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GENERAL RELATIVITY, PARTIAL DIFFERENTIAL EQUATIONS, AND DYNAMICAL SYSTEMS

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0. **Introduction.** In this paper we study two aspects of the Einstein equations of evolution for an empty spacetime. In the first part (§§1–3) we give a simple direct proof that the differentiability of the Cauchy data is maintained for short time. In the second part (§§4–5) we sketch how, on a suitable configuration space, Einstein's equations can be considered as forced geodesics modified by terms which reflect a moving coordinate system equipped with its own system of clocks. Both of these topics will be presented in more detail elsewhere [**15**], [**12**].

Let Ω be a bounded open domain in $\mathbf{R}^3$, let $\mathring{g}_{\mu\nu}(x^i), (x^i) \in \Omega \subset \mathbf{R}^3, 0 \leqq \mu, \nu \leqq 3, 1 \leqq i \leqq 3$, be a Lorentz metric of signature $(-, +, +, +)$, and let $\mathring{k}_{\mu\nu}(x^i)$ be a symmetric 2-covariant tensor field on Ω. The first proof that Cauchy data $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ of Sobolev class (H^s, H^{s-1}), $s \geqq 4$, evolves for small time into a Ricci zero ($R_{\mu\nu} = 0$) spacetime $g_{\mu\nu}(t, x^i)$ which is also of class H^s was given by Choquet-Bruhat [**3**], [**4**], based on earlier work by herself [**2**], and Lichnerowicz [**21**]. Her method of proof is to normalize the Ricci tensor by using harmonic coordinates so that the resulting system is a quasilinear strictly hyperbolic system (no multiple characteristics). The result then follows by quoting a theorem of Leray [**20**, p. 230] about quasilinear strictly hyperbolic systems, as modified by Dionne [**7**, p. 82]. Leray's original version of the theorem loses a derivative (i.e., H^s Cauchy data only has an H^{s-1} evolution), but Dionne remedies this defect.

Our method of proof is based on a simple observation; namely, the Ricci tensor in harmonic coordinates can be reduced to a quasilinear symmetric hyperbolic *first* order system of the form

$$A^0(u)\partial u/\partial t = A^i(u)\partial u/\partial x^i + B(u),$$

where u and $B(u)$ are 50 component column vectors, $A^0(u)$ and $A^i(u)$ are symmetric,

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and $A^0(u)$ is positive-definite. This observation is inspired by the well-known fact that any single second order hyperbolic equation can be reduced to a first order symmetric hyperbolic system.

Let $g_{\alpha\beta}(t, x)$ be an H^s-solution of $R_{\mu\nu} = 0$ with given Cauchy data. Because of the form invariance of the system $R_{\mu\nu} = 0$, $\bar{g}_{\mu\nu} = (\partial x^\alpha/\partial \bar{x}^\mu)(\partial x^\beta/\partial \bar{x}^\nu)g_{\alpha\beta}$ is also an H^s-solution of $R_{\mu\nu} = 0$, if $\bar{x}^\alpha(x^\mu)$ is an H^{s+1}-coordinate transformation. By arranging so that $\bar{x}^\alpha(x^\mu)$ is the identity in a neighborhood of the spacelike hypersurface $t = 0$, $\bar{g}_{\mu\nu}$ has the same Cauchy data as $g_{\mu\nu}$. Hence solutions to the Cauchy problem cannot be functionally unique. However, we prove a uniqueness theorem that says the evolution is unique up to the H^{s+1}-isometry class of the spacetime. This result sharpens, by one degree of differentiability, the uniqueness theorem stated in [3].

In [15] we shall show how our existence and uniqueness theorems can be obtained intrinsically for arbitrary manifolds, not necessarily compact, in the class of metrics for which the space-manifold is complete, and which satisfies suitable asymptotic conditions.

In §§4-5 we consider in what sense Einstein's equations in 3-dimensional form are a Lagrangian system of the classical form kinetic energy minus potential energy. We show how on a suitable configuration space (the manifold $\mathscr{G} \times \mathscr{M}$), the evolution equations are a degenerate dynamical system. Various terms in the Einstein system are given a geometrical explanation. In particular, the central role played by certain Lie derivative terms in the presence of a shift vector field is shown to be analogous to the space-body transitions of hydrodynamics (see Ebin-Marsden [9]) or the rigid body (see Marsden-Abraham [23]). These results are a geometrical reinterpretation of the basic work of Arnowitt-Deser-Misner [1], DeWitt [6] and Wheeler [29].

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1. **Existence of an H^s-spacetime for (H^s, H^{s-1}) Cauchy data.** For empty space relativity, one searches for a Lorentz metric $g_{\mu\nu}(t, x^i)$ whose Ricci curvature $R_{\mu\nu}$ is zero; i.e., $g_{\mu\nu}(t, x^i)$ must satisfy the system

$$R_{\mu\nu}\left(t, x^i, g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\alpha}, \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right) = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} - \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} + \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\nu} + H_{\mu\nu}\left(g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right) = 0$$

where $H_{\mu\nu}(g_{\mu\nu}, \partial g_{\mu\nu}/\partial x^\alpha)$ is a rational combination of $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^\alpha$ with denominator $\det g_{\mu\nu} \neq 0$. Note that the contravariant tensor $g^{\mu\nu}$ is a rational combination of the $g_{\mu\nu}$'s with denominator $\det g_{\mu\nu} \neq 0$.

Let $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ be the Einstein tensor, where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature. Then, as is well known, G^0_μ contains only first order time derivatives of $g_{\mu\nu}$. Thus $G^0_\mu(0, x^i)$ can be computed from the Cauchy data $g_{\mu\nu}(0, x^i)$ and $\partial g_{\mu\nu}(0, x^i)/\partial t$ alone, and therefore $G^0_\mu(0, x^i) = 0$ is a necessary condition on the Cauchy data in order that a spacetime $g_{\mu\nu}(t, x^i)$ have the given Cauchy data and satisfy $G_{\mu\nu} = 0$, which is equivalent to $R_{\mu\nu} = 0$.

The existence part of the Cauchy problem for the system $R_{\mu\nu} = 0$ is as follows: *Let $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ be Cauchy data of class $(H^s(\Omega), H^{s-1}(\Omega))$, $s \geqq 4$, such that $\mathring{G}^0_\mu(x^i) = 0$. Let Ω_0 be a proper subdomain, $\bar{\Omega}_0 \subset \Omega$. Find an $\varepsilon > 0$ and a spacetime $g_{\mu\nu}(t, x^i)$, $|t| < \varepsilon$, $(x^i) \in \Omega_0 \subset \Omega$ such that*

(a) *$g_{\mu\nu}(t, x^i)$ is H^s jointly in $(t, x^i) \in (-\varepsilon, \varepsilon) \times \Omega_0$,*

(b) *$(g_{\mu\nu}(0, x^i), \partial g_{\mu\nu}(0, x^i)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$, and*

