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Hamiltonian One Parameter Groups

A Mathematical Exposition of Infinite Dimensional Hamiltonian Systems with Applications in Classical and Quantum Mechanics

J. E. MARSDEN

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Introduction

Our purpose is to give an exposition of the foundations of non-linear conservative mechanical systems with an infinite number of degrees of freedom. Systems we have in mind are the vibrating string, the electromagnetic field and quantum mechanics. These are all linear. We also outline a non-linear example, the coupled Maxwell and Dirac fields. Perfect fluids will be discussed elsewhere.

The general Hamiltonian formalism is motivated by the finite dimensional case; see ABRAHAM [1], although the usual difficulties with unbounded operators prevent an exact analogy.

Although we offer a new existence theorem for non-linear flows (an easy consequence of a theorem of MOSER), our main emphasis is on their Hamiltonian properties and the relationship with their infinitesimal generators (Hamiltonian vectorfield). In fact, the general existence question is complicated and is far from a satisfactory solution. References are YOSIDA [1], BROWDER [1, 2], HÖRMANDER [1], LIONS [1], MOSER [1, 2], SEGAL [1], DERGUZOV & JAKUBOVIC [1].

We feel that our main accomplishment here is a unification of several branches of physics into one formalism. For example, one conservation theorem has as special cases, conservation of angular momentum in classical mechanics, electromagnetic theory, quantum mechanics (including spin in the Dirac case). Again, there is one theory of canonical transformations to cover all these cases.

A one parameter group, or flow, is called Hamiltonian iff it preserves a symplectic structure. In quantum mechanics, for example, this amounts to being unitary. Our point of view is to regard quantum mechanics and conservative classical continuum (or finite dimensional) mechanics as special cases of the more general theory of Hamiltonian systems. Quantum field theory does not, as yet, fall into this category. In fact, SEGAL's work seems to be along slightly different lines; see the Bibliography. Also, this scheme does not include non-conservative systems; see TRUESDELL & NOLL [1].

Since we have striven for clarity of exposition rather than a concise report of new results, many of the theorems are well known, and no claim to originality is made. Several interesting and important problems remain open, such as DARBOUX's theorem: is every symplectic form locally constant in some chart? Sufficient but probably not necessary conditions are given in COOK [1].

The basic philosophy of Hamiltonian mechanics is that a symplectic structure (Lagrange or Poisson brackets, which are commutators in quantum mechanics), and a Hamiltonian function (expectation of the Hamiltonian operator in quantum mechanics), specify a physical system and its time evolution.

The first two chapters cover the basic theory of Hamiltonian systems in the infinite dimensional case. The last chapter studies important illustrations of the theory. Appendix B to § 8 outlines the method of dealing with non-smooth problems; a distributional potential in quantum mechanics for example. Appendix A shows how an infinite dimensional system may be thought of as a limit of finite dimensional ones. In the applications, § 7 is done from the Lagrangian point of view (classical continuum systems), while quantum mechanics (§ 8), not being a Lagrangian system, is done from the Hamiltonian point of view.

We shall assume the reader is familiar with calculus in vector spaces (LANG [1], DIEUDONNÉ [1], or better, FRÖLICHER & BUCHER [1]) and with calculus on manifolds including the infinite dimensional case (ABRAHAM [1], LANG [1]). A knowledge of semi-groups (YOSIDA [1]) and classical mechanics (ABRAHAM [1]) is helpful, but not essential.

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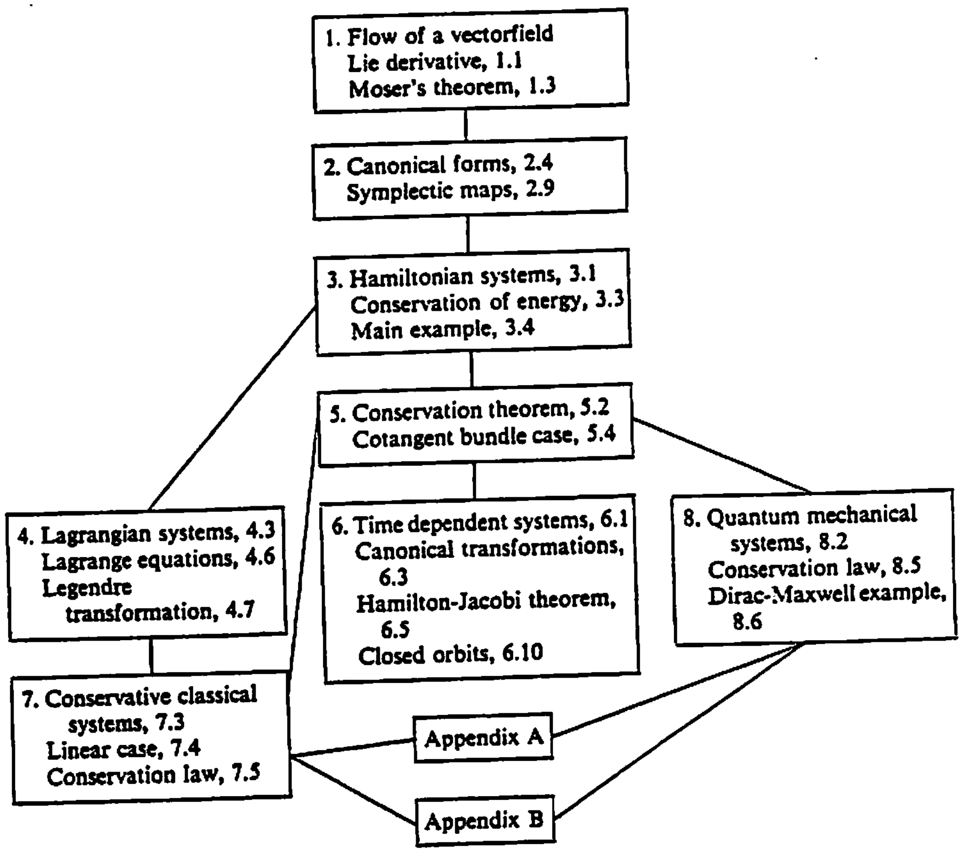

Interdependence of sections and the main results

Glossary of Symbols

Our notation is almost exclusively that of ABRAHAM [1]; the following list may be helpful. The numbers in brackets refer to following sections, while A refers to ABRAHAM [1]. See also LANG [1]. Many of these are reviewed in § 1.

R	the reals,
$E, F, \ldots$	topological vector spaces,
$L(E, F)$	continuous linear maps from E to F,
$Df\colon U \to L(E, F)$	derivative of $f\colon U \subset E \to F$ (A 2.3),
$m \mapsto f(m)$	effect of mapping $f\colon M \to N$,
E^*	$L(E, R)$,
M	manifold (A 3.1),
$\mathscr{F}(M)$	smooth maps $f\colon M \to R$,
TM	tangent bundle of M (A 5.3),
T^*M	cotangent bundle of M (A 6.14),
$\mathfrak{X}(M)$	vectorfields; smooth sections of TM (A 6.15),
$\mathfrak{X}^*(M)$	one forms; smooth sections of T^*M (A 6.15),
$\Omega^k(M)$	exterior k forms (A 10.3),
$\alpha \wedge \beta$	exterior product (A 10.3),

d	exterior derivative (§ 1), (A 10.5),
i_X	inner product (§ 1), (A 10.12),
L_X	Lie derivative (§ 1), (A 8.18),
D_X	domain of vectorfield X (§ 1),
F_t	flow (§ 1); F_{t+s} $F_t \circ F_s$ and $F_t(m) = F(t, m)$,
$F_*: \Omega^k(N) \to \Omega^k(M)$	pull back (A 6.16, 10.7),
$\{f, g\}$	Poisson bracket (2.6), (A 14.23),
$[X, Y]$	Lie bracket (§ 1), (A 8.12),
X_H	Hamiltonian vectorfield of H (2.3), (A 14.23),
ω	symplectic form (2.1), (A 14.8),
ω_L	Lagrangian symplectic form (4.3), (A 17.8),
A	finite dimensional orientable manifold (A 11.4),
$\pi: V \to A$	vector bundle over A (A 4.2),
Ω	volume on A (A 11.4),
μ_Ω	measure of Ω (A 12.9),
$L_2(A)$	square integrable functions on A,
$\langle\, ,\rangle$; $(\, ,)$	inner product,
$\langle\, ,\rangle_0$; $(\, ,)_0$	non degenerate bilinear form
$\operatorname{div}_\Omega K$	divergence (8.2), (A 11.22).

Chapter One: Hamiltonian Systems

§ 1. Preliminaries

In this section we recall some of the basic facts about differential calculus in topological vector spaces, calculus on manifolds and one parameter groups we shall need later.

Let E, F be (locally convex) topological vector spaces and $L(E, F)$ denote the space of continuous linear maps from E to F. For the correct topology on $L(E,F)$ see Frölicher & Bucher [1, p. 65], although we shall not need it explicitly.

If $U \subset E$ is open and $f: U \to F$, then recall that if f is of class C^r, $D^r f: U \to L^r(E, F)$, the r-multilinear maps from E to F (symmetric, in fact). Again see Frölicher & Bucher [1, p. 95]. If $f \in L(E, F)$, then $Df(u) = f$.

One of the basic facts is the **composite mapping theorem**: if f: $U \subset E \to F$ and g: $V \subset F \to G$ are of class C^r and $f(U) \subset V$, then $g \circ f$ is of class C^r and $D(g \circ f)(u) \cdot e = Dg(f(u)) \cdot (Df(u) \cdot e)$.

For f: $U_1 \times U_2 \subset E_1 \times E_2 \to F$ we define the **first partial derivative** by: $D_1 f$: $U_1 \times U_2 \to L(E_1, F)$, where

$$D_1 f(u_1, u_2) = D(f \mid U_1 \times \{u_2\})(u_1) = Df(u_1, u_2) \mid E_1 \times \{0\}.$$

Then f is of class C^r iff $D_1{}^r f$, $D_2{}^r f$ exist and are continuous on $U_1 \times U_2$. We also have $Df = D_1 f + D_2 f$ with the natural identifications. (Frölicher & Bucher [1, p. 91].)

Similarly, if f_1: $U \subset E \to F$, f_2: $U \subset E \to F_2$, then f_1 and f_2 are of class C^r iff $f_1 \times f_2$ is, and in this case, $D(f_1 \times f_2) = Df_1 \times Df_2$.

Leibnitz' rule (product rule) also holds. If f_1: $U \subset E \to F_1$ and f_2: $U \subset E \to F_2$ are of class C^r and B: $F_1 \times F_2 \to G$ is continuous bilinear, then $B \circ (f_1 \times f_2)$ is of class C^r and

$$D\{B \circ (f_1 \times f_2)\}(u) \cdot e = B(Df_1(u) \cdot e, f_2(u)) + B(f_1(u), Df_2(u) \cdot e).$$

Although there is no mean value theorem in general, there are analogues of it. (FRÖLICHER & BUCHER [1, p. 60].) In particular, if $Df=0$ then f is locally constant (globally if U is connected).

If $f: R \to E$ (R is the reals) then Df exists iff

$$df/dt(u) = \lim_{h \to 0} [f(u+h) - f(u)]/h$$

exists, and in this case $Df(u) \cdot r = r\, df(u)/dt$.

In the special case of Banach spaces, much more can be said. For example, we have the **implicit mapping theorem** and the **existence and uniqueness theorem for flows of vectorfields.** (See LANG [1] or DIEUDONNÉ [1].)

Let $U \subset R^n$ be an open set in Euclidean n-space, $R^n = R \times \cdots \times R$ (or a manifold, or manifold with boundary) and F a normed space. Let $C^\infty(U, F)$ denote all C^∞ maps $f: U \to F$ with the topology of uniform convergence (of all derivatives) on compact sets (*cf.* YOSIDA [1, Ch. I]). For $g \in C^\infty(F, G)$ the map $\omega: C^\infty(U, F) \to C^\infty(U, G); f \mapsto g \circ f$ is of class C^∞ and

$$D\omega(f) \cdot f_1(u) = D\, g(f(u)) \cdot f_1(u).$$

This result seems to have been first recognized in ABRAHAM [2]. The general result may be found in FRÖLICHER & BUCHER [1, p. 130].

Next we recall a few facts about manifolds. For the basic definitions, see LANG [1] and ABRAHAM [1]. Our manifolds will be modelled on Banach spaces, or more generally topological vector spaces. For a manifold M, TM denotes the **tangent bundle,** and T^*M denotes the **cotangent bundle.** For $F: M \to N$, the **tangent** of F is denoted $TF: TM \to TN$. The composite mapping theorem becomes $T(F \circ G) = TF \circ TG$.

