A UNIVERSAL FACTORIZATION THEOREM IN TOPOLOGY

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(received December 7, 1965)

1. Introduction. The purpose of this paper is to prove and generalize the following theorem: Given any topological space X, of all the T_2 spaces Z which are continuous images of X, there is a maximal one Y; that is, one over which all others factor, as in Figure 1.

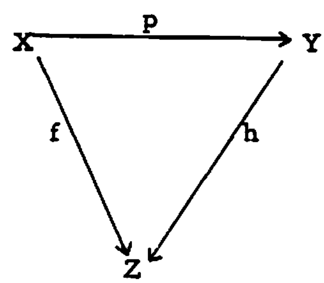

Figure 1.

In pursuit of this result, the authors define a certain species of functors and natural transformations on the category of all topological spaces and maps. A subspecies is singled out which yields the main result. As well it leads to a uniform definition of many separation axioms, and universal proofs for some of the simple properties of these axioms.

The authors are indebted to the referee for many helpful suggestions.

2. Topological Equivalence Relations and Quotients. In this section we introduce the basic machinery in two equivalent forms. The idea is similar in spirit to rewriting the Stone-Čech compactification in terms of an induced natural transformation.

Definition 1. A quotient on Top (the category of topological spaces and maps, the latter meaning continuous mappings) is a pair (F, n) where $F: Top \to Top$ is a covariant functor, $n: I \to F$ is a natural transformation of the identity functor on Top into F and n_X is onto for all X in Top.

If $Q=(F, n)$ is a quotient (on Top) we shall say that a space X (in Top) is Q-invariant when n_X is a homeomorphism.

In the following, if R is an equivalence relation defined on a space X (in Top) we denote the set of equivalence classes endowed with the quotient topology by X/R. Also, X will be called R-discrete when xRx' implies $x=x'$ for all x, x' in X.

Definition 2. An equivalence relation R which is defined on every topological space is called topological when, for any map $f: X \to Y$, xRx' implies $f(x)Rf(x')$.

PROPOSITION 1. There is a 1-1 correspondence between topological equivalence relations and quotients for which the topology on $F(X)$ is the topology induced by n_X.

Proof. Given a topological equivalence relation R, define $F: Top \to Top$ by: $F(X) = X/R$ and $F(f)$ is the map determined uniquely by the commutative diagram in Fig. 2. Here, n_X is the canonical map of X onto X/R.

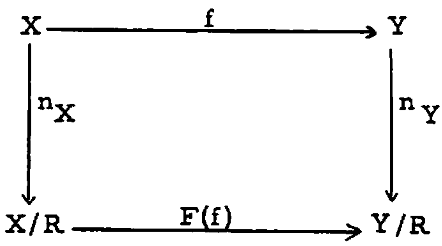

Figure 2.

Uniqueness, together with Figure 3, shows that F is a covariant functor.

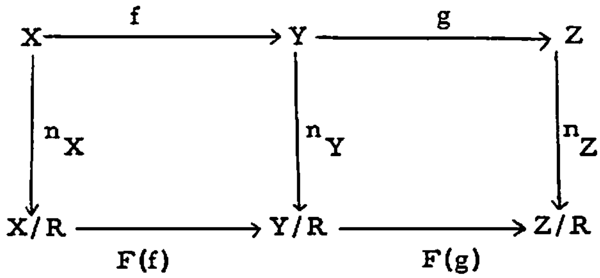

Figure 3.

Conversely, given a quotient $Q = (F, n)$, we define R on X by xRx' if and only if $n_X(x) = n_X(x')$. R is a topological equivalence relation, for, given $f: X \to Y$ and xRx' in X, then $n_Y(f(x)) = F(f) \circ n_X(x) = F(f) \circ n_X(x') = n_Y(f(x'))$. Hence $f(x)Rf(x')$.

Finally, it is easy to see that the above correspondences are the inverse of one another. This completes the proof.

COROLLARY 1. Suppose Q and R induce each other as in Proposition 1. Then X (in Top) is Q-invariant if and only if it is R-discrete.