(c) *$g_{\mu\nu}(t, x^i)$ has zero Ricci curvature.*

The system $R_{\mu\nu} = 0$ is a quasilinear system of ten second order partial differential equations for which the highest order terms involve mixing of the components of the system. As it stands, there are no known theorems about partial differential equations which can be applied to resolve the Cauchy problem. However, as was first noted in 1922 by Lanczos [18] (and in fact in 1916 by Einstein himself for the linearized equations [10]) the Ricci tensor simplifies considerably in harmonic coordinates, i.e., in a coordinate system (x^α) for which the contracted Christoffel symbols vanish, $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0$. (For the existence of such a coordinate system for an arbitrary Lorentz metric see Theorem 3.3.) In fact, an algebraic computation shows that

$$R_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + \frac{1}{2}g_{\mu\alpha}\frac{\partial \Gamma^\alpha}{\partial x^\nu} + \frac{1}{2}g_{\nu\alpha}\frac{\partial \Gamma^\alpha}{\partial x^\mu} + H_{\mu\nu}$$

so that in a coordinate system for which $\Gamma^\mu = 0$,

$$R_{\mu\nu} = R^{(h)}_{\mu\nu} = -\tfrac{1}{2}g^{\alpha\beta}(\partial^2 g_{\mu\nu}/\partial x^\alpha \partial x^\beta) + H_{\mu\nu}.$$

The operator $-\frac{1}{2}g^{\alpha\beta}(\partial^2/\partial x^\alpha \partial x^\beta)$ operates the same way on each component of the system $g_{\mu\nu}$ so that there is no mixing in the highest order derivatives. Thus the normalized system $R^{(h)}_{\mu\nu} = 0$ is considerably simpler than the full system. In fact, the system $R^{(h)}_{\mu\nu} = 0$ has only simple characteristics so that $R^{(h)}_{\mu\nu} = 0$ is a strictly hyperbolic system.

The importance of the use of harmonic coordinates and of the system $R^{(h)}_{\mu\nu} = 0$ is based on the fact that it is sufficient to solve the Cauchy problem for $R^{(h)}_{\mu\nu} = 0$; this remarkable fact discovered by Fourès-Bruhat [2] is based on the observation that the condition $\mathring{\Gamma}^\mu(x^i) \equiv \mathring{g}^{\alpha\beta}(x^i)\mathring{\Gamma}^\mu_{\alpha\beta}(x^i) = 0$ is propagated off the hypersurface $t = 0$ for solutions $g_{\mu\nu}$ of $R^{(h)}_{\mu\nu} = 0$. This is established in the next lemma.

1.1. Lemma. *Let $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ be of Sobolev class (H^s, H^{s-1}) on Ω, $s > \frac{1}{2}n + 2$, $n = 3$, and suppose that $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ satisfies*

(a) $\mathring{\Gamma}^\mu(x^i) = 0$.
(b) $\mathring{G}^0_\mu(x^i) = 0$.

If $g_{\mu\nu}(t, x)$, $|t| < \varepsilon$, $x \in \Omega_0$, Ω_0 *a proper subdomain*, $\bar{\Omega}_0 \subset \Omega$, *is an* H^s-*solution of*

$$(g_{\mu\nu}(0, x), \partial g_{\mu\nu}(0, x)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i)),$$
$$R^{(h)}_{\mu\nu} = -\tfrac{1}{2}g^{\alpha\beta}(\partial^2 g_{\mu\nu}/\partial x^\alpha \partial x^\beta) + H_{\mu\nu} = 0,$$

then $\Gamma^\mu(t, x^i) = 0$ *for* $|t| < \varepsilon$, $x \in \Omega_0$.

PROOF. The case $s > \frac{1}{2}n + 2$ is treated in [14]; here we assume $s > \frac{1}{2}n + 3$. Let $g_{\mu\nu}(t, x^i)$ satisfy (a), (b) and $R^{(h)}_{\mu\nu} = 0$. Then a straightforward computation shows that $\Gamma^\mu(t, x^i) = g^{\alpha\beta}(t, x^i)\Gamma^\mu_{\alpha\beta}(t, x^i)$ satisfies $\partial\Gamma^\mu(0, x^i)/\partial t = 0$. From $G^{\mu\nu}{}_{|\nu} = 0$ (where $_{|\nu}$ means covariant derivative) and $R^{(h)}_{\mu\nu} = 0$, Γ^μ is shown to satisfy the system of linear equations

$$g^{\alpha\beta}\frac{\partial^2\Gamma^\mu}{\partial x^\alpha\partial x^\beta} + A^{\beta\mu}_\alpha\left(g_{\mu\nu}, g^{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)\frac{\partial\Gamma^\alpha}{\partial x^\beta} = 0.$$

This linear system can be reduced to a linear first order symmetric hyperbolic system for which a uniqueness and existence theorem holds. This is exactly analogous to Theorems 1.2 and 3.1 below. But from the uniqueness for this system, $\Gamma^\mu(0, x^i) = 0$ and $\partial\Gamma^\mu(0, x^i)/\partial t = 0$ imply $\Gamma^\mu(t, x^i) = 0$. ∎

According to the lemma, an H^s-solution of $R^{(h)}_{\mu\nu} = 0$ with prescribed Cauchy data is also a solution of $R_{\mu\nu} = 0$ (since $\Gamma^\mu(t, x) = 0 \Rightarrow R^{(h)}_{\mu\nu} = R_{\mu\nu}$), provided that the Cauchy data satisfies (a) $\mathring{\Gamma}^\mu = 0$ and (b) $\mathring{G}^0_\mu = 0$. As mentioned above (b) is a necessary condition on the Cauchy data for a solution $g_{\mu\nu}(t, x)$ to satisfy $R_{\mu\nu} = 0$. If (a) is not satisfied, then a set of Cauchy data can be found whose evolution under $R^{(h)}_{\mu\nu} = 0$ leads to an H^s-spacetime which by an H^{s+1}-coordinate transformation gives rise to a spacetime with the original Cauchy data (see Fischer-Marsden [14]).

From the theorem of Dionne concerning quasilinear strictly hyperbolic systems [7] and Lemma 1.1, Choquet-Bruhat [3], [4] concludes that Cauchy data of class (H^s, H^{s-1}) has an H^s-time evolution. We prove this result directly by reducing the strictly hyperbolic system $R^{(h)}_{\mu\nu} = 0$ to a quasilinear symmetric hyperbolic first order system. Aside from the practical nature of putting the Einstein evolution equations in this form (systems of this type are very well understood: the Cauchy problem has a simple resolution in this form), there is the aesthetic value of bringing relativity into a form which uniformly governs most other equations of mathematical physics, such as Maxwell's equations, the Dirac equation, the Lundquist equations of magnetohydrodynamics, Euler's equations for a compressible fluid, and the equations describing the motion of elastic bodies.

1.2. THEOREM. *Let* Ω *be an open bounded domain in* $\mathbf{R}^3$ *with* Ω_0 *a proper subdomain*, $\bar{\Omega}_0 \subset \Omega$, *and let* $(\mathring{g}_{\mu\nu}(x), \mathring{k}_{\mu\nu}(x))$, $(x^i) \in \Omega$, $0 \leq \mu, \nu \leq 3$, $1 \leq i \leq 3$, *be of Sobolev class*

(H^s, H^{s-1}), $s \geqq 4$. Suppose that $\hat{\Gamma}^\mu(x^i) = 0$ and $\hat{G}^0_\mu(x) = 0$. Then there exists an $\varepsilon > 0$ and a unique Lorentz metric $g_{\mu\nu}(t, x)$, $|t|_1 < \varepsilon$, $(x^i) \in \Omega_0$ such that

(1) $g_{\mu\nu}(t, x^i)$ is jointly of class H^s,

(2) $R^{(h)}_{\mu\nu}(t, x^i) = 0$,

(3) $(g_{\mu\nu}(0, x^i), \partial g_{\mu\nu}(0, x^i)/\partial t) = (\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$.

