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14

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## Methods of Local and Global Differential Geometry in General Relativity

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GENERAL RELATIVITY AS A DYNAMICAL SYSTEM ON THE MANIFOLD  $\mathcal{A}$   
OF RIEMANNIAN METRICS WHICH COVER DiffeOMORPHISMS

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1. Introduction

In this paper we consider the geometrodynamical formulation of general relativity, due most recently to Arnowitt, Deser, and Misner [2], DeWitt [3], and Wheeler [8], from the point of view of manifolds of maps (function spaces) and infinite-dimensional geometry.

Hydrodynamics is approached from this point of view by Arnold [1] and by Ebin-Marsden [4]; in Fischer-Marsden [5, 6] the function spaces appropriate for a dynamical formulation of general relativity are introduced. We hope that our approach will clarify the basic dynamical structure of the Einstein equations and the associated infinite-dimensional geometry in a spirit analogous to that which has been done in hydrodynamics.

The key to our approach is the group  $\mathcal{D} = \text{Diff}(M)$  of smooth ( $C^\infty$ ) diffeomorphisms of a fixed 3-dimensional manifold  $M$ . For hydrodynamics one concentrates on  $\mathcal{D}_\mu$ , the volume preserving diffeomorphisms [4]. For relativity one uses the manifold  $\mathcal{A}$  of Riemannian metrics which cover diffeomorphisms. We begin with a description of  $\mathcal{A}$ .

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## 2. The Manifold $\mathcal{Q}$ and the Einstein System

Let  $M$  be a fixed (no changes in topology) closed (compact without boundary) 3-dimensional oriented smooth manifold, and let

$\mathcal{M} = \text{Riem}(M) =$  manifold of smooth Riemannian (positive-definite) metrics on  $M$ ;

$\mathcal{D} = \text{Diff}(M) =$  the group (under composition) of smooth orientation-preserving diffeomorphisms of  $M$ ; and

$S_2(M) =$  vector space of smooth symmetric 2-covariant tensor fields on  $M$ .

Note that  $S_2(M)$  is a linear space and that in any decent topology,  $M$  is an open convex cone in  $S_2(M)$ .

Let  $\pi : \text{Pos}(M) \rightarrow M$  denote the tensor bundle of symmetric positive definite bilinear forms so that  $\pi^{-1}(m) =$  space of inner products on  $T_m M$ . A Riemannian metric  $g_\eta$  which covers a diffeomorphism  $\eta \in \mathcal{D}$  is a smooth map  $g_\eta : M \rightarrow \text{Pos}(M)$  such that the following diagram commutes:

$$\begin{array}{ccc} & & \text{Pos}(M) \\ & \nearrow g_\eta & \downarrow \pi \\ M & \xrightarrow{\eta} & M \end{array}$$

(that is,  $\pi \circ g_\eta = \eta \in \mathcal{D}$ ). Thus  $g_\eta$  assigns to each point  $m \in M$  an inner product of the tangent space  $T_{\eta(m)} M$ . We let  $\mathcal{Q}$  denote the manifold of all such maps for all  $\eta \in \mathcal{D}$ .  $\mathcal{Q}$  is the manifold of Riemannian metrics which cover diffeomorphisms. One can prove that  $\mathcal{Q}$  has the structure of a smooth infinite dimensional manifold, cf. [4, § 2]; we shall not require this structure.

There is a natural projection  $\bar{\pi} : \mathcal{Q} \rightarrow \mathcal{D}$  defined by  $\bar{\pi}(g_\eta) = \pi \circ g_\eta = \eta \in \mathcal{D}$ . Also, if  $g_\eta \in \mathcal{Q}$ , observe that  $g_\eta \circ \eta^{-1} \in \mathcal{M}$  is an "ordinary" Riemannian metric for  $M$ . Now  $\mathcal{Q}$  is diffeomorphic to  $\mathcal{D} \times \mathcal{M}$  by the map

$$\mathfrak{I}_R : \mathcal{A} \rightarrow \mathcal{D} \times \mathcal{M} ; \quad g_n \mapsto (n, g_n \circ n^{-1}) ,$$

( $\mathfrak{I}_R$  = right translation) with inverse

$$\mathfrak{I}_R^{-1} : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{A} ; \quad (n, g) \mapsto g \circ n .$$

Thus information on  $\mathcal{A}$  can be transferred to  $\mathcal{D} \times \mathcal{M}$  and vice-versa via the mapping  $\mathfrak{I}_R$ . It is convenient to think of  $\mathcal{D} \times \mathcal{M}$  as a realization of  $\mathcal{A}$ .