A (smooth) **vectorfield** is a C^∞ section of the tangent bundle; that is, a map $X: M \to TM$ such that $\tau \circ X$ is the identity, where $\tau: TM \to M$ is the projection. The vectorfields are denoted $\mathfrak{X}(M)$ and the **covectorfields (one forms)** $\mathfrak{X}^*(M)$.

We shall also consider as vectorfields, maps $X: D_X \subset M \to TM$ where $D_X \subset M$ is the **domain** of X, such that

(i) D_X is a manifold modelled on a topological vector space;

(ii) the inclusion $i: D_X \to M$ is smooth;

(iii) D_X is dense in M;

(iv) $X: D_X \to TM$ is smooth.

We can similarly define **k-forms with a domain.**

In practice we often have $X: D_X \to TD_X$ so X is a vectorfield on D_X; see, for example, the appendix to this section.

A **(local) flow** for X is a map, C^∞,

$$F: (-\varepsilon, \varepsilon) \times U \to D_X$$

where U is open in D_X, and $\varepsilon > 0$, such that

$$X(F_s(m)) = \frac{d}{dt} F_t(m) \qquad (t = s),$$

for $m \in D_X$; *the derivative is taken using the topology of M.*

If M is a linear space and X is linear, then F may be extended as a flow in M, but dF_t/dt existing only on D_X. However, in the non-linear case the flow may be defined only on D_X. For the linear case see YOSIDA [1, p. 246]. A non-linear existence theorem is given in the appendix.

To avoid the local character, we shall generally assume the flow to be complete; $F\colon R\times D_X\to D_X$, only for simplicity.

Let $\mathscr{F}(M)$ denote the **smooth real valued functions** $f\colon M\to R$. If $f\colon D_f\to R$; $f\in\mathscr{F}(D_f)$ for a domain D_f, we consider f as densely defined on M. Its derivative makes sense, however, only in D_f.

We let $\Omega^k(M)$ denote the **exterior algebra** of (smooth) k-forms on M and let

$$d\colon \Omega^k(M)\to\Omega^{k+1}(M)$$

denote the **exterior derivative**. Recall that it is given by ($\hat{X}$ denotes that X is omitted)

$$\begin{aligned}(k+1)\,d\omega(X_0,\dots,X_k)=&\sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\hat{X}_i,\dots,X_k))\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\end{aligned}$$

where $[X_i, X_j]$ is the **Lie bracket** of $X_i, X_j\in\mathfrak{X}(M)$ and $X(f)=df\cdot X$ for $f\in\mathscr{F}(M)$. In general, $[X, Y]$ is not defined if X and Y are merely densely defined, but if X has domain D_X and $f\in\mathscr{F}(M)$, $L_X f=X(f)=df\cdot X$ has domain D_X as well.

In addition, locally we have

$$(k+1)\,d\omega(u)\cdot(v_0,\dots,v_k)=\sum_{i=0}^{k}(-1)^i[D\omega(u)\cdot v_i](v_0,\dots,\hat{v}_i,\dots,v_k).$$

The exterior derivative enjoys the usual properties and commutes with **pull backs**

$$F_*\colon \Omega^k(N)\to\Omega^k(M)\quad\text{for}\quad F\colon M\to N\ \text{smooth}.$$

(See 1.1 for the definition.)

Let X be a vectorfield with domain $D_X\subset M$ and $\alpha\in\Omega^k(M)$. Define the **inner product** by

$$i_X\colon \Omega^k(M)\to\Omega^{k-1}(D_X),$$
$$i_X\alpha(m)(v_1,\dots,v_{k-1})=k\,\alpha(m)(X(m),v_1,\dots,v_{k-1})$$

for $m\in D_X$ and $v_i\in T_m D_X$.

Also, define the **Lie derivative** by

$$L_X\colon \Omega^k(M)\to\Omega^k(D);\qquad L_X=d\circ i_X+i_X\circ d.$$

From the properties of d and i_X we deduce at once those for L_X. For example,

$$L_X d=d L_X,\quad L_X(\alpha\wedge\beta)=(L_X\alpha)\wedge\beta+\alpha\wedge(L_X\beta),\quad\text{etc.}$$

In fact, a simple computation from the formulas for d and i_X shows that locally,

$$L_X\alpha(u)\cdot(v_1,\dots,v_k)=D\alpha(u)\cdot X(u)\cdot(v_1,\dots,v_k)+\sum_{i=1}^{k}\alpha(u)\cdot(v_1,\dots,DX(u)\cdot v_i,\dots,v_k)$$

where $v_1,\dots,v_k\in T_u D_X\subset T_u M$; and $k\geqq 0$.

1.1. Theorem. *Suppose X is a vectorfield on M with domain D_X and has flow F_t: $D_X \to D_X$. Then for each $\alpha \in \Omega^k(M)$, we have*

$$F_{\tau *}(L_X \alpha) = \frac{d}{dt}(F_{t*}\alpha), \quad at \quad t=\tau,$$

on D_X (the derivative uses the M topology). In particular, $L_X\alpha = 0$ implies $F_{t*}\alpha = \alpha$ for all $t \in R$ on D_X.

Proof. It is sufficient to prove the result at $\tau = 0$. Let $v_1, \ldots, v_k \in T_u D_X$, so that

$$F_{t*}\alpha(u) \cdot (v_1, \ldots, v_k) = \alpha(F_t(u)) \cdot (TF_t \cdot v_1, \ldots, TF_t \cdot v_k).$$

The result follows immediately by differentiating, using the chain rule, the Leibnitz rule and the local formula for $L_X\alpha$. □

If F: $M \to N$ is a diffeomorphism, and X has domain D_X, we define F^*X with domain $F(D_X)$ by

$$F^* X = TF \circ X \circ F^{-1},$$

and this satisfies $(F_* = (F^*)^{-1} = (F^{-1})^*)$,

$$F^*(i_X \alpha) = i_{F^*X} F^* \alpha$$

and

$$F^*(L_X \alpha) = L_{F^*X} F^* \alpha.$$

Further, if X has flow F_t: $D_X \to D_X$, then F^*X has flow $F \circ F_t \circ F^{-1}$ as is seen at once by differentiation.

We now introduce an important generalization of this.

1.2. Definition. *Suppose F: $D \to D$ is a diffeomorphism on a domain $D \subset M$. We say F is* **admissible** *iff for each $m \in D$, $TF(m)$ extends to a continuous linear map of $T_m M$ into $T_{F(m)} M$. Sometimes by admissible we shall just mean that $TF(m)$ can be extended so that, for example, $TF \cdot X$ makes sense.*

Clearly F^{-1} is also admissible.

If X and F have the same domain and F is admissible, then comments similar to those above apply. In particular, if X has an admissible flow F_t, then $F_t^* X = X$. As we shall see later, admissibility often holds in the Hamiltonian case.

Other facts we shall use freely are the *Poincaré lemma* ($d\alpha = 0$ implies $\alpha = d\beta$ locally), partitions of unity, orientability and integration on finite dimensional manifolds. See ABRAHAM [1, § 11, 12].

Appendix: MOSER's *Theorem on Non-linear Flows*

MOSER's approach [1, 2] seems to be the most promising at present for existence of solutions of non-linear partial differential equations. Here we outline his theorem for the case of flows.

Let A be a finite dimensional (compact) manifold and $C^k(A)$ the space of maps from A to a normed space which are of class C^k with the C^k-norm (ABRAHAM [1, p. 168]). Fix $a > 0$ and ρ an integer. Let $B^k(A)$ denote the maps

$$U: (-a, a) \times C^k(A) \to C^k(A)$$

which are of class C^1 in the second variable and differentiable in the first in some containing space; *e.g.*, $L^2(A)$, with

$$u \in B^{k+\rho}(A) \quad \text{implies} \quad \frac{\partial u}{\partial t} \in B^k(A).$$

1.3. Theorem (MOSER). *Using the above notation, suppose*

$$X: C^{k+\rho}(A) \to C^k(A)$$

is of class C^2 *and for each* $u \in B^{k+\rho}(A)$, $h \in B^k(A)$, *the (time dependent) vectorfield*

$$g \mapsto DX(u) \cdot g + h; \qquad g \in C^{k+\rho}$$

has a C^1 *flow on* $C^{k+\rho}$ *(called the linearized flow).*

Then for each $g_0 \in C^{k+\rho}$ *there is a neighborhood* V *of* g_0 *and* $\varepsilon > 0$ *such that* X *has a flow*

$$F: (-\varepsilon, \varepsilon) \times V \to C^{k+\rho}.$$

If the linearized flows are unique, so is F.

Note that the hypotheses rely heavily on the linear theory. We have assumed A compact so we have the containing space $L_2(A)$ and also smoothing operators (MOSER [1]); however, this is not essential.

The *idea* of the proof is as follows. Consider the map

$$f: B^{k+\rho} \to B^k; \quad u \mapsto \frac{\partial u}{\partial t} - X \circ u.$$

By assumption, we can solve $Df(u) \cdot v = h$ for v. Conditions (11), (12), (14) of MOSER [1] hold locally by continuity and differentiability. For $g_0 \in C^{k+\rho}$ find v so $\partial v/\partial t = DX(g) \cdot v_t(g_0) + X(g_0)$. Choose V, ε so

$$X \circ v_t - [DX(g) \cdot v_t(g) + X(g)]$$

is small for $t \in (-\varepsilon, \varepsilon)$, $g \in V$. With this, the hypothesis of MOSER's theorem are satisfied. Further investigations along these lines with specific applications remain to be done; see also MOSER [2], and Section 8.

§ 2. Symplectic Geometry

The main structure on a manifold which allows us to do Hamiltonian mechanics is a symplectic structure. Much of this is *formally* the same as the finite dimensional case.

2.1. Definition. *A* **symplectic manifold** (M, ω) *consists of a manifold* M *(modelled on a Banach space) and a closed non-degenerate two-form* ω (**symplectic form**); *that is,* $d\omega = 0$, *and for each* $m \in M$, $\omega_\flat: T_m M \to T_m^* M$; $\omega_\flat(v) \cdot w = \omega(m) \cdot (v, w)$ *is an isomorphism (onto). We let* $\omega_\sharp = \omega_\flat^{-1}$.

Note that it is not enough to demand $\omega(m) \cdot (v, w) = 0$ for all w implies $v = 0$.

2.2. Proposition. *In* 2.1, $\omega_\flat: TM \to T^*M$ *is a vector bundle isomorphism.*

The proposition is clear using local charts. See also LANG [1, p. 8].

As an example, let E denote the Banach space of continuous real functions on $[-1, 1]$ and $F = E \times E$. Define $\omega \in \Omega^2(F)$ by

$$\omega(e, f) \cdot ((e_1, f_1), (e_2, f_2)) = \int_{-1}^{1} (f_1 e_2 - f_2 e_1)\, d\mu,$$

where ef denotes the pointwise product. Here ω is closed (it is constant), and $\omega(e,f) \cdot ((e_1, f_1), (e_2, f_2)) = 0$ for all (e_2, f_2) implies $(e_1, f_1) = 0$ ($\omega_\flat$ is a monomorphism). But ω is not symplectic since $\omega_\flat$ is not onto (use a δ-function). The reason for this will be evident below, where we also give positive examples (2.4).

As usual, we write $X_f = (df)^\sharp = \frac{1}{2}\, \omega_\sharp(df)$, for $f \in \mathscr{F}(M)$. More generally, we make the following:

2.3. Definition. *Let* (M, ω) *be a symplectic manifold and* X *a vectorfield on* M *with domain* D_X. *Define* $X^\flat$ *a one-form with domain* D_X *by*

$$X^\flat(m) = 2\,\omega_\flat(m) \cdot X(m).$$

Similarly define

$$\alpha^\sharp(m) = \tfrac{1}{2}\, \omega_\sharp(m) \cdot \alpha(m); \qquad m \in D_\alpha.$$

Let $f \in \mathscr{F}(D_f)$. *We say* f *is a* **Hamiltonian function** *(or is* **admissible***) iff* $df(m)\colon T_m D_f \to R$ *has a (unique) extension to*

$$df \in T_m^* M.$$

Then define $X_f = (df)^\sharp$, *a vectorfield with domain* $D_{X_f} = D_f$, *called the* **Hamiltonian vectorfield** *of* f.