From Lemma 1.1, this $g_{\mu\nu}(t, x^i)$ also satisfies $R_{\mu\nu}(t, x^i) = 0$. Moreover, $g_{\mu\nu}(t, x^i)$ depends continuously on $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ in the (H^s, H^{s-1}) topology. If $(\mathring{g}_{\mu\nu}(x^i), \mathring{k}_{\mu\nu}(x^i))$ is of class (C^∞, C^∞) on Ω, then $g_{\mu\nu}(t, x^i)$ is C^∞ for all t for which the solution exists.

Note. The case $s = 4$ is delicate and is treated in [14]. Here we assume $s \geqq 5$. In [14] we also give a complete discussion for the case of spatial asymptotic conditions.

PROOF. The system $R^{(h)}_{\mu\nu} = 0$ is reduced to a first order system by introducing the ten new unknowns $k_{\mu\nu} = \partial g_{\mu\nu}/\partial t$ and the thirty new unknowns $g_{\mu\nu,i} = \partial g_{\mu\nu}/\partial x^i$ and considering the quasilinear first order system of fifty equations:

$$\partial g_{\mu\nu}/\partial t = k_{\mu\nu},$$

$$\text{(Q)} \qquad g^{ij}(\partial g_{\mu\nu,i}/\partial t) = g^{ij}\frac{\partial k_{\mu\nu}}{\partial x^i},$$

$$-g^{00}\frac{\partial k_{\mu\nu}}{\partial t} = 2g^{0j}\frac{\partial k_{\mu\nu}}{\partial x^j} + g^{ij}\frac{\partial g_{\mu\nu,i}}{\partial x^j} - 2H_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}).$$

We are considering $H_{\mu\nu}$ as a polynomial in $g_{\mu\nu,i}$ and $k_{\mu\nu}$ and rational in $g_{\mu\nu}$ with denominator $\det g_{\mu\nu} \neq 0$. At first, we extend our initial data to all of $\mathbf{R}^3$, say to equal the Minkowski metric outside a compact set, and consider the system (Q) on $\mathbf{R}^3$. Note that the Cauchy data need not satisfy the constraints $\hat{G}^0_\mu = 0$ during the transition.

The matrix g^{ij} has inverse $g_{jk} - (g_{j0}g_{k0}/g_{00})$ (i.e., $g^{ij}(g_{jk} - (g_{j0}g_{k0}/g_{00})) = \delta^i_k$) so that the second set of thirty equations can be inverted to give

$$\text{(1)} \qquad \partial g_{\mu\nu,i}/\partial t = \partial k_{\mu\nu}/\partial x^i.$$

For $g_{\mu\nu}$ of class C^2, (1) implies

$$g_{\mu\nu,i} = \partial g_{\mu\nu}/\partial x^i,$$

so that the system (Q) is equivalent to $R^{(h)}_{\mu\nu} = 0$.

Let

$$u = \begin{pmatrix} g_{\mu\nu} \\ g_{\mu\nu,i} \\ k_{\mu\nu} \end{pmatrix}$$

be a fifty component column vector, where $g_{\mu\nu,i}$ is listed as

$$\begin{pmatrix} g_{00,1} \\ \vdots \\ g_{33,1} \\ \vdots \\ g_{00,3} \\ \vdots \\ g_{33,3} \end{pmatrix},$$

$0^{10} = 10 \times 10$ zero matrix, $I^{10} = 10 \times 10$ identity matrix, and let $A^0(u) = A^0(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})$ and $A^j(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})$ be the 50×50 matrices given by

$$A^0(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}) = \begin{pmatrix} I^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & g^{11}I^{10} & g^{12}I^{10} & g^{13}I^{10} & 0^{10} \\ 0^{10} & g^{12}I^{10} & g^{22}I^{10} & g^{23}I^{10} & 0^{10} \\ 0^{10} & g^{13}I^{10} & g^{23}I^{10} & g^{33}I^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & -g^{00}I^{10} \end{pmatrix},$$

$$A^j(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}) = \begin{pmatrix} 0^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j1}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j2}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j3}I^{10} \\ 0^{10} & g^{1j}I^{10} & g^{2j}I^{10} & g^{3j}I^{10} & 2g^{j0}I^{10} \end{pmatrix},$$

and let $B(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu})$ be the fifty component column vector given by

$$B(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}) = \begin{pmatrix} k_{\mu\nu} \\ 0^{30} \\ -2H_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,i}, k_{\mu\nu}) \end{pmatrix}$$

where 0^{30} is the thirty component zero column vector.

Note that $A^0(u)$ and $A^j(u)$ are symmetric, and that $A^0(u)$ is positive-definite if $g_{\mu\nu}$ has Lorentz signature. A direct verification shows that the first-order quasi-linear symmetric hyperbolic system

$$A^0(u)(\partial u/\partial t) = A^i(u)(\partial u/\partial x^i) + B(u)$$

is just the system (Q). From Theorem 2.1 and its generalizations proven below, we conclude that for Cauchy data

$$\mathring{u}(x^i) = \begin{pmatrix} \mathring{g}_{\mu\nu}(x^i) \\ \mathring{g}_{\mu\nu,i}(x^i) \\ \mathring{k}_{\mu\nu}(x^i) \end{pmatrix}$$

of Sobolev class H^{s-1}, $s - 1 > \frac{1}{2}n + 2$, there exists an $\varepsilon > 0$ and a solution

$$u(t, x^i) = \begin{pmatrix} g_{\mu\nu}(t, x^i) \\ g_{\mu\nu,i}(t, x^i) \\ k_{\mu\nu}(t, x^i) \end{pmatrix}$$

of class H^{s-1}. By Sobolev's lemma, $u(t, x^i)$ is also of class C^2, and so, by the second set of equations of (Q), $g_{\mu\nu,i} = \partial g_{\mu\nu}/\partial x^i$. Since $(g_{\mu\nu,i}, k_{\mu\nu}) = (\partial g_{\mu\nu}/\partial x^i, \partial g_{\mu\nu}/\partial t)$ is of class H^{s-1}, $g_{\mu\nu}(t, x^i)$ is in fact of class H^s. The continuous dependence of the solutions on the initial data follows from the general theory below.

To recover the result for the domain Ω from the result for $\mathbf{R}^n$, we can use the standard domain of dependence arguments; see Courant-Hilbert [5].

Since Ω is bounded, $(\mathring{g}_{\mu\nu}, \mathring{k}_{\mu\nu})$ of class C^∞ implies that the solution is in the intersection of all the Sobolev spaces and hence is C^∞; again we are using a general regularity result about symmetric hyperbolic systems.

From Lemma 1.1, the $g_{\mu\nu}(t, x^i)$ so found satisfy the field equations $R_{\mu\nu} = 0$. For the case in which the Cauchy data does not satisfy $\Gamma^\mu(x^i) = 0$, see [14]. ∎

Although $g_{\mu\nu}(t, x^i)$ is a unique solution of $R^{(h)}_{\mu\nu} = 0$, with prescribed Cauchy data, it is not a unique solution of $R_{\mu\nu} = 0$. We return to this point in §3.

