DARBOUX'S THEOREM FAILS FOR WEAK SYMPLECTIC FORMS

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Abstract. An example of a weak symplectic form on a Hilbert space for which Darboux's theorem fails is given.

Introduction. Let $E$ be a Banach space and $B : E \times E \to \mathbf{R}$ a continuous bilinear form. Let $B^* : E \to E^*$ be defined by $B^*(e) \cdot f = B(e, f)$. Call $B$ nondegenerate if $B^*$ is an isomorphism and call $B$ weakly nondegenerate if $B^*$ is injective. For a symmetric bilinear form $G$ on $E$, define the skew form $\tilde{G}$ on $E \times E$ by
\[
\tilde{G}((e_1, e_2), (f_1, f_2)) = G(f_2, e_1) - G(e_2, f_1).
\]
It is easily seen that $\tilde{G}$ is nondegenerate (resp. weakly nondegenerate) iff $G$ is.

Now let $M$ be a Banach manifold. A symplectic form (resp. weak symplectic form) on $M$ is a smooth closed two form $\omega$ on $M$ such that for each $p \in M$, $\omega$ as a bilinear form on $T_p M$ is nondegenerate (resp. weakly nondegenerate); here $T_p M$ is the tangent space at $p$. Using a technique of Moser, Weinstein ([6], [7]) showed that for each $p \in M$ there is a local chart about $p$ on which $\omega$ is constant. This is a significant generalization and simplification of the classical theorem of Darboux. However, in many physical examples (the wave equation and fluid mechanics for instance) one deals with weak symplectic forms (see [1], [3], [4], [5]).

It is therefore interesting to know if Darboux's theorem remains valid for weak symplectic forms. In this note we give a counterexample.

Symplectic forms induced by metrics. If $M$ is a manifold, its cotangent bundle $T^* M$ carries a canonical symplectic form $\omega$. If $M$ is modeled on a reflexive space the form is nondegenerate; otherwise it is only weakly nondegenerate. See [1], [4]. Now let $\langle \cdot, \cdot \rangle_p$ be a (smooth) weak riemannian metric on $M$. Then it induces a map of $TM$ to $T^* M$. The pull back $\Omega$ of $\omega$ to $TM$ is called the form induced by the metric. It is a weak symplectic
form and in a chart \( U \) for \( M \) it is given by (using principal parts):

\[
2\Omega_{u,e}(\langle e_1, e_2 \rangle, \langle e_3, e_4 \rangle) = D_u(e, e_1) \cdot e_3 - D_u(e, e_3) \cdot e_1 + \langle e_4, e_1 \rangle_u - \langle e_2, e_3 \rangle_u.
\]

Here, \( D_u \) denotes the derivative of the map \( u \mapsto \langle e, e_1 \rangle_u \) with respect to \( u \). In the finite dimensional case this corresponds to the classical formula

\[
\Omega = \sum g_{ij} dq^i \wedge dq^j + \sum \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i dq^j \wedge dq^k.
\]

Observe that in the finite dimensional case if we take new variables \( q^1, \ldots, q^n, p_1, \ldots, p_n \) where \( p_i = \sum g_{ij} \dot{q}^j \), then (as is easy to check) \( \Omega = \sum dq^i \wedge dp_i \) which gives a chart in which \( \Omega \) is constant.

**The example.** The following is a simplification of an earlier example. We thank the referee and Paul Chernoff for suggestions in this regard.

Let \( H \) be a real Hilbert space. Let \( S : H \to H \) be a compact operator with range a dense, but proper subset of \( H \), which is selfadjoint and positive: \( \langle Sx, x \rangle > 0 \) for \( 0 \neq x \in H \). For example if \( H = L^2(\mathbb{R}) \), we can let \( S = (1 - \Delta)^{-1} \) where \( \Delta \) is the Laplacian; the range of \( S \) is \( H^2(\mathbb{R}) \).

Since \( S \) is positive, \(-1\) is clearly not an eigenvalue. Thus, by the Fredholm alternative, \( aI + S \) is onto for any real scalar \( a > 0 \). Define on \( H \) the weak metric \( g(x)(e, f) = \langle A_x e, f \rangle \) where \( A_x = S + \| x \|^2 I \). Clearly \( g \) is smooth in \( x \), and is an inner product. Let \( \Omega \) be the weak symplectic form on \( H \times H = H_1 \) induced by \( g \), as was discussed above.

**PROPOSITION.** There is no coordinate chart about \((0, 0) \in H_1 \) on which \( \Omega \) is constant.

**PROOF.** If there were such a chart, say \( \phi : U \to H \times H \) where \( U \) is a neighborhood of \((0, 0) \), then in particular in this chart, the range \( F \) of \( \Omega^b_x \), as a map of \( H_1 \) to \( H^*_1 \), would be constant. Let \( B_{x,y} \) be the derivative of \( \phi \) at \((x, y) \in H_1 \). Then we obtain that the range of \( \Omega_{x,y} \) equals \( B_{x,y} F \).

Now by the above formula for \( \Omega \), at the point \((x, 0) \) we have

\[
2\Omega_{(x,0)}((e_1, e_2), (e_3, e_4)) = g_x(e_1, e_3) - g_x(e_2, e_3).
\]

But by construction, for \( x \neq 0 \), \( g_x \) is a strong metric (i.e., \( A_x \) is onto for \( x \neq 0 \)), so the range of \( \Omega_{(x,0)}^b \) is all of \( H^*_1 \) for \( x \neq 0 \). Since \( B_{x,y} \) is an isomorphism, this implies that \( \Omega_{(0,0)}^b \) is onto all of \( H^*_1 \) as well. But \( g_0 \) is only a weak metric which is not onto as a map of \( H_1 \) to \( H^*_1 \). Hence \( \Omega_{(0,0)}^b \) cannot be onto as well, a contradiction.

As was pointed out by the referee, the example even shows that \( \Omega \) cannot be made constant on a continuous vector bundle chart on \( T^2M \to TM \), let alone by a manifold chart on \( TM \).
Of course the essence of the example is that the range of $\Omega$ suddenly changed at one point i.e., the topology of the metric suddenly changed. This is perfectly compatible with the smoothness of $\Omega$ as it is only a weak symplectic form. This suggests a possible conjecture pointed out by Paul Chernoff: if $\Omega$ is such that the ranges of $\Omega_u$ are locally equivalent via an isomorphism, then Darboux's theorem should hold. This can be verified directly in case $\Omega$ comes from a metric which has locally equivalent ranges.

REFERENCES


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