Suppose $D_X \supset D_g$. *Define*

$$L_X g = X(g) = dg \cdot X \in \mathscr{F}(D_g)$$

the **Lie derivative.** *We may similarly define* $L_X \alpha$ *for* $D_X \supset D_\alpha$.

Since $g \in \mathscr{F}(D_g)$ need not be smooth on M, we do *not* have

$$L_X g = \frac{d}{dt} F_{t*} g \quad \text{at} \quad t = 0$$

if X has flow F_t. In fact, conservation of energy must be proven by different techniques. See § 3.

We shall see in examples later that functions $H\colon D \to R$ of interest are in fact Hamiltonian.

Next we consider the canonical forms:

2.4. Theorem. *Let* M *be a manifold modelled on a Banach space* E *and* $T^* M$ *its cotangent bundle. Define a one-form* θ *on* $T^* M$ *by*

$$\theta(\alpha_m) \cdot w_{\alpha_m} = -\alpha_m \cdot T\tau^*(w_{\alpha_m})$$

where $\tau^*\colon T^* M \to M$ *is the projection,* $\alpha_m \in T_m^* M$ *and* $w_{\alpha_m} \in T_{\alpha_m}(T^* M)$. *Then* $\omega = d\theta$ *is a symplectic form on* $T^* M$ *iff* E *is reflexive. (That is the map* $j\colon E \to E^{**}$; $j(e) \cdot \alpha = \alpha(e)$ *is an isomorphism.)* semi-

Further, locally, using principal parts, we have

$$\omega(u,\alpha)((e_1,\alpha_1),(e_2,\alpha_2))=\tfrac{1}{2}[\alpha_2(e_1)-\alpha_1(e_2)].$$

Proof. Locally we have $\theta(u,\alpha)\cdot(e,\beta)=-\alpha\cdot e$, so that by the local formula for $d\theta$ (§ 1) we have the formula for ω. From this formula it follows that $\omega_\flat$ is a monomorphism. In fact, if $\alpha_2\cdot e_1-\alpha_1\cdot e_2=0$ for all $e_2\in E$, $\alpha_2\in E^*$ then $\alpha_1=0$ (setting $\alpha_2=0$) and by the Hahn-Banach theorem (YOSIDA [1, p. 107]), $e_1=0$.

Suppose that ω is symplectic. Then for $e\in E^{**}$ there is e_1,α_1 so

$$\omega_\flat(e_1,\alpha_1)\cdot(e_2,\alpha_2)=e(\alpha_2).$$

Hence $j(e_1/2)=e$ or E is reflexive. Conversely if E is reflexive and $\sigma=(\alpha_1,e)\in E^*\times E^{**}$ let $j(f)=e$; then $2\omega_\flat(f,-\alpha)=(\alpha,e)$ since $2\omega_\flat(f,-\alpha)\cdot(e_1,\alpha_1)=\alpha_1 f+\alpha\cdot e_1=e\cdot\alpha_1+\alpha\cdot e_1$. □

From this proof we deduce the following:

2.5. Corollary. *Suppose X_H is a Hamiltonian vectorfield on T^*M [with E reflexive]. Then identifying E and E^{**}, we have locally on $U\times E^*$*

$$X_H(u,\alpha)=(D_2H(u,\alpha),-D_1H(u,\alpha))$$

for (u,α) in the domain of H.

In particular, $c\colon I\to U\times E^*$ is an integral curve iff c maps into the domain of H and

$$dc_1(t)/dt=D_2H(c_1(t),c_2(t)),$$

$$dc_2(t)/dt=-D_1H(c_1(t),c_2(t))$$

using the T^*M-topology. (Hamilton's equations.)

As we shall see later these equations give, in the special cases, equations for the electromagnetic field and SCHRÖDINGER's equation. Of course if E is finite dimensional, they are the ordinary equations of Hamiltonian mechanics.

Poisson brackets of one-forms $\alpha\in\mathcal{X}^*(m)$ (with (M,ω) a symplectic manifold) are defined by

$$\{\alpha,\beta\}=-[\alpha^\sharp,\beta^\sharp]^\flat.$$

Unfortunately, if α or β is defined only on a domain D, this won't make sense. Therefore we restrict ourselves to Poisson brackets of functions.

2.6. Definition. *Let (M,ω) be a symplectic manifold and f,g Hamiltonians defined on a domain $D\subset M$. Define the* **Poisson bracket** *$\{f,g\}\in\mathcal{F}(D)$ by*

$$\{f,g\}(m)=-i_{X_f}i_{X_g}\omega(m)=2\omega(m)\cdot(X_f(m),X_g(m)).$$

Thus if the manifold is T^*M, we have, locally, by 2.5, 2.4,

$$\{f,g\}=D_2g\cdot D_1f-D_2f\cdot D_1g.$$

2.7. Proposition. *In the above,*

$$\{f,g\}=L_{X_g}f=-L_{X_f}g.$$

This follows immediately since

$$df(dg^\sharp)=2\omega(df^\sharp, dg^\sharp)$$

as $df=(df^\sharp)^\flat$.

In case ω is locally given by 2.4 (the usual circumstance), it is easy to check that the Hamiltonian functions *formally* form a Lie algebra under the Poisson bracket, although $\{f, g\}$ need not be a Hamiltonian function.

If $X_H=X_K$ on a domain D, then H and K differ by constants if D is connected. This follows as $d(H-K)=0$ on D.

Finally in this section we study some properties of **symplectic mappings** (*i.e.*, **homogeneous canonical transformations**).

2.8. Definition. *Let (M,ω) be a symplectic manifold and $F\colon D\to D$ a smooth admissible map. We say F is* **symplectic** *iff $F_*\omega=\omega$, on D. That is, $F_*(i_*\omega)=i_*\omega$ with i the inclusion map of D into M.*

We make a similar definition for F mapping between different symplectic manifolds. The main theorem on symplectic maps is as follows:

2.9. Theorem. *Let (M,ω) be a symplectic manifold, $D\subset M$ a domain and $F\colon D\to D$ an admissible diffeomorphism. Then the following are equivalent:*

(i) *F is symplectic;*

(ii) *for all vectorfields, $(F^*X)^\flat=F^*(X^\flat)$ if $D_X\supset D$ (or on any open subset of M);*

(iii) *for all one-forms α with $D_\alpha\supset D$, $(F^*\alpha)^\sharp=F^*(\alpha^\sharp)$ (or on any open subset of M);*

(iv) *for any Hamiltonian H, $D_H\supset D$, $F^*X_H=X_{H\circ F^{-1}}$ (or on any open subset of M);*

(v) *for f, g Hamiltonians with $D_f=D_g\supset D$,*

$$F^*\{f,g\}=\{F^*f, F^*g\}$$

(or on any open subset of M).

(Note that F admissible and f Hamiltonian implies $f\circ F$ is Hamiltonian.)

Classically this theorem is proven in the linear case, where there are no domain problems for F.

Proof. First, (i) implies (ii) since at $m\in D$,

$$\begin{aligned}(F^*X)^\flat\cdot v&=2\omega(F^*X, v)=2(F^*\omega)(F^*X, V)\\&=2\omega(X, TF\cdot v)\circ F\end{aligned}$$

which is just $F^*(X^\flat)\cdot v$. Similarly we see that in fact (i), (ii), (iii) are equivalent.

Second (iii) implies (iv) since

$$F^*(X_H)=F^*(dH)^\sharp=(F^*dH)^\sharp=(dF^*H)^\sharp.$$

Third, (iv) implies (v) since

$$F^*\{f,g\}=F^*(L_{X_g}f)=L_{F^*X_g}F^*f=\{F^*g, F^*f\}.$$

This computation for arbitrary $f\in\mathcal{F}(M)$ also shows (v) implies (iv).

Finally, (iv) implies (iii) since pointwise we can write $\alpha(u)=df(u)$ for some $f\in\mathcal{F}(U)$ on a local chart. □

§ 3. Hamiltonian Systems

A one parameter group F_t is called **Hamiltonian** iff F_t is symplectic for each $t\in R$. In this section we study the basic properties of such systems. *From here on, flow means admissible flow.* See 1.2*ff*.

3.1. Theorem. *Let (M,ω) be a symplectic manifold and X a vectorfield on M with domain D. Suppose X has flow F_t: $D\to D$. Then the following are equivalent.*

(i) $L_X\omega=0$;

(ii) $i_X\omega$ *is closed*;

(iii) *locally on D we may write $X=X_H$ for some Hamiltonian function H,*

(iv) *F_t is symplectic for each $t\in R$*;

(v) *locally there is a Hamiltonian H so that*

$$L_X f=\{f,H\}.$$

If D is a linear space, H in (iii) *and* (v) *may be chosen globally (on all of D).*

Proof. Since ω is closed, $L_X\omega=d\,i_X\omega$ so (i) and (ii) are obviously equivalent. Also, $i_X\omega=X^\flat$ so (ii) implies, by the Poincaré lemma, that locally on D, $X^\flat=dH$. Since $X^\flat$ is a one-form on the whole of T_mM, $m\in D$, it is clear that H is Hamiltonian (see 2.3). Similarly (iii) implies (ii). Clearly, by 2.7, (iii) and (v) are equivalent. Finally, (i) and (iv) are equivalent by 1.1. □

As usual, a vectorfield satisfying 3.1 is called **locally Hamiltonian**, and is **globally Hamiltonian** iff H can be chosen globally on D; *i.e.*, $i_X\omega\in\Omega^1(D)$ is exact.

The next proposition considers the linearized equations at a point. This is useful in verifying the hypotheses of Moser's theorem (1.3) for example.

3.2. Proposition. *Let (M,ω) be a linear symplectic manifold with ω: $M\times M\to R$; (ω is constant). Suppose X is a (non-linear) Hamiltonian vectorfield with linear domain D_X and flow F_t. Then*

(i) *for each $m\in D_X$,*

$$DF_t(m):\ D\to D$$

is a linear symplectic map;

(ii) *for each $m\in D_X$ we have: the linear map*

$$DX(m):\ D\to M$$

satisfies

$$\omega(DX(m)\cdot v,u)=-\omega(v,DX(m)\cdot u)$$

for $u,v\in D_X$ (a skew adjointness condition).

In fact, both (i) *and* (ii) *are equivalent to the condition that X be Hamiltonian.*

Proof. (i) is just $F_{t*}\omega=\omega$ written out while (ii) is just $d(i_X\omega)=0$. See § 1. □

Next we prove **conservation of energy**.

3.3. Theorem. *Let X_H be a Hamiltonian vectorfield with connected domain D and flow F_t. Further, suppose F_t has an invariant point. (For example the origin if the system is linear.) Then*

$$H \circ F_t = H$$

for all $t \in R$ on D.

Proof. As we remarked before, $\{H, H\} = 0$ is not sufficient, as H is not, in general, smooth on M. However, by 3.1, F_t is symplectic, and so by 2.9,

$$F_{t*} X_H = X_H = X_{H \cdot F_t},$$

the first equality following from the fact that any vectorfield is invariant under its own flow (§ 1).

Therefore $H - H \circ F_t$ is constant on D. Going to the invariant point, we see that this constant is zero. □

Thus it is preservation of the symplectic form, rather than a direct argument, that gives us conservation of energy. This same technique will be applied also in § 5. It would be interesting to see if the hypothesis on the fixed point is really essential in non-linear examples. Our examples later all satisfy the hypotheses; only one example is non-linear.

In § 5 we will develop some conservation laws special to function spaces. Therefore it is convenient to discuss the main example of a symplectic manifold at this point. The definitions are of wide enough scope to allow for relativistic and multi-component systems such as the electromagnetic field and the Dirac equation.