2. **First order quasilinear symmetric hyperbolic systems.** The theory of linear first order symmetric hyperbolic systems is due to Friedrichs [17] with some simplifying modifications by Lax [19]. The essential ideas for handling quasilinear equations appear in Schauder [25], Frankl [16], and Petrovskii [24]. Friedrichs [17, p. 352] mentions that these ideas can be used to prove the unique existence of a solution for a quasilinear first order symmetric hyperbolic system, and it is again mentioned in Courant-Hilbert [5, p. 675], but we have been unable to find the details of a complete treatment in the literature. Here we shall outline the methods for $\mathbf{R}^n$ and present an intrinsic version for manifolds elsewhere [15]. The basic idea is to find energy type estimates, use the contraction mapping principle to find an H^{s-1}-solution, and then show that this H^{s-1}-solution is in fact H^s.

2.1. THEOREM. *Let $H^s(\mathbf{R}^n, \mathbf{R}^m)$ denote the H^s maps from $\mathbf{R}^n$ to $\mathbf{R}^m$, and let*

$\mathscr{U}^s \subset H^s(\mathbf{R}^n, \mathbf{R}^m)$ be an open subset. Let $\delta > 0$, and for $(t, x, u) \in (-\delta, \delta) \times \mathbf{R}^n \times \mathscr{U}^s$, let $A^i(t, x, u)$ be a symmetric $m \times m$ matrix, and let $B(t, x, u)$ be an m-component column vector. Suppose that $A^i(t, x, u)$, and $B(t, x, u)$ are H^s-functions of (t, x), and are rational functions of u with nonzero denominators. (More generally, one could use Sobolev's "condition T" on compositions of H^s functions [28].)

Given $u_0 \in \mathscr{U}^s$, $s > \frac{1}{2}n + 2$, there is an $\varepsilon > 0$, $\varepsilon < \delta$, and a unique $u(t, x)$, $|t| < \varepsilon$, $x \in \mathbf{R}^n$, which is H^s in (t, x) and which satisfies

$$u(t, x) = u_0(x),$$
$$\partial u/\partial t = A^i(t, x, u)(\partial u/\partial x^i) + B(t, x, u).$$

Moreover, the solution $u(t, x)$ depends continuously on u_0 in the H^s-topology. If $A^i(t, x, u)$ and $B(t, x, u)$ are functions of (t, x) in $\bigcap_{s > n/2+2} H^s((-\varepsilon, \varepsilon) \times \mathbf{R}^n)$ and u_0 is in $\bigcap_{s > n/2+2} H^s(\mathbf{R}^n)$, then $u(t, x)$ is in $\bigcap_{s > n/2+2} H^s((-\varepsilon, \varepsilon) \times \mathbf{R}^n)$.

Note. For the above applications, we should replace $\partial u/\partial t$ by $A^0(t, x, u)\partial u/\partial t$ where A^0 is a symmetric positive-definite matrix. Here we consider the case $s > \frac{1}{2}n + 2$; the case $s > \frac{1}{2}n + 1$ is obtained in [14]. Moreover, 2.1 can be generalized to the case in which the coefficients and the Cauchy data satisfy asymptotic conditions. This case is more delicate and is discussed following the proof of the present theorem.

PROOF. Let $\| \ \|_s$ denote the H^s-norm for functions $u: \mathbf{R}^n \to \mathbf{R}^m$ in $\mathscr{U}^s$. Let E denote the set of continuous curves $\omega: [-\delta, \delta] \to \mathscr{U}^s$ such that $\omega(0) = u_0$ and $\|\omega(t) - u_0\|_s \leqq 1$, $-\delta \leqq t \leqq \delta$. Thus E is a complete metric space, and we want to define a map $f: E \to E$ by

$$f(\omega)(t) = u_0 + \int_0^t A^i(s, x, \omega(s, x)) \frac{\partial}{\partial x^i} f(\omega)(s, x)\, ds + \int_0^t B(s, x, \omega(s, x))\, ds$$

where the integration is done as a curve in H^{s-1}. From the linear theory of first order symmetric hyperbolic systems, it follows that for δ sufficiently small there is a unique such mapping $f: E \to E$. Indeed, for $\omega \in E$, the unique solution of the linear system

$$\text{(I)} \qquad u(0) = u_0, \qquad \partial u/\partial t = A^i(t, x, \omega)(\partial u/\partial x^i) + B(t, x, \omega)$$

is exactly $f(\omega)$. Moreover, from the usual Leray estimates of the linear theory (see Courant-Hilbert [5, p. 671]), it is easy to show that for $\omega \in E$ and for u satisfying (I), there is a constant β independent of δ, ω, and u such that $\|u(t)\|_s \leqq e^{\beta|t|}\|u_0\|_s$. Thus f maps E to E.

Let F denote the completion of E in the H^{s-1}-norm. We remark that F is, by the Rellich-Garding theorem, a compact set, although we shall not need this fact. Next we note that for δ sufficiently small, $f: E \to E$ is a contraction in the H^{s-1}-norm, i.e., for $\omega_1, \omega_2 \in E$, there exists a k, $0 < k < 1$, such that

$$\|f(\omega_1) - f(\omega_2)\|_{s-1} \leqq k\|\omega_1 - \omega_2\|_{s-1}. \tag{1}$$

This follows from the estimates

$$\|f(\omega_1) - f(\omega_2)\|_{s-1} \leqq \int_0^t \left\| A^i(s, x, \omega_1(s, x)) \frac{\partial}{\partial x^i} f(\omega_1)(s, x) - A^i(s, x, \omega_2(s, x)) \frac{\partial}{\partial x^i} f(\omega_2)(s, x) \right\|_{s-1} ds + \int_0^t \|B(s, x, f(\omega_1)(s, x)) - B(s, x, f(\omega_2)(s, x))\|_{s-1} \, ds$$

and

$$\left\| A^i(s, x, \omega_1) \frac{\partial}{\partial x^i} f(\omega_1) - A^i(s, x, \omega_2) \frac{\partial}{\partial x^i} f(\omega_2) \right\|_{s-1}$$

$$\leqq \left\| A^i(t, x, \omega_1) \frac{\partial}{\partial x^i} [f(\omega_1) - f(\omega_2)] \right\|_{s-1} + \left\| A^i(t, x, \omega_1) \frac{\partial}{\partial x^i} f(\omega_2) - A^i(t, x, \omega_2) \frac{\partial}{\partial x^i} f(\omega_2) \right\|_{s-1}. \tag{2}$$

The first term in (2) is handled as in the proof of the energy estimates; the second term is bounded by $c_1\|\omega_1 - \omega_2\|_{s-1}\|f(\omega_2)\|_s$ (c_1, c_2, c_3 are constants). Thus

$$\sup_{-\delta<t<\delta} \|f(\omega_1) - f(\omega_2)\|_{s-1} \leqq c_2 \sup_{-\delta<t<\delta} \|f(\omega_1) - f(\omega_2)\|_{s-1} + \delta c_3 \sup_{-\delta<t<\delta} \|\omega_1 - \omega_2\|_{s-1}$$

from which (1) follows.