Let  $\mathcal{T} = C^\infty(M; \mathbb{R})$  = the vector space of smooth real-valued functions  $\xi : M \rightarrow \mathbb{R}$  (scalar fields or 0-covariant tensor fields on M).

We will refer to  $\mathcal{T}$  as the relativistic time-translation group. Note that the constant functions on M form a subgroup of  $\mathcal{T}$  which is isomorphic to  $\mathbb{R}$ , the classical time-translation group. The manifold  $\mathcal{T} \times \mathcal{A} \approx \mathcal{T} \times \mathcal{D} \times \mathcal{M}$  is the proper configuration space for a geometrodynamical formulation of general relativity as we now explain. We will be concerned with the propagation of initial Cauchy data  $(g_0, h_0) \in \mathcal{M} \times S_2(M)$  off some 3-dimensional hypersurface M of, a yet to be constructed, Ricci-flat (vacuum) space-time  $V_4$ . Here  $h = \dot{g} = \frac{\partial g}{\partial t}$  is the velocity canonically conjugate to the configuration fields g. As  $g_t$  is determined only up to its isometry class, the evolution is determined only up to an arbitrary curve  $\mathcal{R}_t \in \mathcal{D}$  of diffeomorphisms called the actual shift (with  $\mathcal{R}_0 = \text{id}_M = e$  = the identity diffeomorphism); that is,  $g_t$  and  $(\mathcal{R}_t^{-1})^* g_t$  are isometric evolutions, where  $(\mathcal{R}_t^{-1})^* g_t(m) \cdot (Y_m, Z_m) = g_t \circ \mathcal{R}_t^{-1}(m) \cdot (T_{\mathcal{R}_t^{-1}}(Y_m), T_{\mathcal{R}_t^{-1}}(Z_m))$ ,  $Y_m, Z_m \in T_m M$ , is the "push-forward" of a covariant tensor field. Moreover, one is free to specify on M an arbitrary system of clock rates, or equivalently of clock settings, given as a curve  $\xi_t \in \mathcal{T}$  of time functions (the clock settings) with  $\xi_0 = \underline{0}$  = the zero function on M (all clocks start at high noon). This arbitrariness or degeneracy is reflected in the evolution equations as follows:

The Einstein System: Let M be a closed oriented 3-dimensional manifold.  
Let  $X_t$  be an arbitrary time-dependent vector field called the shift vector field  
and  $N_t$  an arbitrary positive scalar field called the lapse function;  $N_t(m) > 0$

for all  $(t, m) \in \mathbb{R} \times M$ . Let  $g$  be a given Riemannian-metric on  $M$ , and let  $k$  be a given symmetric 2-covariant tensor field on  $M$  such that

$$\delta(k - (\text{Tr } k)g) = 0,$$

$$\frac{1}{2} ((\text{Tr } k)^2 - k \cdot k) + 2R(g) = 0.$$

The problem is to find a time-dependent metric field  $g_t$  on  $M$  such that  $g_t$  and the supplementary variable

$$k_t = \frac{1}{N_t} \left( \frac{\partial g_t}{\partial t} + L_{X_t} g_t \right),$$

satisfy:

- (i) the given initial conditions:  $(g_0, k_0) = (g, k)$ ,  
(ii) the evolution equation

$$\frac{\partial k_t}{\partial t} = S_{g_t}(k_t) - 2N_t \text{ Ric}(g_t) + 2 \text{ Hess}(N_t) - L_{X_t} k_t.$$

Our notation is the following:

$\delta k$  = divergence of  $k = (\delta k)_i = k_i^j{}_{|j}$  ( $|j$  = covariant derivative with respect to the time-dependent metric  $g$ ),

$\text{Tr } k$  = Trace  $k = g^{ij} k_{ij} = k^i{}_i$ ,

$k \cdot k$  = dot product for symmetric tensors =  $k_{ij} k^{ij}$ ,

$k \times k$  = cross-product for symmetric tensors =  $k_{i\ell} k^{\ell}{}_j$ ,

