A BANACH SPACE OF ANALYTIC FUNCTIONS FOR CONSTANT COEFFICIENT EQUATIONS OF EVOLUTION

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The spaces involved in the theory of equations of evolution (that is, the theory of semi-groups) are such that the infinitesimal generators are only densely defined. For the infinitesimal generator to be everywhere defined and smooth (that is, differentiable), one must work with a Frechet space. This is especially important in the non linear theory of Moser (see Moser [3] and Marsden [2]). If the spaces were Banach spaces, the theory would reduce to the classical Picard theory for ordinary differential equations. See Lang [1], for a discussion of the classical theory. In fact in this case the existence theory is both simpler and more comprehensive, because we obtain the important fact that the flow is a diffeomorphism (the solutions depend smoothly, and not merely continuously on the initial data). This has other advantages too, since the theory for smooth flows on Banach manifolds is well developed. (For example, see Marsden [2], theorem 6.10 for an application we have in mind.)

It is often not pointed out that the Banach space theory <u>does</u> apply in the special case of constant coefficient equations; that is, equations of the form

$$(1) \quad \frac{\partial f}{\partial t} = \Sigma\, c_{i_1} \frac{\partial f}{\partial x_1^{i_1}} + \Sigma\, c_{i_1 i_2} \frac{\partial^2 f}{\partial x_1^{i_1}\, \partial x_2^{i_2}} + \dots + \Sigma\, c_{i_1 \dots i_n} \frac{\partial^n f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} + g$$

where $c_{i_1 \dots i_k}$ are constants. Here f and g are functions of $(x_1, \dots, x_n, t)$ and we are supposed to solve for f given the initial function at $t = 0$. Our aim is to give a <u>simple</u> proof of existence and uniqueness for this equation.

Of course there are classical theorems which do apply, but each has some drawback. For example, the classical Cauchy theorem only gives local solutions, and other theorems make restrictions on the coefficients such as ellipticity. What we do is to shrink the space so that such restrictions become unnecessary. As usual f can have components, so that other equations such as the wave equation are also covered.

First we give the spaces and second give the theorem. There will be nothing to prove except to apply the classical flow theorem from Lang [1].

Definition. Let E, F be Banach spaces and $U \subset E$ open. For a map $f: U \subset E \to F$, we let $Df, \ldots, D^k f$ denote the derivatives. Thus $Df: U \to L(E, F)$ where $L(E, F)$ denotes the continuous linear maps from E to F.

Let $B_k(U, F)$ denote those maps $f: U \to F$ whose first k derivatives exist and are bounded on U. For a sequence $s = (r_1, r_2, \ldots)$ of positive numbers satisfying the condition that $r_n/r_{n+1} \leq M$ for some M, let $B_s(U, F)$ denote the infinitely differentiable functions $f: U \to F$ such that

$$\|f\| = \sup\{\|f(u)\|, r_1\|Df(u)\|, \ldots, r_k\|D^k f(u)\|, \ldots\}$$

is finite.

With this norm, $B_s(U, F)$ is easily seen to be a Banach space. Its members are analytic in the sense that their Taylor series converge (the derivatives are rapidly decreasing). The condition on $s = (r_1, r_2, \ldots)$ is imposed so that the derivative map is continuous.

Let $J^n(E, F) = L^n(E, F) \times L^{n-1}(E, F) \times \ldots \times L(E, F) \times F$ where $L^k(E, F)$ denotes the k-multilinear maps from E to F. In equation (1) the right hand side is represented by a linear map $P: J^n(E, F) \to F$, plus the constant term g. In general, this equation can be written as follows:

THEOREM. Let $P: J^n(E, F) \to F$ be continuous linear and $g \in B_s(U, F)$. Then there exists a unique smooth mapping $F: R \times B_s(U, F) \to B_s(U, F)$ (R denoting the reals) such that $F(0, f) = f$ (initial conditions), $F(t, \cdot)$ is a diffeomorphism, $F(t+s, \cdot) = F(t, \cdot) \circ F(s, \cdot)$ and, if $f_t = F(t, f)$, then

$$\frac{\partial f_t}{\partial t} = P(D^k f_t, \ldots, Df_t, f_t) + g \quad .$$

As we mentioned there is very little to prove since the right side represents a smooth vector field on $B_s(U, F)$. One can also allow g to depend on t as long as $\|g_t\|$ remains bounded. One point which must be checked is that the flow is complete; that is, is defined for all $t \in R$. However, if in general a vector field satisfies an estimate of the form

$$\|X(f)\| \leq K\|f\| + L,$$

then the flow is complete, and this criterion applies here.

We conclude with some remarks. First, F may be infinite dimensional, so the equations may have an infinite number of components. Second, $B_s(U, R)$ is not closed under products, so even the simplest non-linear equations are not covered by this method. Finally, the change in the space can alter the qualitative behaviour of solutions. For example, the heat equation

$$\frac{\partial f}{\partial t} = \Delta f + g$$

where Δ is the Laplacian (this equation fits our framework of course), in $B_s(U, F)$ has solutions defined for all t, but in a larger space such as the space of Radon measures or distributions, solutions are defined only for $t \geq 0$ which is a well known fact.

REFERENCES

1. S. Lang, Introduction to differentiable manifolds. (Wiley, New York, 1966).

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3. J. Moser, A new technique for the construction of solutions of non linear differential equations. Proc. Nat. Acad. Sci. (U.S.A.) 47 (1961) 1824.

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