3.4. Definitions. *Let $\pi: V \to A$ be a finite dimensional vector bundle over A with inner product $\langle\,,\,\rangle$ on each fiber (it depends on $a \in A$) and $\langle\,,\,\rangle_0$ a non-degenerate symmetric bilinear form on the fibers of V.*

Let Ω be a volume on A with corresponding measure μ_Ω and $L_2(A, V)$ the Hilbert space of measurable functions $f: A \to V$ which are sections of $\pi: V \to A$ and

$$\int_A \langle f(x), f(x)\rangle \, d\mu_\Omega(x) < \infty$$

with inner product

$$(f, g) = \int_A \langle f(x), g(x)\rangle \, d\mu_\Omega(x).$$

Also, define the bilinear form

$$(f, g)_0 = \int_A \langle f(x), g(x)\rangle_0 \, d\mu_\Omega(x).$$

The complexification of $L_2(A, V)$ is denoted $CL_2(A, V)$ and is identified with $L_2(A, V) \times L_2(A, V)$ using standard notations. For example, if $f, g \in CL_2(A, V)$ $f = f_1 + i f_2$, we let $\bar{f} = f_1 - i f_2$ and

$$(f, g)_0 = \int_A \langle \overline{f(x)}, g(x)\rangle_0 \, d\mu_\Omega(x) \quad \textit{etc.}$$

3.5. Theorem. *In 3.2, $CL_2(A, V)$ is a symplectic manifold with the symplectic form*

$$\omega(f, g) = \tfrac{1}{2} \operatorname{Im}(f, g)_0 = \tfrac{1}{2}\{(g_2, f_1)_0 - (f_2, g_1)_0\}$$

where Im *denotes imaginary part and* $f=f_1+if_2$. *In fact, identifying* $L_2(A, V)$ *with* $L_2(A, V)^*$ *by means of* $\langle\, ,\rangle_0$, *this is the natural symplectic structure* (2.4).

Proof. With the identification indicated, the natural symplectic structure (which is symplectic in view of 2.4, and the fact that every Hilbert space is reflexive) is

$$\omega((f_1,f_2),(g_1,g_2))=\tfrac{1}{2}\{(g_2,f_1)_0-(f_2,g_1)_0\}$$

as stated. □

Observe that the condition 3.2(i) becomes, in this case: for all $f, g\in L_2(A, V)\cap D_X$,

$$(f,g)_0=(G_2 f, G_1 g)_0$$

where $DF_t(m)=G_1\times G_2$; and in particular, if $DF_t(m)$ is *complex* linear ($G_1=G_2$) then it is unitary, so extends as a map of $L_2(A, V)$ onto $L_2(A, V)$. The real linear case corresponds to classical continuum systems (§ 7) and the complex linear case corresponds to quantum mechanics (§ 8).

The condition 3.2 (ii) in the complex linear case (*i.e.*, $DX(m)$ is complex linear) is equivalent to $iDX(m)$ being symmetric:

$$(iDX(m)\cdot f,g)_0=(f,iDX(m)\cdot g)_0$$

for all $f,g\in D_X$. If $iDX(m)$ has a self-adjoint extension, then by STONE's theorem the linearized equation has a flow. This is what 1.3 demands.

Finally, we remark that the symplectic structure in 3.5 is naturally associated with the complex structure of the manifold. (This is a general phenomenon for manifolds with a Kähler structure.)

§ 4. Lagrangian Systems

The basic idea of a Lagrangian system is the same as in the finite dimensional case. See ABRAHAM [1, § 17]. In the general case here, Lagrangian systems are suitable for describing conservative classical continuous systems (see § 7 for details). Quantum mechanics is a Hamiltonian, but **not** a Lagrangian system. Lagrangian systems are a special case of Hamiltonian ones.

We also briefly recapitulate the elements of the Legendre transformation theory (which fails in the quantum mechanical case).

First we recall the definition of second order equations:

4.1. Definition. *Let M be a manifold and X a vectorfield on TM with domain D. Then X is called a* **second order equation** *iff $T\tau\circ X$ is the identity on D where $\tau: TM\to M$ is the projection.*

If $c: R\to D$ is an integral curve of X (that is $F_t(m)$ for m fixed) then $\tau\circ c$ is called a **base integral curve** *of X.*

The basic properties are:

4.2. Proposition. (i) *Suppose X is a vectorfield on TM with domain D and X possesses a flow. Then X is a second order equation iff for all integral curves $c: R\to D$, we have*

$$\frac{d}{dt}(\tau\circ c)=c \quad \text{at} \quad t=\tau$$

(derivative in TM).

(ii) *X is a second order equation iff in local coordinates,*

$$X(u,e)=(e, X_2(u,e))$$

for (u, e) *in the domain, and* X_2 *the second component of* X.

(iii) *If* X *is a second order equation and* $c: R\to M$ *is a base integral curve in the domain,*

$$\frac{d^2c}{dt^2}=X_2\left(c(t), \frac{dc}{dt}(t)\right),$$

locally. (All derivatives in time t*, use the topology of* M *as usual.)*

Proof. $T\tau\circ X$ is the identity iff for all integral curves c in D, $T\tau\circ c'(t)=c(t)$. But by the composite mapping theorem on M, $T\tau\circ c'=(\tau\circ c)'$. Parts (ii) and (iii) follow at once from the definitions. □

Returning to Lagrangians proper, we make

4.3. Definition. *Let* $L\in\mathscr{F}(D)$ *where* $D\subset TM$ *is a (dense) domain, and where* M *is modelled on* $\mathbf{E}$. *We say that* L *is a* **regular Lagrangian** *iff*

(i) $v_m\in D$ *implies* $T_mM\subset D$,

(ii) *if* $L_m=L|T_mM$, *then* L_m *is smooth in the* M*-topology,*

(iii) $FL(v_m)=DL_m(v_m)$: $T_mM\to R$ *has a (unique) extension to a map* FL: $TM\to T^*M$ *which is a local diffeomorphism,*

(iv) *for* $(u, e)\in D$, *locally,* $D_1L(u, e)$ *extends to a map in* $L(E, R)$.

Also, if L *is regular, we set*

$$\omega_L=(FL)_*\omega,$$

where ω *is the symplectic form on* T^*M.

Thus, ω_L will be a symplectic form on TM.

Notice that (iv) just means that L is a "Hamiltonian function" (in view of (iii); see 2.3).

4.4. Proposition. *Suppose* L *is a regular Lagrangian on* TM. *Then locally we have:*

(i) $$FL(u,e)=(u, D_2L(u,e))$$

for $(u, e)\in D$; *(so* $D_2L(u, e)$ *is smooth),*

(ii)
$$\begin{aligned}2\omega_L(u,e)\cdot((e_1,e_2),(e_3,e_4)) &=D_1D_2L(u,e)\cdot e_3\cdot e_1+D_2D_2L(u,e)\cdot e_4\cdot e_1\\ &\quad -D_1D_2L(u,e)\cdot e_1\cdot e_3-D_2D_2L(u,e)\cdot e_2\cdot e_3\end{aligned}$$

for $(u, e)\in D$ *and all* $e_1, e_2, e_3, e_4\in E$.

This proposition follows at once from the definitions and the local formula for ω in 2.4.

4.5. Definition. *Let L be a regular Lagrangian on TM. Define the* **action** *of L by $A: TM \to R$; $A(w_m) = FL(w_m) \cdot w_m$ and the* **energy** *E of L with domain D by $E = A - L$.*

Thus, locally, $A(u, e) = D_2 L(u, e) \cdot e$ and $E(u, e) = D_2 L(u, e) \cdot e - L(u, e)$.

By LEIBNITZ' rule we find that, locally,

$$D E(u, e) \cdot (e_1, e_2) = D_1 D_2 L(u, e) \cdot e_1 \cdot e + D_2 D_2 L(u, e) \cdot e_2 \cdot e - D_1 L(u, e) \cdot e_1 .$$

Thus, in view of the regularity assumptions, E is a Hamiltonian function (2.3).

The main theorem of this section is as follows:

4.6. Theorem. *Let L be a regular Lagrangian on TM, with energy E. Then using the symplectic form ω_L, we have*

(i) *X_E is a second order equation with the same domain as L, say D,*

(ii) *$c: R \to D$ is a base integral curve of X_E in the domain iff*

$$\frac{d}{dt} D_2 L\left(c(t), \frac{dc}{dt}(t)\right) = D_1 L\left(c(t), \frac{dc}{dt}(t)\right)$$

(Lagrange's equations; using the M topology for d/dt).

Proof. By use of the explicit form for ω_L in 4.4 an easy computation shows that, locally,

$$X_E(u, e) = \left(e, D_2 D_2 L(u, e)^{-1} \{D_1 L(u, e) - D_1 D_2 L(u, e) \cdot e\}\right)$$

(just verify $X_E^\flat = DE$, given above).

Thus from 4.2, X_E is a second order equation and c is a base integral curve iff

$$\frac{d^2 c}{dt^2} = D_2 D_2 L(c, c')^{-1} \{D_1 L(c, c') - D_1 D_2 L(c, c') \cdot c'\} .$$

This gives the Lagrange equation, since we may apply the chain rule on M, $D_2 L(u, e)$ being smooth on M (smoothness on D would *not* suffice). □

Of course in the finite dimensional case these are the usual Lagrange equations. For continuum systems they give the standard density Lagrange equations. (See § 7.)

Finally we describe the Legendre transformation. We are brief and omit proofs as they are essentially the same as ABRAHAM [1, § 18] with modifications as indicated above.

4.7. Remarks. A Lagrangian L on TM is called **hyperregular** iff in addition to being regular, FL is a (global) diffeomorphism.

In this case we let $H = E \circ FL^{-1}$, so that obviously the flows of X_H on $T^* M$ and X_E on TM are related by conjugation, as FL is symplectic. The domain of H is $FL(D)$ (and carries the same topological structure).

Thus, *every hyperregular Lagrangian on TM is equivalent to a Hamiltonian system on $T^* M$.*

Conversely, since M is assumed modelled on a reflexive space (see 2.4), we may, given H on $T^* M$, construct $FH: T^* M \to T^{**} M \approx TM$ and demand it extend to a diffeomorphism, as in 4.3; that is, H is **hyperregular** (we also assume that H is a Hamiltonian function).

We then define $E = H \circ (FH)^{-1}$, $A = G \circ (FH)^{-1}$ and $L = A - E$ where $G = \theta(X_H)$ is the action of H; θ the canonical one-form. All functions have the obvious domains.

Then we find that L is hyperregular and in fact, $FL = (FH)^{-1}$. Moreover L has action A and energy E. We call FL the Legendre transformation.

Thus every *hyperregular Hamiltonian system on T^*M is equivalent to a Lagrangian system on TM.*

For a discussion of geodesics in the infinite dimensional case (here things are greatly simplified as the domains are all of M) see LANG [1, p. 109]. (Unfortunately, this includes no examples of physical interest but nevertheless is a good mathematical illustration of a Hamiltonian system. Of course when things are smooth on all of M, the more usual techniques can be used; for example, in 3.3 the fixed point assumption is unnecessary.)

Chapter Two: Hamiltonian Methods

§ 5. Symmetry Groups and Conservation Laws

Symmetry groups provide an important practical method for obtaining conservation laws. For example, angular momentum in quantum mechanics or energy-momentum in electromagnetic theory are obtained by exploiting rotational and translational symmetry respectively.

The motivation is the same as in the finite dimensional case, smooth or not (see ABRAHAM [1, § 22] and MARSDEN [2]).

An important kind of symmetry action in a function space is one generated by an action on an underlying manifold A (see 3.3), and so we develop conservation laws special to this case.

5.1. Definitions. *If G is a finite dimensional Lie group, and M is a manifold (infinite dimensional, say), an* **action** *of G on M (sometimes called a* **transformation group**) *is a homomorphism*

$$\Phi\colon G \to \operatorname{Diff} M; \quad g \mapsto \Phi_g$$

where Diff M *is the group of diffeomorphisms on M. We do* **not** *assume Φ is smooth in $g \in G$ (this is false in the important applications).*

If X is a left invariant vectorfield on G, with flow F_t, we have a corresponding flow F_t' on M;

$$F_t'(m) = \Phi_{F_t(e)}(m)$$

where $e \in G$ is the identity. We let X' denote the infinitesimal generator of F_t' and assume X' has a domain D with a structure making $X'\colon D \to TM$ smooth. X' is an **infinitesimal transformation.**

The action Φ is called **symplectic** *iff M is a symplectic manifold and Φ_g is symplectic for each $g \in G$.*

Thus an action Φ is symplectic iff each F_t' is symplectic iff each X' is locally Hamiltonian. The fundamental conservation theorem is as follows:

5.2. Theorem. *Let (M, ω) be a symplectic manifold, X_H a Hamiltonian vectorfield on M (with domain D), and Φ a symplectic action on M. Suppose that Φ is a symmetry action of H; that is, leaves the domain of H invariant and $H \circ \Phi_g = H$ on D.*

Also, suppose X_H has flow F_t with a fixed point and X_K is an infinitesimal transformation of Φ with connected domain $D_K \supset D$.

Then K is a constant of the motion. That is, $K \circ F_t = K$ on the domain D.

