On Completeness of Homogeneous Pseudo-Riemannian Manifolds

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The following theorem answers a question raised by J. A. Wolf, [6] page 95.

Theorem. Let M be a compact pseudo-riemannian manifold. Let G be a Lie group which acts transitively on M by isometries. Then M is geodesically complete.

This result was proved by Hermann [1] in the special case of a semi-simple compact Lie group carrying a left invariant pseudo-riemannian metric. It should be noted that in the statement of the theorem neither the homogeneity nor the compactness may be dropped. For example it has become well-known to relativists that there are incomplete Lorentz metrics on the two torus. These were constructed by Y. Clifton and W. Pohl. *Cf.* [2], p. 189. An incomplete metric on the noncompact group $SO(2, 1)$ is constructed in [1] although this is a special case of a whole class of incomplete pseudo-riemannian manifolds constructed by J. A. Wolf. See [5] and [6].

To prove the theorem we shall show that the tangent bundle TM of M is the union of compact subsets S_α parametrized by elements α of the dual $\mathfrak{G}^*$ of the Lie algebra of G, with S_α invariant under the geodesic flow. Since a vector field whose integral curves remain in a compact set has a complete flow, this is clearly enough to prove the theorem.

The construction of S_α comes from mechanics; S_α is defined to be a level surface of the conserved moment for the geodesic flow. We now summarize the appropriate definitions.

Let ξ_M be the infinitesimal generator (Killing vector field) on M corresponding to $\xi \in \mathfrak{G}$. Since we have a homogeneous space, for each $m \in M$ the vectors $\xi_M(m)$ span the tangent space T_mM. Define the *moment* for the action by

$$P : TM \to \mathfrak{G}^*; \quad P(v)\cdot\xi = \langle v, \xi_M(m)\rangle, \quad v \in T_mM.$$

It is a fundamental fact from mechanics that P is conserved by any Hamiltonian flow on TM invariant under the tangent action of G and in particular by the

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geodesic flow. See, for example, [3, 4]. This result is one geometric aspect of the classical Noether theorem. Furthermore, P is Ad^*-equivariant; *i.e.*,

$$P \circ T\Phi_g = (Ad_{g^{-1}})^* \circ P \quad \text{for all} \quad g \varepsilon G$$

where Φ_g denotes the action of $g \varepsilon G$ on M, $Ad_g = TR_{g^{-1}}TL_g : \mathfrak{G} \to \mathfrak{G}$ is the adjoint action and $(Ad_g)^*$ is its dual. This is an easy verification; see [3]; or [4], Prop. 4.4.

Set $S_\alpha = P^{-1}(\alpha)$ for each $\alpha \varepsilon \mathfrak{G}^*$. By the above, these sets are invariant under the geodesic flow. Obviously $TM = \bigcup_{\alpha \varepsilon \mathfrak{G}^*} S_\alpha$. The following lemma will thus complete our proof.

Lemma. *Each of the sets S_α is a compact subset of TM.*

Proof. Certainly S_α is closed. Furthermore, the restriction of the canonical projection $\pi : TM \to M$ to S_α is one-to-one because from the fact that the $\xi_M(m)$ span T_mM, we see that S_α intersects each fiber in at most one point.

We claim first of all that $\pi(S_\alpha)$ is closed and hence compact. Indeed $x \notin \pi(S_\alpha)$ means that α is not in the range of the linear map obtained by restricting P to T_xM. Thus α is not in the range of $P|T_yM$ for y in a whole neighborhood of x. Hence $\pi(S_\alpha)$ is closed.

Now let v_x, $v_y \varepsilon S_\alpha$, so $\langle v_x, \xi_M(x)\rangle = \langle v_y, \xi_M(y)\rangle = \alpha(\xi)$ for all $\xi \varepsilon \mathfrak{G}$. From the fact that $\xi_M(m)$ span T_mM and non-degeneracy of $\langle\ ,\ \rangle$, we may conclude that v_x is close to v_y if x is close to y. Hence the inverse $\pi^{-1} : \pi(S_\alpha) \to S_\alpha$ is continuous. Thus S_α is compact. □

Remarks. 1. If dim G = dim M, then S_α is actually a submanifold because $P : TM \to \mathfrak{G}^*$ is a submersion in that case (the derivative of P along the fibers is one-to-one and hence surjective).

2. Of course we have actually proved more. We only require that the infinitesimal generators span at each point, and that we have an invariant Hamiltonian system. Clearly conservation of energy, which is the basis of the proof for the Riemannian case (see [6] p. 89), plays no role here.

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References

1. R. Hermann, *Geodesics and classical mechanics on Lie groups*, J. Math. and Phys. **13** (1972), 460–464.
2. L. Marcus, *Cosmological models in differential geometry*, University of Minnesota, 1963 (mimeographed).
3. J. Marsden, *Hamiltonian one parameter groups*, Arch. Rational Mech. Anal. **28** (1968), 362–396.
4. S. Smale, *Topology and mechanics*, Invent. Math. **10** (1970), 305–331, **11** (1970), 45–64.
5. J. A. Wolf, *Isotropic manifolds of indefinite metric*, Comment. Math. Helv. **39** (1964), 21–64.
6. ———, *Spaces of constant curvature*, Second Edition, J. A. Wolf, Berkeley, Calif., 1972.

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