Thus f extends to a contraction on the complete metric space F so by the contraction mapping principle f has a unique fixed point, a solution in H^{s-1} to the quasilinear system we are studying. Since f depends continuously on u_0, so does the fixed point. We remark that the original argument of Schauder used compactness of F and the Schauder fixed-point theorem. P. Lax has pointed out that the above contraction argument can be replaced by the extraction of a weakly convergent sequence in L_2, and using H^s-boundedness, to deduce convergence in H^k, $k < s$. This again yields a solution in H^{s-1}.

Finally we show that the solution $u(t, x)$ in H^{s-1} is in fact in H^s. The trick is to look at the differential equation satisfied by the second spatial derivative of u, the solution found in H^{s-1}. Now

$$\partial u/\partial t = A^i(t, x, u)(\partial u/\partial x^i) + B(t, x, u)$$

so if Du is the first differential of u,

$$\begin{aligned}\frac{\partial}{\partial t}(Du) = D_2A^i(t,x,u)\frac{\partial u}{\partial x^i} + D_3A^i(t,x,u)\cdot Du\cdot\frac{\partial u}{\partial x^i} \\ + A^i(t,x,u)\frac{\partial}{\partial x^i}Du + DB(t,x,u)\end{aligned} \tag{3}$$

where D_2A^i and D_3A^i are the partial derivatives with respect to the second and third variables, respectively. If we consider (3) as a linear equation in the unknown $v = Du$ of the form

$$\partial v/\partial t = A^i(\partial v/\partial x^i) + C\cdot v + D$$

then we must treat $D_3A^i(t,x,u)(\partial u/\partial x^i)$ as a coefficient. However, since u is only H^{s-1}, $\partial u/\partial x^i$ is only H^{s-2}. However, if we differentiate again it is easy to see that $w = D^2u$ satisfies

$$\partial w/\partial t = A^i(\partial w/\partial x^i) + \tilde{C}\cdot w + \tilde{D}$$

where now A^i, $\tilde{C}$, $\tilde{D}$ are H^{s-2}-functions ($\partial Du/\partial x^i$ is, for example, taken to be part of D^2u). The reason is just that second derivatives do not occur multiplied together as the first ones did. Now if $s > \frac{1}{2}n + 3$, the coefficients are in H^r, $r = s - 2 > \frac{1}{2}n + 1$, so by the linear theory w which is initially in H^{s-2} remains in H^{s-2}. Hence u remains in H^s. (However one only needs $s > \frac{1}{2}n + 2$ here by using the fact that only the lower order terms are affected and $r > \frac{1}{2}n$; see below.

This argument also shows that the map $u_0 \mapsto u_t$ is continuous in H^s and not merely in H^{s-1}. Moreover, if the coefficients $A^i(t,x,u)$, and $B(t,x,u)$ are smooth, then the same argument shows that if we have a solution in H^s whose initial condition is in H^{s+1} then in fact the solution is in H^{s+1} (as long as it is defined in H^s). Hence smooth initial conditions remain smooth. ∎

Now in our application, we are considering a system of the form

$$A^0(t,x,u)(\partial u/\partial t) = A^i(t,x,u)(\partial u/\partial x^i) + B(t,x,u).$$

This case may be handled as follows. We assume as above that A^0, A^i and B are H^s-functions. By using the technique above, we are led to consider first the linear case. We proceed as in [5] to reduce to the case $A^0 = \mathrm{Id}$, by writing $A^0 = TT^*$ and letting $v = T^{-1}u$. However, in the case that A^0 depends on t, x in an H^s manner, this modifies the B term by replacing it by an H^{s-1} term.

Without further conditions on the $A^i(t,x)$, Cauchy data u_0 of class H^s need only have a time evolution $u(t,x)$ of class H^{s-1}. For example, if $A^i(t,x) = 0$, $B(t,x) = B(x)$, then $\partial u/\partial t = B(x)u$ can be integrated explicitly to give $u(t,x) = e^{tB(x)}u_0(x)$. For $B(x)$ of class H^{s-1}, $u(t,x)$ need only be of class H^{s-1}.

The appropriate condition on the matrices $A^i(t,x)$ can be found from the following standard lemma from perturbation theory [30].

2.2. LEMMA. *Let F be a Banach space, $D_A \subset F$ a dense domain, and $A: D_A \subset F \to F$*

a linear operator which is a generator. Let $B:F \to F$ be a bounded operator. Then $A + B:D_A \subset F \to F$ is a generator whose domain is exactly D_A.

Thinking of F as H^{s-1}-functions, D_A as H^s-functions, and $B:F \to F$ as multiplication by an H^{s-1} matrix, we see that D_A will be the domain of the closure of the operator $A = A^i(\partial/\partial x^i)$ from H^s to H^{s-1}. In concrete examples like 3.1 below, the domain of this operator is not hard to work out. Thus in the symmetric hyperbolic case, where we know that A is a generator, we know that solutions to the full system with a B term which is H^{s-1} will remain in D_A if they start out in that set.

The same remarks remain valid in the quasilinear case; that is, if A^i is H^s and B is H^{s-1} then solutions which start out in D_A will remain there. In both this and the linear case, it is important to realize that the H^s energy estimates fail, and so the proof of Theorem 2.1 as given breaks down. This failure occurs because in estimating derivatives of order s of the B term, one runs into H^r for $r < \frac{1}{2}n$ and the requisite ring structure of H^k is no longer available. However, one can use the H^{s-1}-estimates, and the regularity argument of Theorem 2.1 together with the linear theory Lemma 2.2. In addition to applications of Lemma 2.2 to quasilinear systems, in the next section we shall also need to consider the linear first order symmetric hyperbolic systems $A^0(t,x)(\partial u/\partial t) = A^i(t,x)(\partial u/\partial x^i) + B(t,x)\cdot u$ where $A^0(t,x)$ and $A^i(t,x)$ are of class H^s in (t,x) but $B(t,x)$ is only of class H^{s-1}.

3. **Uniqueness for the Einstein equations.** In this section we show that any two H^s-spacetimes which are Ricci flat and which have the same Cauchy data are related by an H^{s+1}-coordinate transformation. The key idea is to show that any H^s-spacetime when expressed in harmonic coordinates is also of class H^s. This in turn is based on an old result of Sobolev [28]; namely, that solutions to the wave equation with (H^s, H^{s-1}) coefficients preserve (H^{s+1}, H^s) Cauchy data. We can give an easy proof of this result by using Lemma 2.2 and the well-known result that any single second order hyperbolic equation can be reduced to a system of symmetric hyperbolic equations.

3.1. THEOREM. *Let $(\psi_0(x), \dot\psi_0(x))$ be of Sobolev class (H^{s+1}, H^s) on $\mathbf{R}^3$. Then there exists a unique $\psi(t,x)$ of class H^{s+1} that satisfies*

$$(\psi(0,x), \partial\psi(0,x)/\partial t) = (\psi_0(x), \dot\psi_0(x)),$$

$$g^{\mu\nu}(t,x)(\partial^2\psi/\partial x^\mu\, \partial x^\nu) + b^\mu(t,x)(\partial\psi/\partial x^\mu) + c(t,x)\psi = 0$$

where $g^{\mu\nu}(t,x)$ is a Lorentz metric of class H^s, $b^\mu(t,x)$ a vector field of class H^{s-1}, and $c(t,x)$ is of class H^{s-1}.