Proof. (As in 3.3, the classical proof using Poisson brackets is fallacious here.) Since F_t is symplectic, $F_{t*} X_K = X_{K \circ F_t}$ by 2.9. We claim that also, $F_{t*} X_K = X_K$ which will prove the result as in 3.3. For this, it suffices to show that $F_t \circ F'_\tau \circ F_{-t} = F'_\tau$ where F'_τ is the flow of X_K. This is equivalent to $F'_\tau \circ F_t \circ F_{-\tau} = F_t$, or $F'_{\tau *} X_H = X_H$. But since F'_τ is symplectic by assumption we have $F'_{\tau *} X_H = X_{H \circ F'_\tau}$. But $H \circ F'_\tau = H$ since H is invariant under the action of Φ. □

Below we shall determine K explicitly in the most important cases.

First, we consider the action of a group in the cotangent bundle case:

5.3. Theorem. *Let M be a manifold and Φ an action on M. Define an action on $T^* M$ by*

$$\Phi_g^*(\alpha_m) = \alpha_m \circ (T_m \Phi_g)^{-1} \in T^*_{\Phi_g(m)} M .$$

Then Φ^* is a symplectic action on $T^* M$ using the natural symplectic structure. (Assume M is modelled on a reflexive space; see 2.4.)

Further, if X' is an infinitesimal generator of Φ, and $X^{*\prime}$ is the corresponding one for Φ^*,

$$X^{*\prime} = X_{P(X')}$$

where $P(X')$ is the Hamiltonian function given by

$$P(X')(\alpha_m) = \alpha_m(X'(m))$$

and domain $\bigcup \{T_m^* M : m$ lies in the domain of $X'\}$.

We call $P(X')$ the momentum corresponding to X'.

In the finite dimensional case P is in fact a Lie algebra homomorphism, and this "correspondance principle" was used in the original transition to quantum mechanics. (See Marsden [2] for details.)

Proof of 5.3. The proof of Abraham [1, 14.16] shows that Φ^* is a symplectic action, and further $(\Phi_g^*)_* \theta = \theta$. Then if F'_t is the flow of X', $(F'_t{}^*)_* \theta = \theta$, or $L_{X^{*\prime}} \theta = 0$. Therefore since $L_X = d \circ i_X + i_X \circ d$ and $i_X \omega = X^\flat$, we find $X^{*\prime} = X_P$ where $P = -\theta(X^{*\prime})$. But $\theta(X^{*\prime}) \circ \alpha_m = \theta(\alpha_m) \circ X^{*\prime}(\alpha_m) = -\alpha_m \cdot T_\tau^* \circ X^*(\alpha_m)$, and $T_\tau^* \circ X^* = X \circ \tau$ from the flows. Hence the result. □

Thus if a symmetry action is of the cotangent bundle variety, the conserved quantities are $P(X')$.

Next we specialize further:

5.4. Theorem. *Consider $M = L_2(A, V)$ described in 3.4. Suppose Ψ is a smooth action on A and leaves the volume Ω invariant. On M, define an action by*

$$\Phi_g(f) = f \circ \Psi_{g^{-1}} .$$

Then if X is an infinitesimal generator on A (we assume X is smooth on A), an infinitesimal transformation on M is

$$X'(f) = -L_X f$$

so the domain of X' contains the smooth functions on A, and iX' has a self-adjoint extension.

Furthermore, we have

(i) *if H is a Hamiltonian on T^*M, H is invariant under Φ^* iff*

$$H(f \circ \Psi) = H(f)$$

for f in the domain of H; $f=(f_1, f_2)=f_1+if_2$;

(ii) $$P(X')(f_1, f_2) = \int_A \langle f_1, L_X f_2\rangle_0 \, d\mu_\Omega = \tfrac{1}{2}(iL_X f, f)_0$$

(the "expectation" of the symmetric operator iL_X), for f_1, f_2 smooth.

If (i) *holds and X_H has a flow with a fixed point, then the functions $P(X')$ in* (ii) *are constants of the motion.*

Proof. Since Ψ_g is volume preserving, it clearly maps L_2 into itself. Now if F_t is the flow of X, that of X' is $F_t^*(f) = f \circ F_{-t}$, so by differentiation $X'(f) = -L_X f$.

Statement (i) is clear from the definitions, as is (ii), using the identifications of 3.4, 3.5 and the fact that, since Ψ_g is volume preserving, iL_X is a symmetric operator. The last statement follows at once from 5.2 and 5.3. □

Thus in these circumstances, we obtain the constants of motion quite explicitly. For example, if Ψ is a group of rotations, then iL_X are just the quantum mechanical angular momentum operators. We use the first expression for classical continuous systems.

Also note that if $P(X')$ and $P(Y')$ are as in 5.4, then $P([X, Y]') = \{P(X'), P(Y')\}$ and $\{P(X'), P(Y')\}(f) = 2(i[L_X, L_Y]f, f)_0$, which is a standard remark in quantum mechanics. That is, if iL_X and iL_Y are constants of the motion, so is $[iL_X, iL_Y]$. For further discussion see § 8.

§ 6. Canonical Transformations

The treatment of canonical transformations is similar to the finite dimensional case, so that we shall be brief here, mainly emphasizing the points of departure. We begin with **time dependent systems.**

6.1. Definitions. *Let (M, ω) be a symplectic manifold, $\mathbf{R}$ the reals, $\mathbf{R} \times M$ the product manifold and $\pi_1: \mathbf{R} \times M \to \mathbf{R}$, $\pi_2: \mathbf{R} \times M \to M$ the projections, and $j^t: M \to \mathbf{R} \times M$ the injection at time t.*

Put $\tilde{\omega} = \pi_{2*}\omega$, called the **contact form.**

A function $f: D \subset \mathbf{R} \times M \to \mathbf{R}$ with dense domain D is called a **Hamiltonian function** *(time dependent) iff for each $t \in \mathbf{R}$,*

$$f_t: D_t \subset M \to \mathbf{R}$$

is a Hamiltonian function (2.3), *where* $D_t=(j^t)^{-1}(D)$, *the domain at time* t, *and* $f_t=f\circ j^t$. *(The situation is somewhat simpler if* D_t *does not depend on* t, *so that* $D=R\times D_0$.)

If $K\colon D\subset R\times M\to R$ *is a Hamiltonian function, put*

$$\omega_K=\tilde{\omega}+dK\wedge d\pi_1$$

a two-form with domain D $(dK\wedge d\pi_1\equiv d_M K\wedge d\pi_1)$, *called the* **Cartan form.**

Let $\underline{t}$ *denote the unit vectorfield:*

$$\underline{t}(t,m)=(t,1,0)\in T(R\times M)\approx TR\times TM\,.$$

A **time dependent vectorfield** *on* M *is a map* $X\colon D\subset R\times M\to TM$ *such that for each* $t\in R$, $X_t=X\circ j^t$ *is a vectorfield with domain* D_t.

We say X **has a flow** *iff the vectorfield* $\tilde{X}\colon D\to T(R\times M)$; $\tilde{X}=\underline{t}+X$ *has a flow.*

If $H\colon D\subset R\times M\to R$ *is a Hamiltonian function, we put*

$$X_H(t,m)=X_{H_t}(m)$$

for $(t,m)\in D$, *and for* f,g *with the same domain,*

$$\{f,g\}(t,m)=\{f_t,g_t\}(m)\,.$$

Of course for a system with time dependent Hamiltonian, energy is not conserved. We leave it to the reader to develop conservation laws in the time dependent case. See ABRAHAM [1, p. 136] and follow the methods of § 5.

The proof in ABRAHAM [1, p. 136] shows that for a Hamiltonian $H\colon D\subset R\times M\to R$, $\tilde{X}_H$ is the unique vectorfield with domain D, so $i_{\tilde{X}_H}\omega_H=0$ and $i_{\tilde{X}_H}d\pi_1=1$. Also, if α is a k-form on $R\times M$ with domain D, then $\alpha=\beta_1+\beta_2\wedge d\pi_1$ where β_1 and β_2 are *vertical forms*; that is, $j^t_*\beta_i=0$ implies $\beta_i=0$. (Choose $\beta_1(t,m)\cdot(v_1,\dots,v_k)=\alpha(t,m)\cdot(v_1^v,\dots,v_k^v)$ where v^v denotes the component along M, and $\beta_2(t,m)\cdot(v_1,\dots,v_{k-1})=k\,\alpha(t,m)\cdot(t,v_1^v,\dots,v_{k-1}^v)$.)

6.2. Definition. *Let* (M,ω) *and* (N,ρ) *be symplectic manifolds. A* **canonical transformation** *is an (admissible) map*

$$F\colon D\to D'$$

where $D\subset R\times M$ *and* $D'\subset R\times N$ *are domains such that*

(i) *for each* $t\in R$, $F_t=\pi_2\circ F\circ j^t$ *is a diffeomorphism of* D_t *onto* D'_t;

(ii) *for each* $(t,m)\in D$, $F(t,m)$ *is differentiable in* t *using the* M,N *topologies*;

(iii) $\pi_1\circ F=\pi_1$ *or* F *preserves time*;

(iv) *there is a Hamiltonian function* K_F *with domain* D *such that on* D,

$$F_*\tilde{\rho}=\tilde{\omega}+dK_F\wedge d\pi_1\,.$$

The main theorem on canonical transformations is the following:

6.3. Theorem. *Let* (M,ω) *and* (N,ρ) *be symplectic manifolds and* $F\colon D\to D'$ *satisfy* (i), (ii), (iii) *of* 6.2. *Then the following are equivalent:*

(i) *F is canonical;*

(ii) *for all Hamiltonians H with domain D', there is a* K^H $(K^H = H \circ F + K_F)$ *so that on D,*

$$F_* \rho_H = \omega_{K^H};$$

(iii) F_t *is symplectic for each t and there is a* K_F *on D so* $F_* \underline{t} = X_{K_F}$ *on D;*

(iv) F_t *is symplectic for each t and for all Hamiltonians H on D', there is a* K^H *so* $F_* \tilde{X}_H = \tilde{X}_{K^H}$ *on D;*

(v) *there is a Hamiltonian* K_F *on D so that for all Hamiltonians H on D',*

$$F_* \tilde{X}_H = \tilde{X}_{K^H}$$

where $K^H = H \circ F + K_F$.

Since the proof is a fairly simple modification of ABRAHAM [1, § 21], together with previous remarks, we omit it. Roughly, a map is canonical if it preserves the form of time dependent Hamiltonian systems with a given domain. We call K_F the **generating function.**

Next we show that the evolution of a time independent Hamiltonian system is given by a canonical transformation.

6.4. Theorem. *Let* (M, ω) *be a symplectic manifold and H a Hamiltonian with domain D. Suppose* X_H *has a flow* F_t*. Then the map*

$$\tilde{F}: \mathbf{R} \times D \to \mathbf{R} \times D; \quad \tilde{F}(t, m) = (t, F(t, m))$$

is a canonical transformation with domain $\mathbf{R} \times D$*. In fact, F transforms H to equilibrium; that is,* $K^H = 0$ *and* $K_{\tilde{F}} = -H$.

Proof of 6.4. Clearly, (i), (ii), (iii) of 6.2 hold, and, by 3.1, $\tilde{F}_t = F_t$ is symplectic. Then by 6.3 it suffices to show $\tilde{F}_* \underline{t} = X_{-H}$. But the flow of $\underline{t}$ is $G_t(s, m) = (t+s, m)$, so that

$$\tilde{F}^{-1} \circ G_t \circ \tilde{F}(s, m) = (s+t, F(-t, m)),$$

by the group property of the flow. But this is exactly the flow of X_{-H}. □

Similarly, the map $\tilde{F}^{-1}$ is a canonical transformation with $K_{\tilde{F}^{-1}} = H$.

Finally we state the **Hamilton-Jacobi theorem.** This does not seem to be in the literature in the infinite dimensional case, but the same proof as in ABRAHAM [1, p. 144–146] holds.