$S_g(k) = k \times k - \frac{1}{2} (\text{Tr } k)k = k_{i\ell} k^{\ell}{}_j - \frac{1}{2} (g^{mn} k_{mn}) k_{ij}$  = DeWitt spray on  $\mathcal{T}^2$ ,

$L_{X_t} g_t$  = Lie derivative of  $g_t$  with respect to the time-dependent vector field  $X_t = X_{i|j} + X_{j|i}$ ,

$L_{X_t} k_t$  = Lie derivative of  $k_t = X^{\ell} k_{ij\ell} + k_{i\ell} X^{\ell}{}_{|j} + k_{j\ell} X^{\ell}{}_{|i}$ ,

$\text{Ric}(g_t)$  = Ricci curvature tensor formed from  $g_t = R_{ij} =$

$$\Gamma_{ij,k}^k - \Gamma_{ki,j}^k + \Gamma_{ij}^k \Gamma_{k\ell}^{\ell} - \Gamma_{ik}^{\ell} \Gamma_{\ell j}^k,$$

$R(g_t)$  = Scalar curvature =  $R^k{}_k$ ,

$\text{Hess}(N)$  = Hessian of  $N$  = double covariant derivative =  $N_{|i|j}$ .

We now explain how the Einstein system, the lapse function  $N_t$ , the shift vector field  $X_t$ , and the configuration space  $\mathcal{T} \times \mathcal{D} \times \mathcal{M}$  are interrelated (see Fischer-Marsden [5] for more details).

### 3. The Geometry of the Shift Vector Field

Let  $\mathcal{D} = \text{Diff}(M)$ , the group of all smooth orientation preserving diffeomorphisms of  $M$ . Now  $\mathcal{D}$  is a manifold modeled on a Frechet space; (see Ebin-Marsden [4] and related references for the structure of  $\mathcal{D}$ ). The tangent space  $T_{\mathcal{D}}$  at a point  $\eta \in \mathcal{D}$  is the manifold of smooth maps  $X_{\eta} : M \rightarrow TM$  which cover  $\eta$ , that is, such that the following diagram commutes:

$$\begin{array}{ccc} & & TM \\ & \nearrow X_{\eta} & \downarrow \tau_M \\ M & \xrightarrow{\eta} & M \end{array}$$

where  $\tau_M$  denotes the canonical projection of  $TM$  to  $M$ . To see this let  $\eta_t \in \mathcal{D}$  be a curve in  $\mathcal{D}$ ,  $\eta_0 = \eta$ , so that  $\left. \frac{d\eta_t}{dt} \right|_{t=0}$  represents a tangent vector in  $T_{\mathcal{D}}$ . But for  $m \in M$  fixed,  $\sigma(t) = \eta_t(m)$  is a curve in  $M$  with  $\sigma(0) = \eta(m)$  and with tangent  $\sigma'(0) = \left. \frac{d\eta_t}{dt}(m) \right|_{t=0} \in T_{\eta(m)}M$ . Thus  $\frac{d\eta_t}{dt}$  is a map from  $M$  to  $TM$  covering  $\eta$ .

We refer to  $X$  as a vector field which covers  $\eta$ , so that  $T\mathcal{D}$  is the manifold of vector fields covering diffeomorphisms. In particular,  $T_e\mathcal{D} = \mathfrak{X}(M)$  = the vector space of smooth vector fields on  $M$  = the Lie algebra of  $\mathcal{D}$ .

As with the manifold  $Q$ , there is a natural projection  $\bar{\pi} : T\mathcal{D} \rightarrow \mathcal{D}$  defined by

$$\bar{\pi}(X_{\eta}) = \tau_M \circ X_{\eta} = \eta \in \mathcal{D}.$$

Let  $R_{\eta_1} : \mathcal{D} \rightarrow \mathcal{D}$  denote right translation by  $\eta_1$ ; ( $R_{\eta_1}(\eta) = \eta\eta_1$ ).