PROOF. As in the proof of Theorem 1.2, this single equation can be reduced to a first order symmetric hyperbolic system

$$A^0(t,x)(\partial u/\partial t) = A^i(t,x)(\partial u/\partial x^i) + B(t,x)\cdot u$$

where $A^0(t, x)$ and $A^i(t, x)$ are of class H^s, $B(t, x)$ is of class H^{s-1}, and u is the 5 component column vector

$$u = \begin{pmatrix} \psi \\ \psi_{,i} \\ \psi_{,0} \end{pmatrix}.$$

In fact the A^0 and A^i are exactly as in Theorem 1.2 with $0^{10} \to 0$ and $I^{10} \to 1$ and

$$B(t, x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c(t, x) & b^1(t, x) & b^2(t, x) & b^3(t, x) & b^0(t, x) \end{pmatrix}.$$

Regard $A = A^i\, \partial/\partial x^i$ as a densely defined operator on H^{s-1} with domain H^s. A simple check using positive-definiteness of g_{ij} (ellipticity of the operator $g^{ij}(\partial^2/\partial x^i\, \partial x^j)$) shows that the conditions of Lemma 2.2 on the A^i are met with the domain of $\bar{A} = \mathrm{cl}(A^i\partial/\partial x^i)$ in H^{s-1} being at least as large as $H^{s-1} \oplus H^s \oplus H^{s-1}$ on the three blocks of u. Thus if u_0 is in D_A, then

$$u = \begin{pmatrix} \psi \\ \psi_{,i} \\ \psi_{,0} \end{pmatrix}$$

remains in D_A, which means that ψ is H^{s+1} in x and H^s in t. From the differential equation itself we see that in fact ψ is also H^{s+1} in t. ∎

We remark that this proof "works" because we use the symmetric hyperbolic system in u and thus the coefficients need only be of class H^s, H^{s-1}.

From Theorem 3.1, we can now prove that when one transforms an H^s-spacetime to harmonic coordinates, it stays H^s.

3.2. THEOREM. *Let $g_{\mu\nu}(t, x)$ be an H^s-spacetime. Then there exists an H^{s+1}-coordinate transformation $\bar{x}^\lambda(x^\mu)$ such that*

$$\bar{g}_{\mu\nu}(\bar{x}^\lambda) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}(\bar{x}^\lambda)\frac{\partial x^\beta}{\partial \bar{x}^\nu}(\bar{x}^\lambda) g_{\alpha\beta}(x^\mu(\bar{x}^\lambda))$$

is an H^s-spacetime with $\bar{\Gamma}^\mu(\bar{t}, \bar{x}) = \bar{g}^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta}(\bar{t}, \bar{x}) = 0$.

PROOF. To find $\bar{x}^\lambda(x^\mu)$ consider the wave equation

$$\Box\psi = -g^{\alpha\beta}(\partial^2\psi/\partial x^\alpha\, \partial x^\beta) + g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}(\partial\psi/\partial x^\mu) = 0.$$

and let $\bar{t}(t, x)$ be the unique solution of the wave equation with Cauchy data $t(0, x) = 0$, $\partial\bar{t}(0, x)/\partial t = 1$, and let $\bar{x}^i(t, x)$ be the unique solution of the wave equation with Cauchy data

$$\bar{x}^i(0, x) = x^i, \qquad \partial\bar{x}^i(0, x)/\partial t = 0.$$

For $g_{\mu\nu}$ of class H^s, Γ^μ is of class H^{s-1}, so by Theorem 3.1, $\bar{t}(t, x)$ and $\bar{x}(t, x)$ are H^{s+1}-functions and in fact by the inverse function theorem for H^s-functions, $(\bar{t}(t, x), \bar{x}(t, x))$ is an H^{s+1} diffeomorphism in a neighborhood of $t = 0$.

Since $\Box\bar{x}^\mu(t, x) = 0$ is an invariant equation,

$$\bar{\Box}\bar{x}^\mu = -\bar{g}^{\alpha\beta}\frac{\partial^2\bar{x}^\mu}{\partial\bar{x}^\alpha\,\partial\bar{x}^\beta} + \bar{g}^{\alpha\beta}\bar{\Gamma}^\nu_{\alpha\beta}\frac{\partial\bar{x}^\mu}{\partial\bar{x}^\nu} = \bar{g}^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta} = 0$$

in the barred coordinate system, so $\bar{x}^\mu$ is a system of harmonic coordinates. ∎

As a simple consequence of Theorem 3.2 we have the following uniqueness result for the Einstein equations:

3.3. THEOREM. *Let $g_{\mu\nu}(t, x)$ and $\bar{g}_{\mu\nu}(t, x)$ be two H^s-spacetimes with zero Ricci tensor and such that $(g_{\mu\nu}(0, x), \partial g_{\mu\nu}(0, x)/\partial t) = (\bar{g}_{\mu\nu}(0, x), \partial\bar{g}_{\mu\nu}(0, x)/\partial t)$. Then $g_{\mu\nu}(t, x)$ and $\bar{g}_{\mu\nu}(t, x)$ are related by an H^{s+1}-coordinate change in a neighborhood of $t = 0$.*

PROOF. From Theorem 3.2 there exist H^{s+1}-coordinate transformations $y^\mu(x^\alpha)$ and $\bar{y}^\mu(x^\alpha)$ such that the transformed metrics $(\partial x^\alpha/\partial y^\mu)(\partial x^\beta/\partial y^\nu)g_{\alpha\beta}$ and $(\partial x^\alpha/\partial\bar{y}^\mu)(\partial x^\beta/\partial\bar{y}^\nu)\bar{g}_{\alpha\beta}$ satisfy $R^{(h)}_{\mu\nu} = 0$. Since the Cauchy data for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are equal the transformed metrics also have the same Cauchy data. By the uniqueness part of Theorem 1.2, $(\partial x^\alpha/\partial y^\mu)(\partial x^\beta/\partial y^\nu)g_{\alpha\beta} = (\partial x^\alpha/\partial\bar{y}^\mu)(\partial x^\beta/\partial\bar{y}^\nu)\bar{g}_{\alpha\beta}$. Since the composition of H^{s+1}-coordinate changes is also H^{s+1}, $\bar{g}_{\alpha\beta}$ is related to $g_{\alpha\beta}$ by an H^{s+1}-coordinate change in a neighborhood of $t = 0$. ∎

4. The Einstein system on the manifold $\mathcal{M}$. We now consider a dynamical formulation of general relativity from the 3-dimensional point of view of Arnowitt, Deser and Misner [1], DeWitt [6], and Wheeler [29]. All tensor fields, such as g, k, X are referred to a fixed oriented smooth 3-dimensional manifold M.

Let

$\mathcal{M}$ = Riem(M) = manifold of all smooth Riemannian metrics (positive-definite) on M;

$S_2(M)$ = the linear space of all smooth symmetric 2-covariant tensor fields on M; and

$\mathcal{D}$ = Diff(M) = the group of smooth orientation-preserving diffeomorphisms of M.