6.5. Definition. *Let E be a reflexive Banach space and* $S: \mathbf{R} \times E \times E \to \mathbf{R}$ *a map. Let*

$$S_t: E \times E \to \mathbf{R}; \quad S_t(e_1, e_2) = S(t, e_1, e_2),$$

and suppose there is a domain $D \subset \mathbf{R} \times E \times E$ *such that* $\partial S / \partial t$ *exists on D and also suppose*

$$F_1 S: \mathbf{R} \times E \times E \to \mathbf{R} \times T^* E; \quad (t, e_1, e_2) \mapsto (t, e_1, D_1 S_t(e_1, e_2))$$

and

$$F_2 S: \mathbf{R} \times E \times E \to \mathbf{R} \times T^* E; \quad (t, e_1, e_2) \mapsto (t, e_2, D_2 S_t(e_1, e_2))$$

are diffeomorphisms. Under these circumstances we call S a **principal function.** *(We may also suppose S is defined only on D.)*

6.6. Theorem. *If $S: R \times E \times E \to R$ is a principal function, then $G = -F_1 S \circ (F_2 S)^{-1}; R \times T^*E \to R \times T^*E$ is a canonical transformation with domain $(F_2 S)(D)$ and*

$$K_G = -\frac{\partial S}{\partial t} \circ (F_2 S)^{-1}$$

on $(F_2 S)(D)$. Furthermore, if H is a Hamiltonian with domain $G(F_2 S(D)) = F_1 S(D)$, G transforms H to equilibrium iff, on D,

$$H \circ F_1 S + \frac{\partial S}{\partial t} = 0.$$

As in the finite dimensional case, this is a specialized method for finding the flow of X_H which can be expected to work in only certain problems (*e.g.*, quantum mechanical harmonic oscillator).

Some main theorems of the finite dimensional case do not hold in general. We have in mind the Hamiltonian flow box and closed orbit theorems (ABRAHAM [1, p. 142, 178]). To recover these we need to assume the domain of H is all of M, and H is smooth on M. These are severe restrictions in practice, but we sketch the theorems in the Appendix anyway.

Appendix: Hamiltonian Flow Box Theorem

For the theorems here to be applied, the crucial assumption is smoothness on M. This rarely occurs in examples of physical interest. However, here we don't require the symplectic form to be an isomorphism, but only that it be non-degenerate, so the spaces can be adjusted. For example, by use of somewhat artificial spaces, the theorem applies to the wave equation which is, in fact, Hamiltonian; see MARSDEN [1].

6.7. Lemma. *Let M be a Banach manifold, $X \in \mathfrak{X}(M)$ and $X(m) \neq 0$. Then there is a chart at m, $U = V \times I$, $I = (-a, a)$ for $a > 0$ such that for each $v \in V$, the mapping $t \mapsto (v, t)$ is an integral curve of X.*

See ABRAHAM [1, p. 79] or ABRAHAM [3, p. 58] for the proof.

6.8. Theorem. *Let M be a symplectic manifold modelled on a Banach space, $H \in \mathcal{F}(M)$ and $dH(m) \neq 0$. Then there is a chart U at m such that $U = I \times J \times W$; $I = (-a, a)$, $J = (-\varepsilon, \varepsilon)$ and the following hold:*

(i) *for $e, w \in J \times W$; $t \mapsto (t, e, w)$ is an integral curve of X_H;*

(ii) $H(t, e, w) = e - H(m)$;

(iii) $\omega = \omega_0 + dt \wedge dH$

where ω_0 is a two-form on $J \times W$.

Proof. Using 6.7, write $U = I \times V$ locally. Now $H(t, v) = H(t', v)$ for all $t, t' \in I$, by conservation of energy. Hence H defines a smooth mapping on V and $dH \neq 0$. By the implicit mapping theorem we may write $V = J \times W$ where $H(t, e, w) = H(e)$.

Now define $\tilde{F}: I \times U \to I \times U$; $(t', u) \mapsto (t', F_t'(u)) = (t', t + t', e, w)$ if $u = (t, e, w)$. By 6.4, $\tilde{F}$ is a canonical transformation with $\tilde{F}_* \tilde{\omega} = \tilde{\omega} - dH \wedge dt'$.

Define $i: U \to I \times U$; $(t, e, w) \mapsto (t, (0, e, w))$ and note that $F \circ i(t, e, w) = (t, (t, e, w))$, and

$$(\bar{F} \circ i)_* \bar{\omega} = (\bar{F} \circ i)_* \pi_{2*} \omega = \omega,$$

as $\pi_2 \circ F \circ i$ is the identity.

Hence $\omega = i_* \bar{F}_* \bar{\omega} = i_* \bar{\omega} - i_*(dH \wedge dt')$. But $H \circ i = H$ (conservation of energy) and $t' \circ i = t$ (abuse of notation).

Therefore let $\omega_0 = i_* \bar{\omega}$ which is given by

$$\omega_0(t, e, w)(T_1, E_1, W_1), (T_2, E_2, W_2)) = \omega(0, e, w) \cdot ((0, E_1, W_1), (0, E_2, W_2)). \quad \square$$

As an application of this theorem we consider the global geometric problem of closed orbits. It is the analogue of a more refined version in the finite dimensional case (ABRAHAM [1, p. 178]).

We first recall a few facts. A **closed orbit** of a vectorfield, is an integral curve $c: (-a, a) \to M$ such that $c(\tau) = c(0)$ for some $\tau > 0$, but $c(t) \neq c(0)$ for $0 < t < \tau$. Then c may be extended to R and $c(R)$ is compact.

A **transversal section** of X at $m \in M$ is a submanifold S of M such that $T_s S \oplus X(s) = T_s M$ for $s \in S$.

If c is a closed orbit of X and S a transversal section of $m \in c$, a **Poincaré map** is a diffeomorphism $\Theta: W_0 \to W_1$ such that

(i) $W_0, W_1 \subset S$ are open (in S) neighborhoods of m;

(ii) there is a $\delta \in \mathscr{F}(W_0)$ so

$$\Theta(s) = F(s, \tau - \delta(s))$$

where F is the flow of X; and

(iii) if $0 < t < \tau - \delta(s)$, $F(t, s) \notin W_0$.

We may now prove local existence and uniqueness of Poincaré maps exactly as in ABRAHAM [1, p. 159]. Finite dimensionality is not required.

We may also prove (ABRAHAM [1, 28.4]):

6.9. Proposition. *Let M be a symplectic manifold modelled on a Banach space, $H \in \mathscr{F}(M)$ and c a closed orbit of X_H. Then there is a local transversal section S and a Poincaré map $\Theta: W_0 \to W_1$ such that*

(i) $\Theta_* \omega_1 = \omega_0 - d\delta \wedge dH$, *$\delta$ as above and $\omega_1 = i_* \omega$ $i: W_1 \to M$ (ω_0 similarly defined);*

(ii) *there are submanifolds $\Sigma_{e'}$ on which H is constant, as in* 6.8 *such that $S \cap \Sigma_{e'}$ are submanifolds (W of* 6.8) *and $\Theta_*(\omega_1)_0 = (\omega_0)_0$ where $(\omega_1)_0$ is ω_0 of* 6.8 *on $S \cap \Sigma_{e'}$.*

With a slight change of hypotheses we can also recover the closed orbit theorem. (The proof is essentially the same.)

6.10. Theorem (ABRAHAM). *Let M be a symplectic manifold modelled on a Banach space, $H \in \mathscr{F}(M)$ and c a closed orbit of X_H. Locally, in the notation of* 6.8 *suppose $\psi: J \times W \to J \times W$; $\psi(e, w) = \Theta(e, w) - w$ has $D_2 \psi(0, 0)$ an isomorphism. Then for some $\varepsilon > 0$ and all $e' \in (e - \varepsilon, e + \varepsilon)$, where $H(c) = e$ there is a closed orbit of energy e'. Moreover, this collection of closed orbits is diffeomorphic to a cylinder.*

In the finite dimensional case the condition is one on the spectrum of the flow. Presumably something similar holds in general.

Chapter Three: Applications

§ 7. Some Conservative Classical Continuum Systems

Here we study some *conservative* classical continuum systems from the Lagrangian point of view. Typical systems are vibrating plates and the electromagnetic field.

7.1. Definitions. *Let* $\pi\colon V \to A$ *be a vector bundle over an orientable manifold* A *and consider the space*

$$M = TL_2(A, V) = L_2(A, V) \times L_2(A, V)$$

discussed in 3.4, *with bilinear form on* $L_2(A, V)$;

$$(f, g)_0 = \int_A \langle f(a), g(a)\rangle_0 \, d\mu_\Omega(a).$$

The **kinetic energy function** *is defined by*

$$T\colon M \to R; \qquad T(f_1, f_2) = \tfrac{1}{2}(f_1, f_2)_0.$$

Let W *denote the vector bundle over* $A \times V$ *whose fiber over* (m, v) *is* $L(T_m A, T_v V)$ *and for* $f\colon A \to V$ *define* $Tf\colon A \to W$ *as* $Tf(a)$ *is the linear map* Tf *on the fiber over* a *to the fiber over* $f(a)$. *We often write* Df *for* Tf *considered this way.* (W *is called a jet bundle; see* ABRAHAM [3, p. 19].)

A **potential density** *is a smooth map*

$$h\colon A \times V \times W \to R, \qquad \text{bounded on } A,$$

and the corresponding potential energy V_h, *defined on smooth functions, is given by*

$$V_h(f_1, f_2) = \int_A h(x, f_1(x), Df_1(x))\, d\mu_\Omega(x).$$

The **Lagrangian** L_h *corresponding to a potential density is defined by*

$$L_h = T - V_h.$$

How one might relax the smoothness assumptions is indicated in Appendix B, following § 8.

In coordinate language, h is considered a function of $x^i, f^j(x^i), \partial f^j/\partial x^i$. To maintain the coordinate free spirit it is necessary to introduce some more notation at this point.

7.2. Lemma. *In the above notation, suppose* $K\colon W \to R$ *is a smooth map, linear on fibers. Then there is a unique smooth map*

$$\operatorname{div}_\Omega K\colon V \to R$$

linear on fibers such that

$$\int_A K \cdot Df(x)\, d\mu_\Omega(x) = -\int_A \operatorname{div}_\Omega K \cdot f(x)\, d\mu_\Omega(x)$$

for every smooth section $f\colon A \to V$ *with compact support* ($f \in L_2(A, V)$). *We call* $\operatorname{div}_\Omega K$ *the* **divergence** *of* K.

Proof. Uniqueness is clear. For existence, using a partition of unity, it is enough to work in a coordinate chart. Write $f = (f^1, \ldots, f^m)$ and $K \cdot Df = K_1 \cdot Df^1 + \cdots + K_m Df^m$. Thus each component K_j represents a vectorfield, and $K_j \cdot Df^j = L_{K_j} f^j$. But

$$\int_A K_j \cdot Df^j\, d\mu_\Omega = -\int_A f^j L_{K_j} \Omega$$

by STOKES' theorem (see MARSDEN [2, § 3]), and $L_{K_j}\Omega = \operatorname{div}_\Omega K_j \Omega$ by definition of divergence. Therefore we may take

$$\operatorname{div}_\Omega K \cdot f = \sum_{j=1}^{m} (\operatorname{div}_\Omega K_j) \cdot f^j,$$

proving the assertion. □

(It is also easy to see that these "multivectors", or derivations on the sections of V are isomorphic to the sections of the bundle whose fibers are $L(T_m{}^* M, \pi^{-1}(m) = V_m)$ over M.)

From the above proof we see that $\operatorname{div}_\Omega K$ may be computed locally like the usual divergence.

The main theorem of this section is as follows:

7.3. Theorem. *In* 7.1, *suppose* V_h *is smooth on a domain* D *consisting of smooth functions with compact support, or which vanish sufficiently rapidly at infinity so* STOKES' *theorem (integration by parts) in* 7.2 *applies. (If* D *consists of smooth functions (say with compact support) with the topology of uniform convergence on compact sets this is guaranteed by the composition theorem quoted in* § 1.)

Then the following hold:

(i) L_h *is a regular and hyperregular Lagrangian with energy* $E = T + V$,

(ii) ω_L *is the symplectic form of* 3.5,

(iii) $f_1(t, x)$ *is a base integral curve of* X_E *in* D *iff*

$$\frac{\partial^2 f_1}{\partial t^2} = \{D_2 h(x, f_1(t,x), Df_1(t,x)) - \operatorname{div}_\Omega D_3 h(x, f_1(t,x), Df_1(t,x))\}^\sharp$$

where $k^\sharp$ *is defined by*

$$(k^\sharp, g)_0 = \int_A k \cdot g(x)\, d\mu_\Omega(x)$$

($\langle k^\sharp(x), g(x)\rangle_0 = k \cdot g(x)$; *for* $k\colon V \to R$).

(iv) *if* $f_1(t, x)$ *is a base integral curve in* D, *then*

$$\frac{1}{2}\int_A \left\langle \frac{\partial f_1}{\partial t}, \frac{\partial f_1}{\partial t} \right\rangle_0 d\mu_\Omega + \int_A h(x, f_1(t,x), Df_1(t,x)\, d\mu_\Omega(x)$$

is constant in t *(assume the flow has a fixed point, or is linear; see* 3.3).