Then

$$TR_{\eta_1} : T\mathcal{D} \rightarrow T\mathcal{D} ; X_{\eta} \mapsto X_{\eta} \circ \eta_1 ,$$

so that for  $X_{\eta} \in T_{\mathcal{D}}$ ,  $TR_{\eta_1}(X_{\eta}) = X_{\eta} \circ \eta_1^{-1} \in T_e\mathcal{D}$  is an "ordinary" vector field on  $M$ , called the pull-back of  $X_{\eta}$  by right translation.

Now let  $X_t : M \rightarrow TM$  be a time-dependent vector field on  $M$ . Then the flow  $\mathcal{N}_t$  of  $X_t$  with  $\mathcal{N}_0 = \text{identity}$  is a smooth curve in  $\mathcal{D}$  (as  $X_t$  is time-dependent,  $\mathcal{N}_t$  is not a one-parameter subgroup of  $\mathcal{D}$ ) which satisfies

$$\frac{d\mathcal{N}_t}{dt} = X_t \circ \mathcal{N}_t, \quad \text{or} \quad \frac{d\mathcal{N}_t}{dt} \circ \mathcal{N}_t^{-1} = X_t.$$

Conversely, given a smooth curve  $\mathcal{N}_t \in \mathcal{D}$  with  $\mathcal{N}_0 = \text{identity}$ ,  $\frac{d\mathcal{N}_t}{dt} \circ \mathcal{N}_t^{-1} = X_t$  is a time-dependent vector field which generates  $\mathcal{N}_t$  as its flow.

Thus in the Einstein system, if one gives the shift vector field  $X_t$ , then the actual shift of  $M$  is its flow  $\mathcal{N}_t \in \mathcal{D}$ , a curve in  $\mathcal{D}$ . Equivalently one may specify the actual shift  $\mathcal{N}_t \in \mathcal{D}$  and compute the shift vector field as above. It is because of the presence of the shift vector field that the group must be included in the configuration space.

The relationship between the Lie derivative terms and the shift vector field can be explained geometrically as follows. Suppose that for  $\bar{N}_t = 1, \bar{X}_t = 0$ ,  $(\bar{g}_t, \bar{k}_t) \in \mathcal{M} \times S_2(M)$  is a solution to the Einstein system with initial conditions  $(\bar{g}_0, \bar{k}_0)$ ; that is,

$$\frac{\partial \bar{g}_t}{\partial t} = \bar{k}_t,$$

$$\frac{\partial \bar{k}_t}{\partial t} = S_{\bar{g}_t}(\bar{k}_t) - 2 \text{Ric}(\bar{g}_t).$$

Now let  $X_t$  be an arbitrary shift vector field with flow  $\mathcal{N}_t$ ,  $\mathcal{N}_0 = \text{identity}$ . Then  $(g_t, k_t) = ((\mathcal{N}_t^{-1})^* \bar{g}_t, (\mathcal{N}_t^{-1})^* \bar{k}_t)$  are solutions to the evolution equations with  $N_t = 1, X_t = \text{given shift vector field}$ , and the same initial data as before. This follows by a direct verification:

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= \frac{\partial (\mathcal{N}_t^{-1})^* \bar{g}_t}{\partial t} \\ &= (\mathcal{N}_t^{-1})^* \frac{\partial \bar{g}_t}{\partial t} - L_{X_t} ((\mathcal{N}_t^{-1})^* \bar{g}_t) \\ &= k_t - L_{X_t} g_t, \end{aligned}$$

where we have used the fact that

$$\frac{d}{dt} (\eta_t^{-1})^* g = -L_{X_t} (\eta_t^{-1})^* g ; \text{ see [7] , p. 32 .}$$

Similarly,

$$\begin{aligned} \frac{\partial k_t}{\partial t} &= \frac{\partial (\eta_t^{-1})^* \bar{k}_t}{\partial t} \\ &= (\eta_t^{-1})^* \frac{\partial \bar{k}_t}{\partial t} - L_{X_t} (\eta_t^{-1})^* \bar{k}_t \\ &= S_{g_t} (k_t) - 2\text{Ric} (g_t) - L_{X_t} k_t \end{aligned}$$

since  $S_{\bar{g}} (\bar{k})$  and  $\text{Ric} (\bar{g})$  are tensors and hence commute with  $(\eta_t^{-1})^*$ ; that is,  $(\eta_t^{-1})^* (\text{Ric} (\bar{g})) = \text{Ric} ((\eta_t^{-1})^* \bar{g}) = \text{Ric} (g)$ .

