parameter subgroup $g(t) \in G$ such that for all $t \in \mathbf{R}$, $F_t(p) = \Phi_{g(t)}(p)$, where F_t is the flow of X_H and Φ is the action of G.*

(ii) *$p \in P$ is a relative periodic point iff there is a $g \in G$, and $\tau > 0$ such that for all $t \in \mathbf{R}$, $F_{t+\tau}(p) = \Phi_g(F_t(p))$.*

Proof: (i) p is a relative equilibrium iff $\pi_\mu(p)$ is a fixed point for the induced flow on P_μ iff $\pi_\mu(F_t(p)) = \pi_\mu(p)$. If this holds there is a unique curve $g(t) \in G_\mu$ such that $F_t(p) = \Phi_{g(t)}(p)$ since the action of G_μ on $\psi^{-1}(\mu)$ is free. The flow property $F_{t+s}(p) = F_t \circ F_s(p)$ immediately gives $g(t+s) = g(t)\, g(s)$, so $g(t)$ is a one-parameter subgroup of G_μ. Conversely, if $F_t(p) = \Phi_{g(t)}(p)$ where $g(t)$ is a one-parameter subgroup of G, we must show $g(t) \in G_\mu$. But this follows from invariance of $\psi^{-1}(\mu)$ under F_t and equivariance (see § 2 above).

One proves (ii) in a similar way. □

As a result of our definition we have the following theorem of Smale, whose proof has also been simplified by Robbin [15] and Souriau. We present another proof.

THEOREM 5. *Let μ be a regular value of ψ. Then $p \in \psi^{-1}(\mu)$ is a relative equilibrium iff p is a critical point of $\psi \times H$: $P \to \mathfrak{G}^* \times \mathbf{R}$.*

Proof: By our definition (and Theorem 1), p is a relative equilibrium iff $\pi_\mu(p)$ is a critical point of $\tilde{H}$, the reduced Hamiltonian. Since we have invariance under G, this is equivalent to p being a critical point of $H|\psi^{-1}(\mu)$. This in turn is equivalent to p being a critical point of $\psi \times H$ (Lagrange multiplier theorem.) □

Thus the advantage of passing to P_μ is that relative equilibria really become equilibria and, moreover, we have a Hamiltonian system on P_μ with a (non-degenerate) symplectic form.

In Theorem 5, it is necessary that μ be a regular value. For example, in the n-body problem (where $G = SO(3)$), if all the bodies are lined up with velocities headed towards the center of mass, we have a critical point of $\psi \times H$ but the bodies do not travel in circles (Theorem 4 (i) fails). This was pointed out by J. Robbin.

§ 7. Stability of relative equilibria

DEFINITION. Let $p \in P$ be a relative equilibrium of the Hamiltonian vector field X_H as above. We call p *relatively stable* if the point p is (Liapunov) stable on the quotient space P/G, where p appears as a fixed point.

THEOREM 6. *Let the conditions of Theorem 1 and Corollary 3 hold and let $p \in P$ be a relative equilibrium. Let $\tilde{H}$ be the induced Hamiltonian on P_μ. If $d^2\tilde{H}$ is definite at $\pi_\mu(p)$, then p is relatively stable.*

Proof: The condition tells us that $\pi_\mu(p)$ is a stable fixed point on P_μ, by conservation of energy. Thus we conclude that within each $\psi^{-1}(\mu)/G_\mu$, p is stable. But by openness of the conditions, the same is true of nearby reduced phase spaces $P_{\mu'}$, μ' near μ. Thus p is actually relatively stable. □

If G is a Lie group with a left invariant metric, a relative equilibrium represents a fixed point v in the Lie algebra, or a one-parameter subgroup of G. We can use Theorem 6 to test its stability. If we do so, we recover a result of V. Arnold [2] (who proved it directly by an apparently more complicated procedure) as follows. The quadratic form $d^2\tilde{H}$ at $v \in \mathfrak{G}$ is, in this case, worked out to be—after a short straightforward computation:

$$Q_v(w_1, w_2) = \langle B(v, w_1), B(v, w_2)\rangle + \langle [w_1, v], B(v, w_2)\rangle,$$

where $\langle B(u, v), w\rangle = \langle [u, w], v\rangle$. Thus the condition requires Q_v to be definite. In case of a rigid body ($G = SO(3)$) this yields the classical result that a rigid body spins stably about its longest and shortest principal axes, but unstably about the middle one. For fluids (G=group of volume-preserving diffeomorphisms) the situation is complicated by the fact that the metric is weak so the criterion is not directly applicable. In celestial mechanics stability of the relative equilibria often depends on stability criteria much deeper than that above, such as Moser's "twist stability theorem"; cf. [1].

Note. The authors have learned that some results similar to those in this paper were obtained by K. Meyer [23], However the approach and the applications are rather different.

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