In the dynamical formulation of general relativity, one is concerned with the evolution of initial Cauchy data $(\mathring{g}, \mathring{h}) \in \mathcal{M} \times S_2(M)$ on some 3-dimensional hypersurface M of a yet to be constructed Ricci-flat (vacuum) spacetime V_4.

As one is interested in finding the evolution g_t of Riemannian metrics only up to the isometry class $\{(\eta_t^{-1})^* g_t | \eta_t \in \mathscr{D}\}$ of g_t (here $(\eta_t^{-1})^*$ is the "push forward" of covariant tensor fields), the evolution is determined only up to an arbitrary curve $\eta_t \in \mathscr{D}$ with η_0 = the identity diffeomorphism. In other words, only the orbit class or geometry of g_t is determined (see DeWitt [6] and Fischer [11] for the structure of the orbit space $\mathscr{M}/\mathscr{D}$). Moreover, one is free to specify on M an arbitrary system of clock rates.

These degeneracies are reflected in the evolution equations as follows:

THE EINSTEIN SYSTEM. *Let* $(\mathring{g}, \mathring{k}) \in \mathscr{M} \times S_2(M)$ *satisfy the constraints:*

$$\delta(\mathring{k} - \mathring{g}(\mathrm{Tr}\, \mathring{k})) = 0, \qquad \tfrac{1}{2}((\mathrm{Tr}\, \mathring{k})^2 - \mathring{k} \cdot \mathring{k}) + 2R(\mathring{g}) = 0.$$

Let X_t *be an arbitrary time-dependent vector field on* M *(the shift vector field) and* N_t *an arbitrary time-dependent scalar field on* M *(the lapse function) such that*

$$N_t(m) > 0,$$

$$N_0^2 - \|X_0\|^2 > 0 \quad \textit{for all } (t, m) \in R \times M.$$

The problem is to find a curve $(g_t, k_t) \in \mathscr{M} \times S_2(M)$ *which satisfies the evolution equations*

$$\partial g_t/\partial t = N_t k_t - L_{X_t} g_t,$$

$$\partial k_t/\partial t = N_t S_{g_t}(k_t) - 2N_t \,\mathrm{Ric}(g_t) + 2\,\mathrm{Hess}(N_t) - L_{X_t} k_t,$$

and which has initial conditions $(g_0, k_0) = (\mathring{g}, \mathring{k})$.

Our notation is the following:

δk = divergence of $k = (\delta k)_i = -k^j_{i|j}$.
$\mathrm{Tr}\, k$ = Trace $k = g^{ij} k_{ij} = k^i_i$.
$k \cdot k$ = dot product for symmetric tensors $= k_{ij} k^{ij}$.
$k \times k$ = cross product for symmetric tensors $= k_{il} k^l_j$.
$S_g(k) = k \times k - \tfrac{1}{2}(\mathrm{Tr}\, k) k = k_{il} k^l_j - \tfrac{1}{2}(g^{mn} k_{mn}) k_{ij}$.
$\|X\|^2$ = norm of $X = g_{ij} X^i X^j$.
$L_{X_t} g_t$ = {Lie derivative of g_t with respect to the time-dependent vector field X_t} $= X_{i|j} + X_{j|i}$ ($_{|j}$ = covariant derivative with respect to the time-dependent metric).
$L_{X_t} k_t$ = Lie derivative of $k_t = X^l k_{ij|l} + k_{il} X^l{}_{|j} + k_{jl} X^l{}_{|i}$.
$\mathrm{Ric}(g_t)$ = (Ricci curvature tensor formed from g_t) $= R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^l_{ik}\Gamma^k_{lj}$.
$R(g_t)$ = scalar curvature $= R^k_k$.
$\mathrm{Hess}(N)$ = Hessian of N = double covariant derivative $= N_{|i|j}$.

In the case that we choose $N_t = 1$ and $X_t = 0$, the proper configuration space for the Einstein system is the manifold $\mathscr{M}$. We equip $\mathscr{M}$ with a metric $\mathscr{G}$, referred to as the *DeWitt metric*, by setting for $g \in \mathscr{M}$,

$$\mathscr{G}_g : T_g\mathscr{M} \times T_g\mathscr{M} \approx S_2(M) \times S_2(M) \to \mathbf{R},$$

$$\mathscr{G}_g(h, k) = \int_M ((\mathrm{Tr}\, h)(\mathrm{Tr}\, k) - h \cdot k)\mu_g$$

where $\mu_g = (\det g)^{1/2}\, dx^1 \wedge dx^2 \wedge dx^3$ is the usual volume element.

The following is a straightforward computation (see [12]):

4.1. PROPOSITION. *The Lagrangian $L_0(g, h) = \frac{1}{2}\mathscr{G}_g(h, h)$ is nondegenerate and the associated Lagrangian vector field exists and is given by the second order system*

$$\text{(Z)} \qquad \begin{aligned} \partial g_t/\partial t &= k_t, \\ \partial k_t/\partial t &= k_t \times k_t - \tfrac{1}{2}(\mathrm{Tr}\, k_t)k_t - \tfrac{1}{8}((\mathrm{Tr}\, k_t)^2 - k_t \cdot k_t)g). \end{aligned}$$

For each $(\mathring{g}, \mathring{k}) \in \mathscr{M} \times S_2(M)$ there exists a unique smooth curve $(g_t, k_t) \in \mathscr{M} \times S_2(M)$ defined for short time with initial conditions $(g_0, k_0) = (\mathring{g}, \mathring{k})$ and which satisfies (Z).

Now one adds a potential term to L_0; set

$$L(g, k) = \tfrac{1}{2}\mathscr{G}_g(k, k) - 2\int_M R(g)\mu_g$$

where $R(g)$ is the scalar curvature of g. Adding this potential term adds a gradient term to the equations of motion. For the potential $V = 2\int_M NR(g)\mu_g$ (where N is a positive scalar on M included for later use), a computation gives

$$-\mathrm{grad}\, V = -2N(\mathrm{Ric}(g) - \tfrac{1}{4}R(g)g) + 2\,\mathrm{Hess}(N),$$

where the gradient has been computed with respect to the DeWitt metric. Using the pointwise conservation law $\frac{1}{2}((\mathrm{Tr}\, k)^2 - k \cdot k) + 2R(g) = 0$ (see [12]), we have

4.2. PROPOSITION. *$(g_t, k_t) \in \mathscr{M} \times S_2(M)$ is an integral curve of the second order system determined by $L = \frac{1}{2}\mathscr{G}_g(k, k) - 2\int_M R(g)\mu_g$ iff*

$$\begin{aligned} \partial g_t/\partial t &= k_t, \\ \partial g_t/\partial t &= S_{g_t}(k_t) - 2\,\mathrm{Ric}(g_t). \end{aligned}$$

Eardley, Liang, and Sachs [8] have given conditions for which the velocity terms $S_g(k)$ dominate the Ric(g) term (for example near a singular hypersurface) so that the latter can with some justification be neglected. In this case the integral curves can be given explicitly, and are the geodesics of $\mathscr{M}$ with respect to the metric $\mathscr{G}$.

5. **The evolution equations with a shift vector field and space-body transitions.** Now suppose we consider the equations with an arbitrary shift vector field X_t. We assert that there is a simple method for solving these equations if the solution for $X = 0$ is known.