The significance of this result may be clarified as follows: Besides the realization of  $\mathcal{A}$  as  $\mathcal{D} \times \mathcal{M}$  by "right translations," there is a realization of  $\mathcal{A}$  as  $\mathcal{D} \times \mathcal{M}$  by "left translations" defined as follows:

$$\tilde{\mathcal{I}}_L : \mathcal{A} \rightarrow \mathcal{D} \times \mathcal{M} ; \quad g_n \mapsto (\eta^{-1})^* (g_n \circ \eta^{-1}) .$$

These two realizations of  $\mathcal{A}$  are entirely analogous to the two realizations of  $\text{TSO}(3)$  for the rigid body into body and space coordinates respectively; see Arnold [1]. Thus the introduction of a shift may be viewed merely as shifting from body to space coordinates by use of the coordinate change  $\eta_t$ .

#### 4. The Lapse Function and the Intrinsic Shift Vector Field

To discuss the lapse we assume that the shift vector field  $Y_t = 0$ . (They can be handled simultaneously by using the semi-direct product on  $\mathcal{T} \times \mathcal{D}$ .) If we choose the lapse  $N_t = 1$ , then the evolution of  $g$  is parameterized by a canonical evolution parameter, the proper time  $\tau$ . But suppose that  $g$  is a solution of the Einstein system for an arbitrary lapse  $N$ . One constructs a space-time on  $\mathbb{R} \times \mathcal{M}$  in a tubular neighborhood of  $\mathcal{M}$  by the Lorentz metric (in coordinates)



$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij} dx^i dx^j.$$

The proper time function  $\tau(t, m) = \tau_t(m) = \tau(t, x^k)$  (in this tubular neighborhood of  $M$ ) is then just the time coordinate in Gaussian normal coordinates  $(\tau(t, x^k), \bar{x}^i(t, x^k))$ , where  $\bar{x}^i(t, x^k)$  is the space part of the Gaussian coordinates. To find the relation between the lapse  $N_t$  and  $\tau_t$ , we consider the transformation of  $g_{\mu\nu}$  to Gaussian normal coordinates; writing out  $\bar{g}^{00} = g^{\mu\nu} \frac{\partial \tau}{\partial x^\mu} \frac{\partial \tau}{\partial x^\nu}$  yields

$$-1 = -\frac{1}{N^2} \left( \frac{\partial \tau}{\partial t} \right)^2 + g^{kl} \frac{\partial \tau}{\partial x^k} \frac{\partial \tau}{\partial x^l},$$

which is solved for  $N_t$  to give

$$N_t = \frac{d\tau_t}{dt} \frac{1}{\sqrt{1 + \|\text{grad } \tau_t\|^2}},$$

where  $\|\text{grad } \tau\|^2 = g^{kl} \frac{d\tau}{dx^k} \frac{d\tau}{dx^l}$  is computed with respect to the inverse  $g^{kl}$  of the time-dependent metric  $g_{kl}$  ( $= g^{ij}$  since the shift is zero). The factor  $\frac{1}{\sqrt{1 + \|\text{grad } \tau\|^2}}$  takes into account the fact that in general the lapse depends on space coordinates and therefore pushes up the hypersurface  $M$  through  $\mathbb{R} \times M$  unevenly.

The single first order partial differential equation for  $\tau$

$$\left( \frac{d\tau}{dt} \right)^2 - N^2 g^{kl} \frac{d\tau}{dx^k} \frac{d\tau}{dx^l} = N^2$$

can be reduced to a system of eight first-order ordinary differential equations by the Cauchy method of characteristics. Of course this system of ordinary differential equations is just the system of geodesic equations of the Lorentz metric  $g_{\mu\nu}$  (for unit timelike geodesics) in Hamiltonian form. If we choose on the non-characteristic hypersurface  $t = 0$  the initial condition:  $\tau(0, m) = 0$  (corresponding to geodesics normal to  $t = 0$ ), then we are assured of a unique  $\tau(t, m)$  that satisfies the above equation with the initial condition  $\tau(0, m) = 0$ . Note that  $\frac{d\tau}{dt} = N$  on this initial hypersurface.