Proof. Suppose X is Q-invariant. Then xRx' means that $n_X(x) = n_X(x')$. But n_X is a homeomorphism. Conversely, if X is R-discrete, $X = X/R = F(X)$ and the canonical map is the identity.

Definition 3. Given a topological space X, a pair (Y, g) consisting of a space Y (in Top) and a map $g: X \to Y$, is said to have the R-factorization property for X when for any R-discrete space Z together with a map $f: X \to Z$ there is a map $h: Y \to Z$ so that $f = h \circ g$.

The observation that Fig. 2 collapses to a triangle when Y is R-discrete, gives

COROLLARY 2. Given a space X and a topological equivalence relation R, then $(X/R, n_X)$ has the R-factorization property for X.

Now that we have established the relation between quotients and topological equivalence relations, we will discuss only the latter.

3. The Limit Relation. Given a topological equivalence relation R, we define a new relation limR as follows. For X a topological space containing x and x', we have $x(\lim R)x'$ if and only if for all R-discrete spaces Z together with maps f from X to Z, then $f(x) = f(x')$. Note that such pairs (Z, f) always exist.

PROPOSITION 2. (i) limR is a topological equivalence relation;

(ii) $X/(\lim R)$ is R-discrete;

(iii) X is R-discrete if and only if it is limR-discrete.

Proof. (i) If $f:X \to Y$ is a map and $f(x)$ and $f(x')$ are not limR related then there is a map $g:Y \to Z$ with Z R-discrete and $g(f(x)) \neq g(f(x'))$. Since $g \circ f$ is a map, x and x' are not limR related.

(ii) Suppose that x and x' in X are not limR related. Then there is a map f from X into an R-discrete space Z with $f(x) \neq f(x')$. From Corollary 2 of Proposition 1 there is an h such that $f = h \circ p$, where p is the canonical map of X onto X/(limR). Thus $h(p(x)) \neq h(p(x'))$, and hence p(x) and p(x') are not R related.

(iii) If X is R-discrete and x(limR)x', the identity map on X gives $x = x'$. Conversely, if X is limR-discrete then $X = X/(\text{limR})$ is R-discrete by (ii).

COROLLARY. X/(limR) is limR-discrete.

The result we mentioned in the introduction can be stated as follows:

THEOREM. Given a topological space X and a topological equivalence relation R, there is a pair (Y, p) consisting of an R-discrete space Y and a map p of X onto Y, which has the R-factorization property for X. This pair is unique up to homeomorphism.

Proof. We take X/(limR) for Y, with p the natural map. By (ii) of Proposition 2, X/limR is R-discrete. The R-factorization property is an immediate consequence of (iii) of Proposition 2 and Corollary 2 of Proposition 1. For uniqueness, if (Y, p) and (Y', p') both satisfy the conditions of the theorem we have a diagram as in Fig. 4. Since the diagram commutes and p and p' are onto, we have that (Y, p) and (Y', p') are related by a homeomorphism.

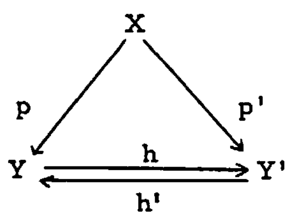

Figure 4.

An example will be given in Section 5 to show that X/R will not suffice for Y in this theorem.

4. Subspace and product theorems.

PROPOSITION 3. If $f:X \to Y$ is 1-1 and Y is R-discrete, then X is R-discrete.

Proof. If xRx' then $f(x)Rf(x')$. Hence $f(x) = f(x')$ and thus $x = x'$ since f is 1-1.

COROLLARY. A subspace of an R-discrete space is R-discrete.

PROPOSITION 4. Given a family of topological spaces Y_i, let Y denote the product space. Let R be a topological equivalencerelation. Then yRy' in Y implies $y_i R y_i'$ in each Y_i. The converse holds if the family is finite.

