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NEW THEORETICAL TECHNIQUES IN THE STUDY OF GRAVITY

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ABSTRACT

Using new methods based on first order techniques, it is shown how sharp theorems for existence, uniqueness, and continuous dependence on the Cauchy data for the exterior Einstein equations can be proved simply and directly. Our main tools are obtained from the theory of quasi-linear first order symmetric hyperbolic systems of partial differential equations. Einstein's equations in harmonic coordinates are cast into this form, thus achieving a certain uniformity of the description of gravity with other systems of partial differential equations occurring frequently in mathematical physics. In this symmetric hyperbolic form, the Cauchy problem for the exterior equations is easily resolved. Similarly, using first order techniques, a uniqueness theorem can be proved which increases by one the degree of differentiability of the coordinate-transformation between two solutions of Einstein's equations with the same Cauchy data. Finally it is shown how the theory of first order symmetric hyperbolic systems admits a global intrinsic treatment on manifolds.

§(0): *INTRODUCTION*

The theorems concerning existence of solutions to the exterior Einstein equations are due to Fourès-Bruhat [*8*], Choquet-Bruhat [*1*,*2*], and Lichnerowicz [*12*]. Their methods involved applications of the theory of strictly hyperbolic systems of *second* order partial differential equations due to Leray [*11*] and as modified by Dionne [*4*]. Actually there is a much simpler theory of *first* order symmetric hyperbolic systems which is applicable here. In this paper we give an exposition of how these considerably less complicated first order methods can be brought to bear to resolve the Cauchy problem for the exterior Einstein equations; for details of these techniques, see Fischer-Marsden [*5*,*6*,*7*], and Marsden, Ebin, Fischer [*13*].

§(1): *BACKGROUND FOR THE MAIN IDEA*

When one is dealing with a *single second order* hyperbolic equation, say the wave equation,

$$\Box\psi = g^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\frac{\partial\psi}{\partial x^\mu} = 0, \qquad 0 \leqslant \mu,\nu \leqslant 3,$$

(where $g^{\mu\nu}$ is a spacetime of signature $(-+++)$), a common technique to resolve the Cauchy problem is to introduce four new unknowns $\psi_{,\mu}$ and consider the *first order system* of five equations

$$\left.\begin{aligned} \frac{\partial\psi}{\partial t} &= \psi_{,0}, \\ g^{ij}\frac{\partial\psi_{,i}}{\partial t} &= g^{ij}\frac{\partial\psi_{,0}}{\partial x^i}, \\ -g^{00}\frac{\partial\psi_{,0}}{\partial t} &= g^{ij}\frac{\partial\psi_{,i}}{\partial x^j} + 2g^{0j}\frac{\partial\psi_{,0}}{\partial x^j} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\psi_{,\mu} \end{aligned}\right\}$$

$1 \leqslant i,j \leqslant 3.$

As is easy to see (e.g. as in Courant-Hilbert [3], p. 594) this first-order system can be put into the form of a linear symmetric hyperbolic system,

$$A^0\frac{\partial u}{\partial t} = A^i\frac{\partial u}{\partial x^i} + B\cdot u.$$

Here $u = \begin{bmatrix}\psi \\ \psi_{,i} \\ \psi_{,0}\end{bmatrix}$ is a 5-component column vector, $A^0(t,x^i)$ a symmetric positive-definite 5×5 matrix, and the $A^i(t,x^i)$ are symmetric 5×5 matrices. Systems of this type have been studied and applied rather extensively, as their importance in mathematical physics is well-established. We refer in particular to Friedrichs [9], Lax [10], and Courant-Hilbert [3].

Writing the second order wave equation as a system of first order symmetric hyperbolic equations considerably simplifies the analysis. In fact, from the existence and uniqueness theorems for such linear systems, it is an easy matter to conclude that there exist unique solutions to the wave equation with prescribed Cauchy data $\psi(0,x^i)$, $\partial\psi(0,x^i)/\partial t$.