The condition

$$\bar{g}^{oi} = -\frac{1}{N^2} \frac{\partial \bar{x}^i}{\partial t} \frac{\partial \tau}{\partial t} + \frac{\partial \bar{x}^i}{\partial x^n} \left( g^{mn} \frac{\partial \tau}{\partial x^m} \right) = 0$$

gives an equation for the space part  $\bar{x}^i(t, x^k)$  of the Gaussian normal coordinate system,

$$\frac{\partial \bar{x}^i}{\partial t}(t, x^k) = \frac{N(t, x^k)}{\sqrt{1 + \|\text{grad} \tau\|^2}} \frac{\partial \bar{x}^i}{\partial x^n}(t, x^k) \left( g^{mn}(t, x^k) \frac{\partial \tau}{\partial x^m}(t, x^k) \right)$$

Set  $Y_t = -\frac{N(t, x^k)}{\sqrt{1 + \|\text{grad} \tau\|^2}} \text{grad} \tau$ ; then the above equation can be written as

$$\frac{d\varphi_G}{dt} = -D\varphi_G \cdot (Y)$$

where  $\varphi_G$  is the spatial part of the Gaussian normal coordinates and  $D\varphi_G$  is, in coordinates, the Jacobian matrix of  $\varphi_G$ . But the identity

$$\frac{d}{dt} (f_t^{-1} \circ f_t) = \frac{df_t^{-1}}{dt} \circ f_t + Df_t^{-1} \cdot \frac{df_t}{dt} = \frac{df_t^{-1}}{dt} \circ f_t + Df_t^{-1} \cdot Y_t \circ f_t = 0$$

then shows that this equation is solved by  $\varphi_G = f_t^{-1}$  if  $f_t$  is the flow of  $Y_t$ . We call  $Y_t$  the intrinsic shift of the lapse since it describes the "tilting" of the Gaussian normal coordinates due to the space dependence of the lapse function. The above argument shows that the partial differential equation for the space part of the Gaussian normal coordinate system can be solved by an ordinary differential equation, namely finding the flow of the intrinsic shift. Finally, the inverse to the contravariant metric

$$\begin{aligned} \bar{g}^{ij}(\tau(t, x^k), \bar{x}^i(t, x^k)) &= \frac{\partial \bar{x}^i}{\partial x^m}(t, x^k) \frac{\partial \bar{x}^j}{\partial x^n}(t, x^k) g^{mn}(t, x^k) - \frac{1}{N^2} \frac{\partial \bar{x}^i}{\partial t}(t, x^k) \frac{\partial \bar{x}^j}{\partial t}(t, x^k) \\ &= \frac{\partial \bar{x}^i}{\partial x^m}(t, x^k) \frac{\partial \bar{x}^j}{\partial x^n}(t, x^k) \left( g^{mn}(t, x^k) - \frac{1}{1 + \|\text{grad} \tau\|^2} g^{ml}(t, x^k) \frac{\partial \tau}{\partial x^l}(t, x^k) g^{nr}(t, x^k) \right. \\ &\quad \left. \cdot \frac{\partial \tau}{\partial x^r}(t, x^k) \right) \end{aligned}$$

solves the evolution equations with  $N = 1$  (and the same initial data) if  $g_{ij}(t, x^k)$  solves the Einstein equations with an arbitrary  $N$ . Writing  $g^{-1}$  for the contravariant components of  $g$ , the above equation can be written intrinsically as

$$\bar{g}^{-1}(\tau(t, m), \varphi_0(t, m)) = D\varphi_0(t, m) \otimes D\varphi_0(t, m) \left( g^{-1}(t, m) - \frac{\text{grad } \tau(t, m)}{\sqrt{1 + \|\text{grad } \tau\|^2}} \otimes \frac{\text{grad } \tau(t, m)}{\sqrt{1 + \|\text{grad } \tau\|^2}} \right).$$