Proof. First we check that L_h is a Hamiltonian function. Now we have (see §1)

$$\begin{aligned} DV_h(f_1, f_2)\cdot(g_1, g_2) &= \int D_2\, h(x, f_1(x), Df_1(x))\cdot g_1\, d\mu_\Omega(x) + \int D_3\, h(x, f_1(x), Df_1(x))\cdot Dg_1\, d\mu_\Omega(x)\\ &= \int D_2\, h(x, f_1(x), Df_1(x))\cdot g_1\, d\mu_\Omega(x) - \int \operatorname{div}_\Omega D_3\, h(x, f_1(x), Df_1(x))\cdot g_1 d\mu_\Omega, \end{aligned}$$

by the lemma. Thus for $f_1 \in D$, it is clear that this has an extension to all $g_1 \in L_2(A, V)$. Also, $DT(f_1, f_2)\cdot(g_1, g_2) = (f_2, g_2)_0$, by Leibnitz' rule, which also extends. Thus L_h is Hamiltonian.

Clearly $FL_h = DT$, so that (i) and (ii) are obvious by definition.

Finally, (iii) is just Lagrange's equations (4.6), and (iv) is conservation of energy (3.3). □

In a coordinate chart with the Euclidean metric, the equations become the usual Lagrange density equations:

$$\frac{\partial^2 f^i}{\partial t^2} = \frac{\partial h}{\partial f^i} - \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \frac{\partial h}{\partial\left(\dfrac{\partial f^i}{\partial x^j}\right)}.$$

(However, 7.3 is more general, holding in general relativity, for example, in which case the indices in the above equation must be correctly positioned.)

For example, choosing potential density

$$h(x^j, f^j(x^i), \partial f^j/\partial x^i) = \tfrac{1}{2} \sum_{i,j} (\partial f^j/\partial x^i)^2$$

results in the classical wave equation. For the proper choice of domains and the existence of the flow, we refer to Yosida [1, Ch. XIV].

Another example of importance is the electromagnetic field, which we briefly sketch: Here we take V to be a four dimensional bundle over A (with A considered as a space like 3-surface) with a metric g, and take $\langle\,,\rangle$ to be a bilinear form on V of Lorentz type. Elements of $L_2(A, V)$ are called the **"four potentials"**. For h, take

$$h(x, f(x), Df(x) = \tfrac{1}{2} G(Df, Df) - (f, J)_0 \qquad \langle\ \rangle_0$$

where J is some fixed element of $L_2(A, V)$ called the **"four current"** and G is the natural metric on linear maps induced by g and $\langle\,,\rangle_0$.

In the flat case the equations reduce to the usual wave equation with source

$$\frac{\partial^2 f^i}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 f^i}{\partial x^{j2}} + J^i(x).$$

In electromagnetic theory the flow preserves the symplectic structure $\langle\,,\rangle_0$, called **Lorentz invariance**, and conservation of energy is known classically as **Poynting's theorem.**

The motion of a charged particle in an electromagnetic field without radiation reaction is a Hamiltonian system, but the general case is *not* (even the coupled system of the particle and fields). For details and more physics, see Rohrlich [1]. The conservation theorems again yield standard results in this case. The electromagnetic field will be coupled with the Dirac wave equation in the next section.

In 3.2 we examined infinitesimally the Hamiltonian case. The real linear case occurs often for classical continuum systems, so we restate that result in this special case.

7.4. Theorem. *In 3.4, 3.5 suppose X is a real linear vectorfield on the symplectic manifold*

$$M = TL_2(A, V) = L_2(A, V) \times L_2(A, V)$$

with domain D and flow $F_t: D \to D$, also real linear. Then the following are equivalent:

(i) *X is Hamiltonian;*

(ii) *$X = X_H$, where $X = (X_1, X_2)$ and on D,*

$$H(f_1, f_2) = \tfrac{1}{2}\{(X_1(f_1, f_2), f_2)_0 - (X_2(f_1, f_2), f_1)_0\};$$

(iii) *X satisfies*

$$(f_2, X_1(g_1, g_2))_0 + (X_2(f_1, f_2), g_1)_0 = (X_2(g_1, g_2), f_1)_0 + (g_2, X_1(f_1, f_2))_0$$

for all $(f_1, f_2), (g_1, g_2) \in D$;

(iv) *F_t is symplectic;*

(v) *F_t satisfies; if $F_t = (F_t^1, F_t^2)$,*

$$(f_1, f_2)_0 = (F_t^2 f_1, F_t^2 f_2)_0$$

for all $(f_1, f_2) \in D$.

Further, if X is a second order equation, (ii) *and* (iii) *become, respectively*

(ii′) *$X = X_H$, where*

$$H(f_1, f_2) = \tfrac{1}{2}(f_2, f_2)_0 - \tfrac{1}{2}(X_2(f_1, f_2), f_1)_0$$

and (iii′) *X satisfies*

$$(X_2(g_1, g_2), f_1)_0 = (X_2(f_1, f_2), g_1)_0 .$$

Notice that in (ii′), H automatically has a kinetic energy term. The theorem follows at once from 3.2 and the definitions. These conditions are in fact quite useful. For example, we can see at once that the wave equation is Hamiltonian and use (ii′) to compute the Hamiltonian (conserved), but that the heat equation is not Hamiltonian.

Although the basic conservation laws were given in section 5, we repeat the theorem in this special case for reference:

7.5. Theorem. *Let X be a real linear Lagrangian system on $L_2(A, V) \times L_2(A, V)$ with domain D and flow F_t. Let Φ be a volume preserving smooth action on A leaving D invariant. Let Y be some infinitesimal generator of Φ with domain $D_Y \supset D$. Then if X_2 denotes the second component of X and we have*

$$X_2(f_1, f_2) \circ \Phi = X_2(f_1 \circ \Phi, f_2 \circ \Phi)$$

for all f_1, f_2 in D, then the following function is a constant of the motion:

$$P(Y)(f_1, f_2) = \int_A \langle f_1, L_Y f_2 \rangle_0 \, d\mu_\Omega .$$

This follows at once from 5.4 and 7.4.

Of course the conservation laws are valid in the non-linear case as well, but the conditions are not so easy to state as transformation laws on the differential equations.

§ 8. Quantum Mechanical Systems

Here we study quantum mechanical systems in the sense of SCHRÖDINGER and DIRAC as special cases of Hamiltonian systems. These systems are *not* of Lagrangian type (§ 4) in contrast to the conservative classical continuum systems of the preceeding section. Nevertheless, all the basic theorems about Hamiltonian systems do apply.

We are mainly interested in the abstract case, although the coupled Maxwell and Dirac systems will be outlined.

As we saw before, Poisson bracket techniques (here commutators) are of limited use, principally because of domain and smoothness problems. This difficulty is well known. However, we can recover the basic conservation laws using the methods of § 5.

8.1. Definition. *In* 3.4 *consider* $CL_2(A, V)=M$ *with the natural symplectic structure and* $\langle\, ,\, \rangle_0$ *positive definite. A* **quantum mechanical system** *is a Hamiltonian vectorfield* X *on* M *such that* X *is complex linear on a linear domain* D *and* X *has a complex linear flow* F_t.

The basic characterization of quantum mechanical systems is as follows:

8.2. Theorem. *Suppose* X *is a complex linear vectorfield on* $CL_2(A, V)$ *and has a complex linear flow. Then the following are equivalent:*

(i) X *is Hamiltonian;*

(ii) $X=X_H$ *where for* $f\in D$, *the domain of* X,

$$H(f)=\tfrac{1}{2}(i\,X(f), f)_0;$$

(iii) $i\,X$ *is symmetric; for all* $f, g\in D$,

$$(i\,X(f), g)_0=(f, i\,X(g))_0;$$

(iv) F_t *is symplectic;*

(v) F_t *is unitary; for all* $f, g\in D$,

$$(F_t(f), F_t(g))_0=(f, g)_0.$$

In each case, F_t *extends as a continuous (unitary) map* $F_t\colon M\to M$.

Although this is a special case of 3.2, it is instructive to see the details.

Proof. That (i) and (iv) are equivalent was proven in 3.1. Now (iv) and (v) are equivalent, for (v) asserts that $(F_t f, F_t g)_0=(f, g)_0$, so in particular the imaginary part is preserved (see 3.4). Since F_t is complex linear, preserving the imaginary part implies the inner product is preserved (replace f by if).

Now $X^\flat(f)\cdot g=\mathrm{Im}(X(f), g)_0$ so that $dX^\flat=0$ iff

$$2\,dX^\flat(f)\cdot(g, h)=\mathrm{Im}(X(g), h)_0-\mathrm{Im}(X(h), g)_0=0.$$

Thus (i) is equivalent to

$$\mathrm{Im}(X(g), h)_0-\mathrm{Im}(X(h), g)_0=0$$

that is, $i\,X$ is symmetric.

From the expression for $X^\flat$, it is clear that a suitable Hamiltonian is

$$H(f)=\tfrac{1}{2}(i\,X(f), f)_0$$

which, by symmetry of iX is clearly a Hamiltonian function (2.4). This proves the theorem. □

8.3. Corollaries. (i) *If X is a Hamiltonian vectorfield, X (on a possibly larger domain) possesses a flow iff there exists a self-adjoint extension of iX;*

(ii) *In 8.2, $\psi(t, x)$ is an integral curve of X in its domain iff*

$$i\frac{\partial\psi}{\partial t}=H_{op}(\psi)$$

where $H_{op}=iX$ is the energy operator ($\partial/\partial t$ in M);

(iii) *in 8.2 if $\psi(t, x)$ is an integral curve in the domain, then*

$$(H_{op}\psi, \psi)_0$$

is constant in time.

Note: 1. With the appropriate selection of V, H_{op}, the equation in (ii) is the Schrödinger or Dirac equation.

2. Conservation of energy (iii) does not require any assumptions about the spectrum of H_{op} (*e.g.*, that it be discrete).

The proof of 8.3 is clear. (i) is just STONE's theorem, (ii) is just HAMILTON's equations and (iii) is conservation of energy 3.2. For the standard cases, (i) is proven in KATO's basic paper [1]. Appendix B indicates how to deal with cases in which X cannot have a self adjoint extension, corresponding to singular, that is, distributional potentials.

Next we consider the relation between Poisson brackets and commutators.

8.4. Proposition. *Suppose R is a symmetric linear operator in $CL_2(A, V)$ with domain D; define*

$$R_E\colon D\to R;\quad R_E(f)=\tfrac{1}{2}(R(f), f)_0$$

its expectation function. Then if R_E is continuous in D, R_E is a Hamiltonian function, and for two such symmetric operators with the same domain,

$$i\{R_E, S_E\}=[R, S]_E.$$

Proof. From 8.2 we have $X_{R_E}=-iR$, so that

$$\begin{aligned}\{R_F, S_F\}(f)&=2\omega(X_{R_E}, X_{S_E})(f)\\ &=\frac{i}{2}\{(iR(f), iS(f))_0-(iS(f), iR(f))_0\}\\ &=-\frac{i}{2}(RS(f)-SF(f), f)_0.\end{aligned}$$

The last step is really formal as $S(f)$ need not be in the domain of R, although this may often be assumed. In general the expression is defined by the previous line. □

Note: 8.4 should not be confused with any correspondence principle between Poisson brackets and commutators. (See MARSDEN [2, § 9].) Here the Poisson brackets *are* commutators. To apply Poisson brackets to the equations of motion it is necessary to assume R and S are bounded operators.

For a discussion of the existence of flows in the time dependent case, see DERGUZOV & JAKUBOVIC [1].

Although the basic conservation laws were treated in § 5, we repeat the theorem in this case:

8.5. Theorem. *Let X be a quantum mechanical system on $M=CL_2(A, V)$ with domain D and flow F_t. Let Φ be a volume preserving smooth action on A and leaving D invariant. Let Y be some infinitesimal generator of Φ with domain $D_Y \supset D$.*

Then if, for each $f \in D$, we have

$$X(f)\circ\Phi_s = X(f\circ\Phi_s),$$

the expectation of the symmetric operator iL_Y (in fact, a self-adjoint operator) is a constant of the motion.

Next we briefly consider a non-linear Hamiltonian system, the coupled Dirac and Maxwell fields.