We now wish to describe how a similar idea can be applied to the Einstein empty space equations $R_{\mu\nu} = 0$.

§(2): *THE EINSTEIN EVOLUTION EQUATIONS AS A QUASI-LINEAR FIRST ORDER SYMMETRIC HYPERBOLIC SYSTEM*

As is well known, in a system of coordinates in which $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0$ (known as *harmonic* coordinates), the empty space field equations are

$$R^{(h)}_{\mu\nu} = -\tfrac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + H_{\mu\nu} = 0,$$

where $H_{\mu\nu}$ is an algebraic function of $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^\alpha$ only (e.g. see Lichnerowicz [*12*] for this computation). Now $R^{(h)}_{\mu\nu} = 0$ is a quasi-linear *second order system* of partial differential equations. The key point to remark about this system is that its principal part, namely the operator $-\tfrac{1}{2}g^{\alpha\beta}(\partial^2/\partial x^\alpha\partial x^\beta)$, is the same for each of the components $g_{\mu\nu}$. Such systems in which the highest order derivatives do not involve mixing of the components are said to be *weakly coupled* and are a particular case of the strictly hyperbolic systems of Leray [*11*].

Because of the complete uncoupling in the highest order terms, we expect that the second order system $R^{(h)}_{\mu\nu} = 0$ should behave, very much like a *single second order* equation. (An example of this phenomenon is in Courant Hilbert [*3*], see p. 139, where it is shown how a system of first order equations with the same principal part behaves like a single first order equation). Because the principal part of $R^{(h)}_{\mu\nu} = 0$ and the wave equation are the same, we thus expect that $R^{(h)}_{\mu\nu} = 0$ can be reduced to a first order quasi-linear symmetric hyperbolic system of the form

$$A^0(u)\frac{\partial u}{\partial t} = A^i(u)\frac{\partial u}{\partial x^i} + B(u).$$

This expectation is in fact correct. Introducing the 40 new components $g_{\mu\nu,\alpha}$ the system $R^{(h)}_{\mu\nu} = 0$ can be reduced to the first order system of 50 equations,

$$\left.\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial t} &= g_{\mu\nu,0}, \\ g^{ij}\frac{\partial g_{\mu\nu,i}}{\partial t} &= g^{ij}\frac{\partial g_{\mu\nu,0}}{\partial x^i}, \\ -g^{00}\frac{\partial g_{\mu\nu,0}}{\partial t} &= g^{ij}\frac{\partial g_{\mu\nu,i}}{\partial x^j} + 2g^{0j}\frac{\partial g_{\mu\nu,0}}{\partial x^j} - 2H_{\mu\nu}. \end{aligned}\right\} \quad \text{(F)}$$

Let $u = \begin{bmatrix} g_{\mu\nu} \\ g_{\mu\nu,i} \\ g_{\mu\nu,0} \end{bmatrix}$ be a 50-component column vector, and let

$$A^0(g_{\mu\nu};g_{\mu\nu,i};g_{\mu\nu,0}) = \begin{bmatrix} I^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & g^{11}I^{10} & g^{12}I^{10} & g^{13}I^{10} & 0^{10} \\ 0^{10} & g^{12}I^{10} & g^{22}I^{10} & g^{23}I^{10} & 0^{10} \\ 0^{10} & g^{13}I^{10} & g^{23}I^{10} & g^{33}I^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & -g^{00}I^{10} \end{bmatrix},$$

$$A^j(g_{\mu\nu};g_{\mu\nu,i};g_{\mu\nu,0}) = \begin{bmatrix} 0^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j1}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j2}I^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & g^{j3}I^{10} \\ 0^{10} & g^{1j}I^{10} & g^{2j}I^{10} & g^{3j}I^{10} & 2g^{j0}I^{10} \end{bmatrix},$$