Our prescription shows how, given a solution to the Einstein equation with an arbitrary  $N$ , to find the solution to the Einstein equations with  $N = 1$  and the same initial data by solving ordinary differential equations only. A similar prescription is available to go from solutions for  $N = 1$  to solutions for arbitrary  $N$ ; see [5]. To take into account the lapse function we introduce the relativistic time translation group  $\mathcal{T} = C^\infty(M; \mathbb{R})$  (a group under pointwise addition of functions). As  $\mathcal{T}$  is a vector space,  $\tau\mathcal{T} = \mathcal{T} \times \mathcal{T}$ . For a given lapse  $N_t$  and a solution  $g_t$  to Einstein's equations with this lapse, we construct a curve  $\tau_t \in \mathcal{T}$  such that

$$\left( \frac{d\tau}{dt} \right)^2 - N^2 \|\text{grad } \tau\|^2 = N^2$$

and  $\tau_0 = 0$ . Thus to find the curve in  $\mathcal{T}$  corresponding to a given lapse  $N$  we must first solve Einstein's equations with this particular lapse.

In the case that  $N$  depends only on the time coordinate, then  $\tau_t$  and  $N_t$  are simply related by  $\tau_t = \int_0^t N_\lambda d\lambda$ . Moreover, if  $(\bar{g}_t, \bar{k}_t)$  is a solution to the Einstein system with initial conditions  $(\bar{g}_0, \bar{k}_0)$  and lapse  $\bar{N}_t = 1$ , then the solution with  $N_t = f(t)$  (and  $X_t = 0$ ) and the same initial conditions is just the reparameterized curve  $(g_t, k_t) = (\bar{g}_{\tau(t)}, \bar{k}_{\tau(t)})$ . This is easily seen, as

$$\frac{\partial g_t}{\partial t} = \frac{\partial \bar{g}_{\tau(t)}}{\partial t} = \frac{\partial \bar{g}_{\tau(t)}}{\partial \tau} \frac{d\tau(t)}{dt} = N_t \bar{k}_{\tau(t)} = N_t k_t$$

and

$$\begin{aligned} \frac{\partial k_t}{\partial t} &= \frac{\partial \bar{k}_{\tau(t)}}{\partial t} = \frac{\partial \bar{k}_{\tau(t)}}{\partial \tau} \frac{d\tau(t)}{dt} = N_t \left( S_{\bar{g}_{\tau(t)}}(\bar{k}_{\tau(t)}) - 2 \text{Ric}(\bar{g}_{\tau(t)}) \right) \\ &= N_t S_{g_t}(k_t) - 2N_t \text{Ric}(g_t) . \end{aligned}$$

### 5. The Einstein Lagrangian on $T^* \mathcal{A} \approx T^* \mathcal{D} \times \mathcal{M}$

Since  $\mathcal{M}$  is an open convex cone in  $S_2(M)$ ,  $T\mathcal{M} = \mathcal{M} \times S_2(M)$ . On  $\mathcal{M}$  we define the DeWitt metric  $\mathcal{H}$  (see DeWitt [3], and Fischer-Marsden [5]) by

$$\mathcal{H}_g : T_g \mathcal{M} \times T_g \mathcal{M} = S_2(M) \times S_2(M) \rightarrow \mathbb{R}$$

$$\mathcal{H}_g(h_1, h_2) = \int_M ((T_r h_1)(T_r h_2) - h_1 \cdot h_2) \mu_g,$$

where  $\mu_g$  is the volume element associated with the metric  $g$  (in coordinates  $\mu_g = \sqrt{\det g} \, dx^1 \wedge dx^2 \wedge dx^3$ ).  $\mathcal{H}$  is a non-degenerate but weak metric on  $\mathcal{M}$ ; here weak means that the map  $\mathcal{H}_g^* : T_g \mathcal{M} \rightarrow T_g^* \mathcal{M}$ , defined by  $\mathcal{H}_g^*(h_1, h_2) = \mathcal{H}_g(h_1, h_2)$  is an injection, by the non-degeneracy, but is not an isomorphism.

We now introduce a potential  $V : \mathcal{M} \rightarrow \mathbb{R}$  defined by

$$V(g) = 2 \int_M R(g) \mu_g$$

(twice the integrated scalar curvature). If on  $T\mathcal{M}$  we consider the Lagrangian

$$L = T - V : T\mathcal{M} = \mathcal{M} \times S_2(M) \rightarrow \mathbb{R},$$

defined by

$$L(g, h) = \frac{1}{2} \mathcal{H}_g(h, h) - V(g),$$

then a computation shows that Lagrange's equations give the Einstein system with lapse  $N_t = 1$  and shift  $X_t = 0$ .