This example is an illustration of a *coupled system.* Namely if M_1 and M_2 are symplectic manifolds, consider $M_1 \times M_2$ with symplectic form

$$\pi_{1*}\omega_1 + \pi_{2*}\omega_2; \quad \pi_i\colon M_1\times M_2 \to M_i$$

π_i being the projection. On $M_1 \times M_2$ we have

$$H = H_1 + H_2 + H_{12}$$

where H_i is a Hamiltonian on M_i and H_{12} is an "interaction term".

In our example, H_1 is quantum mechanical and H_2 is classical, and the coupled system is Hamiltonian.

8.6. Definition *(Dirac-Maxwell system in the flat case). Let $A=\mathbf{R}^3$ and $V=\mathbf{R}^4$ with $\langle\,,\rangle$ the Euclidean metric and $\langle\,,\rangle_0$ the Lorentz metric. As usual, let $L_2=L_2(A,V)$ be the square integrable maps $f\colon A\to V$ and $CL_2=L_2\times L_2$ the complexification. Let*

$$M = CL_2 \times L_2 \times L_2$$

with symplectic structure

$$\omega((\psi_1, f_1, g_1), (\psi_2, f_2, g_2)) = \mathrm{Im}(\psi_1, \psi_2) + \tfrac{1}{2}(g_2, f_1)_0 - (f_2, g_1)_0$$

where $(\psi_1, f_1, g_1)\in M$ and

$$(f_1, g_1)_0 = \int \langle f, g\rangle_0 \, d\mu.$$

Let $\alpha^2, \alpha^2, \alpha^3, \beta \in L(\mathbf{R}^4, \mathbf{R}^4)$ satisfy

$$\alpha^k\beta + \beta\alpha^k = 0, \quad \alpha^k\alpha^l + \alpha^l\alpha^k = 0$$

for $k \neq l$ and $(\alpha^k)^2 = \beta^2 = 1$ and $\alpha^{k*} = \alpha^k$, $\beta^* = \beta$ where $*$ denotes the adjoint with respect to $\langle\,\rangle$.

Consider the following vectorfield on M:

$$X(\psi, f, g) = (\psi', f', g')$$

where

$$\psi' = \sum_{k=1}^{3}\alpha^k\frac{\partial\psi}{\partial x^k} + \beta\psi + i\left(f_4\psi + \sum_{k=1}^{3}\alpha^k f_k\psi\right), f' = g$$

and

$$g' = \sum_{k=1}^{3} \partial^2 f/(\partial x^k)^2 + j$$

where $f=(f^1, f^2, f^3, f^4)$, *with indices raised and lowered according to the Lorentz metric, and where*

$$j=(j^1, j^2, j^3, j^4) \in L_2$$

is defined as

$$j^k = \langle \alpha^k \psi, \psi \rangle, \quad j^4 = \langle \psi, \psi \rangle .$$

This vectorfield is called the **Dirac-Maxwell system**; *we leave the domain unspecified; see below.*

Notice that X is non-linear. These equations do not take radiation into account. See SCHWEBER [1, Ch. 4] for the physics. The basic theorem from our point of view is the following:

8.7. Theorem. *The Dirac-Maxwell system is Hamiltonian with Hamiltonian function*

$$H(\psi, f, g) = (H_d \psi, \psi) - (j, f)_0 + \frac{1}{2}(g, g)_0 + \frac{1}{2}\left(\frac{\partial f^k}{\partial x^k}, \frac{\partial f^j}{\partial x^j}\right)_0$$

where

$$i H_d \psi = \sum_{k=1}^{3} \alpha^k \psi + \beta \psi .$$

Then also X *possesses a local flow and the following quantities are invariant under the flow:*

(i) *energy:* $H(\psi, f, g)$;

(ii) *linear momentum:*

$$P_1(\psi, f, g) = (i\,\partial\psi/\partial x^1, \psi) + \tfrac{1}{2}(\partial f/\partial x^1, g)_0 ;$$

(iii) *angular momentum (including spin):*

$$M_3(\psi, f, g) = (i[x^1\,\partial\psi/\partial x^2 - x^2\,\partial\psi/\partial x^1], \psi) + \tfrac{1}{2}(\alpha_1 \alpha_2 \psi, \psi) + \tfrac{1}{2}([x^1\,\partial f/\partial x^2 - x^2\,\partial f/\partial x^1], g)_0 .$$

The hard part of the theorem is existence of the flow. See GROSS [1]. Here MOSER's theorem (1.3) can also be used. It is especially simple in the case of periodic functions (replace R^3 by the three torus). The details will be left to another place.

The rest of the theorem is straightforward. To show X is Hamiltonian we compute $X^\flat = dH$ using Leibnitz' rule and symmetry of H_d. The last part follows from the conservation theorems, since the flow has a fixed point (0, 0, 0), and is invariant under the translation and rotation groups.

Appendix A: Transition from a Discrete to a Continuous System

Here we outline briefly the setting for a rigorous transition from a "discrete" to a "continuous" system. Roughly, the number of degrees of freedom becomes infinite and individual particles become smeared into a continuum; the change in the classical case is from ordinary differential equations to partial differential equations. Examples are classical systems with a finite number of degrees of freedom converging to a continuum system [and quantum mechanical systems converging to a field theory]. For further motivation, see GOLDSTEIN [1, Ch. 11].

8.8. Definition. *Let E be a Hilbert space (real or complex, finite or infinite dimensional) with inner product $(\ ,\ )$ which identifies E with E^*. Let $E\times E\approx E\times E^*$ have the natural symplectic structure given by*

$$\omega((e_1,e_2),(f_1,f_2))=\tfrac{1}{2}\{(f_2,e_1)-(e_2,f_1)\}.$$

Let A be a finite dimensional orientable manifold with volume Ω and let $B\subset A$ be a subset. Suppose μ_B is a measure on B such that $f\colon A\to R$, Ω-integrable implies f is μ_B integrable. Let $\mu_\Omega=\mu_A$. Let $L_2(B,E\times E)$ be the maps $f\colon B\to E\times E$ with $\int (f(x),f(x))\,d\mu_B(x)<\infty$, called the **phase space** *for card(B) (cardinality) particles at points of B and motions in E, with symplectic form:*

$$\omega_B((f_1,f_2),(g_1,g_2))=\tfrac{1}{2}\{\int (g_2(x),f_1(x))\,d\mu_B(x)-\int (f_2(x),g_1(x))\,d\mu_B(x)\}.$$

If B is finite, say card $B=n$ and $E=R^\nu$, then $L_2(B,R^\nu\times R^\nu)$ is just $R^{n\nu}\times R^{n\nu}$, the phase space for n-particles in R^ν, and the symplectic structure is the usual one.

More generally, if T^*M has the natural symplectic structure and also a Riemannian metric we can consider the manifold of L_2 maps $f\colon A\to T^*M$. The integrated symplectic structure coincides with that above. This is the most natural setting but we use that in 8.8 to be specific.

8.9. Proposition. *Let B_α be an increasing sequence (or net) of subsets of A with measures μ_α such that $\mu_\alpha\to\mu_A$ uniformly. That is, for each $\varepsilon>0$ there is an α_0 so $\alpha>\alpha_0$ implies $|\mu_\alpha(A'\cap B_\alpha)-\mu_A(A')|<\varepsilon$ for all $A'\subset A$ measurable.*

Then $\omega_\alpha\to\omega_A$ in the sense that for each $f,g\in L_2(A,E\times E)$, and bounded,

$$\omega_\alpha((f_1,f_2),(g_1,g_2))\to\omega_A((f_1,f_2),(g_1,g_2)).$$

Proof. For $f\colon A\to R$ integrable, and bounded, it suffices to show $\int f\,d\mu_\alpha\to\int f\,d\mu$. We may assume $f\geqq 0$. The assertion is clear for simple functions. Suppose f_i are simple functions and $f_i\uparrow f$. Then

$$|\int f\,d\mu_\alpha-\int f\,d\mu|\leqq|\int (f-f_i)\,d\mu_\alpha|+|\int (f-f_i)\,d\mu|+|\int (f\,d\mu_\alpha-\int f_i\,d\mu|.$$

For $\alpha\geqq\alpha_0$ the last term is bounded by $\varepsilon\cdot(\sup|f|)$ uniformly in i. Now choose i large such that the first two terms are small. □

8.10. Definition. *Under the conditions of* 8.9 *we say that the phase spaces $L_2(B_\alpha,E\times E)$* **converge** *to $L_2(A,E\times E)$. If H_α are Hamiltonians on $L_2(B_\alpha,E\times E)$ we say they* **converge** *to a Hamiltonian H with domain $D\subset L_2(A,E\times E)$ iff for each $f\in D$, $H_\alpha(f|D)\to H(f)$ and $dH_\alpha\to dH$ similarly.*

Then we have $X_{H_\alpha}\to X_H$ or the equations of motion for the approximating systems converge to that for H. In general, this is not enough to guarantee convergence of the flows. For this, one can use the Trotter-Kato theorem; for example, see YOSIDA [1, p. 269].

Appendix B: Distributional Hamiltonians

In mechanics one is generally given the Hamiltonian and not the flow. Unfortunately, in the non smooth case the infinitesimal generator of the flow need not coincide with the Hamiltonian. Therefore, there is required a new definition of the solution (of partial differential equations with distributional coefficients).

Physical examples are commonplace. For example a quantum mechanical particle with a δ-function potential (the flow consisting of partial reflection and partial transmission), or the vibration of a plate with non-uniform density.

In the case of (classical) Hamiltonian systems with a finite number of degrees of freedom the problem has been satisfactorily solved. See MARSDEN [2]. There we even have a general existence theorem. In the general case, the existence question is much more delicate. The basic theorem applicable here is the Trotter-Kato theorem (YOSIDA [1, p. 249]). We suppose, for simplicity that we are in the linear case.

8.11. Definition. *Consider the symplectic manifold* (*see* 3.4) $M=L_2(A, V)\times L_2(A, V)$ *and suppose* X *is a (real or complex linear) map with domain* $D\subset M$ *and with range in the distributional sections of* V *(see* MARSDEN [2, § 1.3] *for details). We call* X *a* **generalized vectorfield** *on* M. *We say* X *is Hamiltonian iff there exists Hamiltonian vectorfields* X_{H_t} *defined on* D *so* $f\in D$ *implies*

$$X_{H_t}f\to Xf$$

in the sense of distributions, and there is a continuous map $H: D\to R$ *so*

$$H_t(f)\to H(f)$$

for $f\in D$. *In this case we write* $X=X_H$.

For example, if $V_t\to\delta$ the delta function on R, then

$$p^2/2+V_t\to p^2/2+\delta$$

in this sense in both the classical and quantum mechanical cases.

8.12. Definition. *In* 8.11, *we say* X **has a flow** F_t *iff* X_{H_t} *have flows* $F_t^{\,t}$ *which are equicontinuous (this will hold if they are all unitary for example; see* 8.2) *and for each* $f\in D$, $F_t^{\,t}(f)\to F_t(f)$ *on* D.

8.13. Proposition. *In* 8.12, F_t *so defined is a strongly continuous flow:* $F_{t+s}=F_t\circ F_s$; *moreover if each* $F_t^{\,t}$ *is unitary, so is* F_t.

Remark. The infinitesimal generator of F_t is not in general X. (This is analogous to what happens in the finite dimensional case.) The Trotter-Kato theorem, however is the basic tool used to verify 8.12. (If it applies, then 8.13 is well known.)

Proof. The assertion is clear from the following inequality:

$$\begin{aligned}\|F_{t+s}(f)-F_t\circ F_s(f)\|&\leqq\|F_{t+s}(f)-F_{t+s}^{(n)}(f)\|\\&+\|F_t^{(n)}(F_s^{(n)}(f))-F_t^{(n)}(F_s(f))\|+\|F_t^{(n)}(F_s(f))-F_t(F_s(f))\|.\quad\square\end{aligned}$$

8.14. Proposition. *In the above, suppose* $H_n\to H$ *uniformly on* D. *Then on* D:

$$H\circ F_t=H.$$

Proof.

$$|H_n\circ F_t^{(n)}-H\circ F_t|\leqq|H_n\circ F_t^{(n)}-H\circ F_t^{(n)}|+|H\circ F_t^{(n)}-H\circ F|.\quad\square$$

In a similar way we recover the other conservation laws. (Assume each H_t has conserved quantity $P(X)$, then so will H.)

Because of the lack of measure theory, these techniques lack the delicacy of the finite dimensional case. See MARSDEN [2]. We shall leave a fuller treatment to another place.

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Department of Mathematics
Princeton University

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