$$B(g_{\mu\nu};g_{\mu\nu,i};g_{\mu\nu,0}) = \begin{bmatrix} g_{\mu\nu,0} \\ 0^{30} \\ -2H_{\mu\nu}(g_{\mu\nu};g_{\mu\nu,i};g_{\mu\nu,0}) \end{bmatrix},$$

where 0^{10} is the 10×10 zero matrix, 0^{30} is the 30 component zero column vector, and I^{10} is the 10×10 identity matrix. Then it is easy to check that the system (F) can be written as

$$A^0(u)\,\frac{\partial u}{\partial t} = A^i(u)\,\frac{\partial u}{\partial x^i} + B(u) \qquad \text{(S)}$$

If $g_{\mu\nu}$ is a Lorentz metric, $A^0(u)$ will be symmetric and positive-definite and the $A^j(u)$ are symmetric. Thus the system (S) is *symmetric hyperbolic* and we have succeeded in writing the system $R^{(h)}_{\mu\nu} = 0$ in the symmetric hyperbolic form. Thus the Cauchy problem for $R^{(h)}_{\mu\nu} = 0$:

Given $g_{\mu\nu}(0,x^i)$, $\partial g_{\mu\nu}(0,x^i)/\partial t$, *find a* $g_{\mu\nu}(t,x^i)$ *such that* $R^{(h)}_{\mu\nu} = 0$ *and* $g_{\mu\nu}(t,x^i)$ *has the prescribed Cauchy data*

is equivalent to the Cauchy problem for the symmetric hyperbolic system (S).

§(3): *EXISTENCE THEOREM FOR QUASI-LINEAR FIRST ORDER SYMMETRIC HYPERBOLIC SYSTEMS*

Now we turn our attention to the Cauchy problem for abitrary quasi-linear first order symmetric hyperbolic systems of the form

$$\left.\begin{aligned} &u(0,x^i) = u_0(x^i), \\ &A^0(u)\,\frac{\partial u}{\partial t} = A^0(u)\,\frac{\partial u}{\partial x^i} + B(u). \end{aligned}\right\} \quad \text{(S)}$$

The essential idea is to use the linear theory to define a contraction mapping $f\colon E \to E$ from a complete metric space to itself. Thus, let H denote a space of k-component fields in n variables, $u\colon \mathbb{R}^n \to \mathbb{R}^k$, and let Ω be an open subset of H. Let E denote the set of continuous curves $\omega\colon[-\delta,\delta] \to \Omega$ such that $\omega(0) = u_0 \in \Omega$. We want to define a map $f\colon E \to E$. For each $\omega \in E$, let $f(\omega)$ be the unique solution of the linear system

$$\left.\begin{aligned} &u(0,x^i) = u_0(x), \\ &A^0(\omega)\,\frac{\partial u}{\partial t} = A^i(\omega)\,\frac{\partial u}{\partial x^i} + B(\omega). \end{aligned}\right\}$$

Then f maps E to E and using Leray type energy estimates, f can be shown to a be a contraction mapping in a suitable norm; i.e. there exists k, $0 < k < 1$, such that for $\omega_1,\omega_2 \in E$,

$$\|f(\omega_1) - f(\omega_2)\| \leq k\|\omega_1 - \omega_2\|.$$

By the contraction mapping principle, f has a unique fixed point, a solution to the quasi-linear system we are studying. Moreover, since f depends continuously on u_0, so does the fixed point.

In essence the technique is similar to the usual Picard iteration method used in ordinary differential equations, although it differs in several important technical respects.

From the existence, uniqueness, and continuous dependence on initial data outlined here for systems of the type (S), together with the reduction of the system $R^{(h)}_{\mu\nu} = 0$ to a system of this type, it is an easy matter to prove that there exists a unique solution to the Cauchy problem for the system $R^{(h)}_{\mu\nu} = 0$ which depends continuously on the Cauchy data. We have uniqueness here because we have a specified set of partial differential equations; see, however, section 5.