The DeWitt metric  $\mathcal{H}$  on  $\mathcal{M}$  is extended to  $\mathcal{D} \times \mathcal{M} \approx \mathcal{A}$  by defining on each fiber  $T_{(n, g)}(\mathcal{D} \times \mathcal{M}) = T_n \mathcal{D} \times S_2(M)$

$$\mathcal{H}(n, g) : (T_n \mathcal{D} \times S_2(M)) \times (T_n \mathcal{D} \times S_2(M)) \rightarrow \mathbb{R}$$

$$\mathcal{H}(n, g)((X_{n_1}, h_1), (X_{n_2}, h_2)) = \mathcal{H}_g(h_1 + L_{X_{n_1} \circ n_1^{-1}} g, h_2 + L_{X_{n_2} \circ n_2^{-1}} g).$$

The Lagrangian  $L$  on  $T\mathcal{M}$  is now extended to a Lagrangian on  $T(\mathcal{D} \times \mathcal{M})$  by

$$\bar{L} : T(\mathcal{D} \times \mathcal{M}) = T\mathcal{D} \times \mathcal{M} \times S_2(M) \rightarrow \mathbb{R}$$

$$\bar{L}(X_n, g, h) = L(g, h + L_{X_n \circ n^{-1}} g)$$

$$= \frac{1}{2} \mathcal{L}_g (h + L_{X_{\eta} \circ \eta^{-1}} g, h + L_{X_{\eta} \circ \eta^{-1}} g) - v(g).$$

Note that the factor  $\mathfrak{D}$  is now essential as  $X_{\eta}$  is explicitly involved in  $L$ .

Now  $\mathcal{L}$  is a degenerate metric on  $\mathfrak{D} \times \mathcal{M}$  since if

$$\mathcal{L}(\kappa, g) (h + L_{X_{\eta} \circ \eta^{-1}} g, k + L_{X_{\eta} \circ \eta^{-1}} g) = 0 \text{ for all } (Y_{\eta}, k) \in T_{\eta} \mathfrak{D} \times S_2(M),$$

then

$$h + L_{X_{\eta} \circ \eta^{-1}} g = 0,$$

but  $h$  and  $X_{\eta}$  need not be zero independently. This degeneracy has the effect of introducing some ambiguity into the equations of motion. However, the degeneracy of  $\mathcal{L}$  is such that we are free to specify a curve of diffeomorphisms  $\eta_t \in \mathfrak{D}$ ; thus the ambiguity in the equations of motion is completely removed by the specification of the shift vector field  $X_t$ .

Using  $\bar{L} : T(\mathfrak{D} \times \mathcal{M}) \rightarrow \mathbb{R}$ , we construct on  $T(\mathcal{T} \times \mathfrak{D} \times \mathcal{M})$  the

### Einstein Lagrangian

$$L_E : T(\mathcal{T} \times \mathfrak{D} \times \mathcal{M})$$

defined by

$$L_E(S, N, X_{\eta}, g, h) = \int_M \left\{ \left( \frac{h + L_{X_{\eta} \circ \eta^{-1}} g}{N} \right) \cdot \left( \frac{h + L_{X_{\eta} \circ \eta^{-1}} g}{N} \right) - \left[ \text{Tr} \left( \frac{h + L_{X_{\eta} \circ \eta^{-1}} g}{N} \right) \right]^2 \right\} \mu_g - 2 \int_M N R(g) \mu_g.$$

$L_E$  now picks up a degeneracy in the  $\mathcal{T}$  direction, as well as in the  $\mathfrak{D}$  direction, allowing for the arbitrary specification of  $N_t$  as well as  $X_t$ . However, once  $N_t$  and  $X_t$  are specified, the degeneracy of  $L_E$  is completely removed and the evolution equations are well-defined. A computation then shows that Lagrange's equations in the "non-degenerate direction", together with the arbitrarily specified lapse function  $N_t$  and shift vector field  $X_t$ , are the Einstein equations of evolution (see [5] for details).

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