Proof. The first part comes immediately by considering the projection maps. For the converse, define $j_i:Y_i \to Y$ for $i=1,2,\ldots,n$ by $j_i(z) = (y_1, y_2, \ldots, y_{i-1}, z, y'_{i+1}, \ldots, y'_n)$. Now $y_i R y'_i$ implies $j_i(y_i) R j_i(y'_i)$ for $i=1,2,\ldots,n$. The result now follows by applying the transitivity of R a finite number of times.

PROPOSITION 5. With the notation of Proposition 4, Y is R-discrete if and only if each Y_i is R-discrete. (The family need not be finite).

Proof. Proposition 4 gives the "if" part at once. Conversely, suppose Y is R-discrete and $y_i R y'_i$ in each Y_i. Define $j_i:Y_i \to Y$ by $j_i(z)(k) = y_k$ if $i \neq k$ and equal to z if $i = k$. We have $j_i(y_i) R j_i(y'_i)$ and hence $j_i(y_i) = j_i(y'_i)$. Thus we have $y_i = y'_i$. This proves the result.

5. Separation axioms. To construct topological equivalence relations corresponding to separation axioms, we make

Definition 4. An elementary topological relation is a symmetric, reflexive relation defined on every topological space which is preserved under maps.

An elementary topological relation E_0 induces an equivalence relation E as follows: xEx' when there are points $z_1, z_2, \ldots, z_n$ with $z_1 = x$, $z_n = x'$ and $z_k E_0 z_{k+1}$ for $k = 1, 2, \ldots, n-1$. The following is clear:

PROPOSITION 6. (i) E is a topological equivalence relation;

(ii) X is E_0-discrete if and only if it is E-discrete.

<u>Examples</u>. The following are examples of topological equivalence relations and how they are formed.

(1) T_0 defined by: xT_0x' when every open set containing one of x, x' contains them both. In this case R=limR.

(2) T_1 induced by the elementary topological relation E_0 defined by: xE_0x' when every open set containing x contains x' <u>or</u> every open set containing x' contains x.

(3) P_1 induced by: xE_0x' when there is a sequence which converges to both x and x'.

(4) T_2 induced by: xE_0x' when every pair of open sets containing x and x' respectively overlap.

(5) $T_{3/2}$ defined by: $xT_{3/2}x'$ when for every map $f:X \to [0, 1]$ we have $f(x) = f(x')$.

Then, for example, a space X is a T_2 space if and only if it is T_2-discrete in the sense defined by (4). Similar statements hold for the other examples; they can, if desired, be taken as definitions.

Next we come to the question of what separation axioms are not definable by topological equivalence relations. We shall show that T_3 and T_4 fall into this class.

First of all T_4 (normality) is not product invariant and so would contradict Proposition 5. For T_3 we will get a contradiction with Proposition 3. Let I denote $[0, 1]$ with the usual topology. Let I' denote [0, 1] with topology generated by the following subbasis: (i) all open sets of I ; (ii) the complement of K, which is the union of $[1/(2n+1), 1/(2n)]$ $n = 1, 2, \ldots$. Now I' is not regular (T_3) since 0 cannot be separated from the closed set K. However, we have a 1-1 map $I' \to I$. This is not compatible with Proposition 3.

Finally, we give an example where R and limR are not the same. We do this by giving an example of a space X for which X/T_2 is not Hausdorff. Let X be the union of:

$S = \{\ldots, p_{-n}, \ldots, p_{-2}, p_{-1}, p_1, p_2, \ldots\}$ and

$S' = \{\ldots, q_{-2}, q_{-1}, q_1, q_2, \ldots\}$. The topology is generated by the following basis: (i) all subsets of S' ; (ii) complements of sets of the form $\{p_{a_1}, \ldots, p_{a_r}, q_{e_1 a_1}, \ldots, q_{e_r a_r}\}$ where e_i is 1 or -1. It is readily verified that points of S belong to the same class while the subspace S' is T_2. Also, points in S can be separated from those of S'. Thus X/T_2 is not T_2, for the only open set containing the class S is X/T_2.

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