§(4): *SUFFICIENCY OF THE USE OF HARMONIC COORDINATES TO RESOLVE THE CAUCHY PROBLEM*

The importance of the use of harmonic coordinates and of the system $R^{(h)}_{\mu\nu} = 0$ is based on the fact that it is sufficient to solve the Cauchy problem for $R^{(h)}_{\mu\nu} = 0$. This remarkable fact, apparently discovered by Fourès-Bruhat [*8*], follows from the observation that the condition $\Gamma^{\mu}(0,x^i) = 0$ is propagated off the $t = 0$ hypersurface for solutions $g_{\mu\nu}(t,x^i)$ of $R^{(h)}_{\mu\nu} = 0$ under the hypothesis that the Cauchy data $g_{\mu\nu}(0,x^i)$, $\partial g_{\mu\nu}(0,x^i)/\partial t$ satisfy the conditions:

(a) $G^0_{\mu}(0,x^i) = 0$ (where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$);

(b) $\Gamma^{\mu}(0,x^i) = 0$.

Thus for such $g_{\mu\nu}$, $\Gamma^{\mu}(t,x^i) \equiv 0$, so that $R^{(h)}_{\mu\nu} = R_{\mu\nu} = 0$.

We remark that condition (a) is *a posteriori* necessary for $g_{\mu\nu}$ to satisfy $R_{\mu\nu} = 0$ (which is equivalent to $G_{\mu\nu} = 0$), as G^0_{μ} is well known to depend only on first order time derivatives and therefore can be computed from the Cauchy data alone. If condition (b) is not satisfied, then a related set of Cauchy data $\bar{g}_{\mu\nu}(0,\bar{x}^i)$, $(\partial/\partial t)\bar{g}_{\mu\nu}(0,\bar{x}^{\mu})$ can be found which satisfies $\bar{G}^0_{\mu}(0,\bar{x}^i) = 0$ and $\bar{\Gamma}^{\mu}(0,\bar{x}^i) = 0$. The evolution of this Cauchy data under $\bar{R}^{(h)}_{\mu\nu} = 0$ then leads to a spacetime $\bar{g}_{\mu\nu}$ which satisfies $\bar{R}_{\mu\nu} = 0$. A coordinate transformation then gives rise to a spacetime which is also Ricci flat and which has the original Cauchy data.

§(5): *UNIQUENESS*

As $R_{\mu\nu} = 0$ is a tensor system of partial differential equations, solutions to the Cauchy problem for the system cannot be functionally unique. Thus, if $g_{\mu\nu}$ satisfies $R_{\mu\nu} = 0$, and $\bar{x}^{\alpha}(x^{\beta})$ is a coordinate transformation which is the identity in a neighbourhood of the $t = 0$ hypersurface, then the coordinate transformed metric $\bar{g}_{\mu\nu} = (\partial x^{\alpha}/\partial\bar{x}^{\mu})(\partial x^{\beta}/\partial\bar{x}^{\nu})g_{\alpha\beta}$ also satisfies $\bar{R}_{\mu\nu} = 0$ and has the same Cauchy data as $g_{\mu\nu}$.

A proper uniqueness theorem for a tensor system of partial differential equations is the converse of the above remark; namely, to show that two solutions of $R_{\mu\nu} = 0$ with the same Cauchy data are related by a coordinate-transformation, i.e. are isometric.

The essential ideas of the proof are in Fourès-Bruhat [*8*]. The first step is to show that any spacetime can, by a coordinate transformation, be transformed into a spacetime for which $\Gamma^{\mu} = 0$. Now suppose two spacetimes have the same Cauchy data and both satisfy $R^{(h)}_{\mu\nu} = 0$. Bring them both into harmonic form. Then they both satisfy $R^{(h)}_{\mu\nu} = 0$ and their transformed Cauchy data are also equal. But by uniqueness of solutions to the Cauchy problem for $R^{(h)}_{\mu\nu} = 0$, the two spacetimes in harmonic form must be functionally the same, and therefore the original spacetimes are isometric.