5.1. PROPOSITION. *Let g_t, k_t be a solution of the Einstein system with $N = 1$, $X = 0$.*

Then given X_t we construct its flow η_t. Then the solution of the Einstein system with $N = 1$, shift X_t, and the same initial conditions g_0, k_0 is given by

$$\tilde{g}_t = (\eta_t^{-1})^* g_t, \qquad \tilde{k}_t = (\eta_t^{-1})^* k_t.$$

PROOF. The extra terms involving the Lie derivatives are picked up as follows:

$$\begin{aligned} \partial \tilde{k}_t/\partial t &= (\eta_t^{-1})^*(\partial k_t/\partial t) - L_{X_t}(\eta_t^{-1})^* k_t \\ &= (\eta_t^{-1})^*(S_{g_t}(k_t) - 2\,\mathrm{Ric}(g_t)) - L_{X_t}\tilde{k}_t \\ &= S_{\tilde{g}_t}(\tilde{k}_t) - 2\,\mathrm{Ric}(\tilde{g}_t) - L_{X_t}\tilde{k}_t, \end{aligned}$$

where we have used the fact that $\partial(\eta_t^{-1})^* k/\partial t = -L_{X_t}(\eta_t^{-1})^* k$. ∎

Proposition 5.1 shows that even though the evolution equations with a shift involve extra terms which are nonlinear and involve derivatives, the more general system can be solved merely by solving an ordinary differential equation; namely, by finding the flow of X_t.

In order to take into account the shift vector field X_t we enlarge the configuration space $\mathscr{M}$ to $\mathscr{D} \times \mathscr{M}$. For $\eta \in \mathscr{D}$, it is easy to see that $T_\eta \mathscr{D}$ is the set of maps $X \circ \eta$ where X is a vector field on M. The Lagrangian of the preceding section is transferred to $\mathscr{D} \times \mathscr{M}$ by setting, for $(\eta, g) \in \mathscr{D} \times \mathscr{M}$,

$$\begin{aligned} &\mathcal{L}: T_\eta \mathscr{D} \times T_g \mathscr{M} \to \mathbf{R}, \\ &(X \circ \eta, h) \mapsto \tfrac{1}{2}\mathscr{G}_g(h + L_X g, h + L_X g) - 2\int_M R(g)\mu_g. \end{aligned}$$

We observe that for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, $\mathcal{L}(\lambda X \circ \eta, \lambda h) = \lambda^2 \mathcal{L}(X \circ \eta, h)$ so that $\mathcal{L}$ is quadratic in the velocities $(X \circ \eta, h)$. On $T\mathscr{M}$, of course, this is not true.

On $T(\mathscr{D} \times \mathscr{M})$, $\mathcal{L}$ is, roughly speaking, degenerate in the direction of $\mathscr{D}$. This degeneracy has the effect of introducing some ambiguity into the equations of motion, which is, however, precisely removed by the specification of a curve $\eta_t \in \mathscr{D}$. A direct computation proves the following:

5.2. PROPOSITION. *For any smooth curve η_t with generator the vector field $X_t = (d\eta_t/dt) \circ \eta_t^{-1}$, a possible Lagrangian vector field for the degenerate Lagrangian $\mathcal{L}(\eta, g; X \circ \eta, h) = L(g, h + L_X g)$ is given by the equations*

$$\begin{aligned} \partial g_t/\partial t &= k_t - L_{X_t} g_t, \\ \partial k_t/\partial t &= S_{g_t}(k_t) - 2\,\mathrm{Ric}(g_t) - L_{X_t} k_t. \end{aligned}$$

There is a natural action of the group $\mathscr{D}$ on $\mathscr{D} \times \mathscr{M}$ given by

$$\begin{aligned} &\Phi_\eta: \mathscr{D} \times \mathscr{M} \to \mathscr{D} \times \mathscr{M} \\ &(\zeta, g) \mapsto (\eta \circ \zeta, (\eta^{-1})^* g). \end{aligned}$$

This action leads to a natural symmetry and consequent conservation laws for our system.

5.3. PROPOSITION. *Let $\Phi_\eta : \mathscr{D} \times \mathscr{M} \to \mathscr{D} \times \mathscr{M}$ be as above with tangent action $T\Phi_\eta : T(\mathscr{D} \times \mathscr{M}) \to T(\mathscr{D} \times \mathscr{M})$. Then $\mathcal{L} \circ T\Phi_\eta = \mathcal{L}$ for each $\eta \in \mathscr{D}$ and $\delta(k - g(\mathrm{Tr}\, k)) \otimes \mu_g$ ($\otimes$ = tensor product) is a constant of the motion.*

We remark that the standard conservation theorems (cf. [22], [23]), used to prove this proposition, have to be modified to take into account the degeneracy of $\mathcal{L}$. The infinite dimensionality of the symmetry group leads to a differential rather than an integral identity; see [12] for details.

The Lie derivative terms that appear in 5.2 have a natural geometric interpretation related to changing from space to body coordinates in a manner similar to that of the rigid body and hydrodynamics (cf. Marsden-Abraham [23] and Ebin-Marsden [9]). More specifically we consider the manifold M to be the body, and the flow η_t of the shift vector field X, as being a rotation of M. We then make the convention that an observer is in *body coordinates* if he is *on* the manifold, and is in *space coordinates* if he is *off* the manifold.

Now let g_t be a time-dependent metric field on M. We assume that the field is rigidly attached to M as it moves so that we set $g_t = g_{\mathrm{body}}$. An observer in body coordinates then finds $(\partial g_{\mathrm{body}}/\partial t) = k_{\mathrm{body}}$ as the "velocity" of the metric. An observer in space coordinates sees the metric field g_{body} as it is dragged past him by the moving manifold; he sees the metric field $g_{\mathrm{space}} = (\eta_t^{-1})^* g_{\mathrm{body}}$ and computes

(1) $\partial g_{\mathrm{space}}/\partial t = k_{\mathrm{space}} - L_X g_{\mathrm{space}}$,

(2) $\partial k_{\mathrm{space}}/\partial t = S_{g_{\mathrm{space}}}(k_{\mathrm{space}}) - 2\,\mathrm{Ric}(g_{\mathrm{space}}) - L_X k_{\mathrm{space}}$,

where $k_{\mathrm{space}} = (\eta_t^{-1})^* k_{\mathrm{body}}$. But (1) and (2) are just the evolution equations (with $N_t = 1$).

Finally, we remark that the Hessian term in the evolution equations can be accounted for by introducing the general relativistic time translation group $\mathscr{T} = C^\infty(M; \mathbf{R})$ and defining an extended Lagrangian on $\mathscr{T} \times \mathscr{D} \times \mathscr{M}$. The pointwise conservation of the Hamiltonian is closely related to the invariance of the Lagrangian under the action of the group of general relativistic time translations $\mathscr{T}$. Moreover, as with the shift vector field, there is associated with an arbitrary lapse function and a solution of the evolution equations with $N = 1$ and with given Cauchy data, a proper time function τ and an intrinsic shift vector field Y. From the integration of Y, together with τ, we determine from the solution for $N = 1$ the solution with the arbitrary lapse function and with the same Cauchy data. See [12] for a detailed analysis of these topics.

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