First order methods can also be brought into play here. To show that any spacetime can be brought into harmonic form, one must solve the wave equation for four linearly independent functions. As in section 1, this can be done by reducing the wave equation to a linear first order symmetric hyperbolic system. In fact, using this method, it is possible to show that the coordinate transformation relating two solutions of $R_{\mu\nu} = 0$ with the same Cauchy data is differentiable to one more degree of differentiability than the solutions. This is a technical result, but natural geometrically, since one loses on degree of differentiability when a metric is transformed.

§(6): *SYMMETRIC HYPERBOLIC SYSTEMS ON MANIFOLDS*

So far we have been working locally; our methods, however, can be globalized to an arbitrary manifold M, possibly non-compact. This is accomplished by giving an intrinsic treatment of first order symmetric hyperbolic systems.

So let $\pi:E \to M$ be a vector bundle over a 3-manifold M (the starting manifold on which the Cauchy data is given) and let $J^1(E) \to M$ be its first jet bundle; the fiber over $x \in M$ is

$$J^1(E)_x = L(T_xM;E_x) \oplus E_x.$$

A section $u:M \to E$ can then be extended to a section $j(u):M \to J^1(E)$, the *first jet extension of* u. We assume E has a connection and let ∇u be the horizontal part of the tangent Tu; then the two components of $j(u)$ are the derivative ∇u and u itself.

A first order linear operator may be regarded as a map

$$\begin{array}{ccc} J^1(E) & \xrightarrow{\tilde{A}} & E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Then $\tilde{A}\circ j(u)$ reduces in the case of $\mathbb{R}^n$ to $A^i(\partial u/\partial x^i) + B\cdot u$. Let us write, conforming to earlier notation,

$$\tilde{A}\circ j(u) = A\cdot\nabla u + B\cdot u$$

(the two components of $\tilde{A}$) so $A:L(TM,E) \to E$.

Suppose now that we have an inner product $\langle\ ,\ \rangle_x$ on each fiber E_x compatible with the connection. We call

$$\frac{\partial u}{\partial t} = \tilde{A}\cdot j(u) = A\cdot\nabla u + B\cdot u$$

a *symmetric hyperbolic system* if $\tilde{A}\cdot j - (\tilde{A}\cdot j)^*$ is a zero order operator where $(\tilde{A}\cdot j)^*$ is the adjoint of $\tilde{A}\cdot j$. Then in $\mathbb{R}^n$ this condition reduces to the condition that the A^i be symmetric matrices (the A^0 term may be included separately). The point is that most of the theory works well in this context provided M is *complete* Riemannian and has curvature bounded above. For the quasi-linear case we require $\tilde{A}$ to be linear in the first factor ∇u, and that the associated linearized system be symmetric hyperbolic.

§(7): *CONCLUSIONS*

The theory of first order symmetric hyperbolic systems applies to a wide variety of equations of mathematical physics, amongst which are Maxwell's equations, the Dirac equation, the Lunquist equations of magnetohydrodynamics, compressible fluid equations, and the equations of elasticity. As emphasized in Courant-Hilbert [3], p. 592, the essential geometric reason that equations of this type play such a central role in mathematical physics is that these equations are often the Euler-Lagrange equations of a variational problem in which a symmetric bilinear form is varied.

Putting Einstein's equations into a symmetric hyperbolic form achieves a certain uniformity in the description of gravity with other physical systems. Moreover, there are several technical advantages for writing Einstein's equations in this form. For one thing, the proof of the existence of a Ricci flat spacetime with given Cauchy data is considerably simplified. For another, one can use similar first order methods to sharpen previously known uniqueness theorems. Finally, in this form the equations can be written intrinsically and thus globalized to a manifold.

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