STRUCTURAL CONTROL EXPERIMENTS USING AN $\mathcal{H}_\infty$ POWER FLOW APPROACH

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Experimental results are presented comparing velocity feedback with a new technique for designing guaranteed stable control laws for uncertain, modally dense structures with collocated sensors and actuators. A de-reverberated mobility model is used, which is similar in many respects to a wave-based model, but can treat more general structures. The power dissipated by the controller can be maximized in either an $\mathcal{H}_2$ or an $\mathcal{H}_\infty$ sense. The $\mathcal{H}_\infty$ approach guarantees that the controller is positive real, and thus that the system will remain stable for any uncertainty, provided that the power flow is correctly modelled. The experimental results indicate that the controllers designed with this approach are much more effective than simple collocated rate feedback.

1. INTRODUCTION

Broadband active control of flexible structures is difficult for several reasons. Structures tend to be very lightly damped, modally rich and difficult to model in detail, due to their large sensitivity to parameter variations. For many applications, there are likely to be many flexible modes within the desired bandwidth of a structural control system [1], and these modes are likely to be poorly known. Models of structures with closely spaced modes in particular tend to be extremely sensitive to small parameter changes, in their prediction of natural frequencies, and especially in their prediction of mode shapes.

Typically, structural modelling is done with state space based methods, which were originally applicable only to structures with a few flexible modes in the bandwidth. Various tools have been developed in an attempt to increase both the number of modes within the bandwidth, and the extent of uncertainty that the control design techniques are capable of dealing with. Alternatively, an acoustic approach suitable for modally dense structures could be used. With this philosophy, there has been much research on the use of wave-based models for use in structural control. (See for example references [2–5], and the references therein.) Here the assumption is that the local dynamics of the structure near an actuator can be accurately modelled, and that an effective control system can be derived based only on this information. A more general approach with similar philosophy is to use the de-reverberated driving point mobility of the structure [6, 7]. Only that part of the response which is due to the local dynamics is retained in the model. This model will be discussed briefly in the next section. Wave or de-reverberated mobility approaches deal with the modally dense frequency region of a structure well, but do not take advantage of additional information about well known low-frequency modes. If necessary, a HAC/LAC architecture [8] could be used to combine a compensator designed for these modes with one designed using a wave or de-reverberated model.

The control design approach must be suitable for the local model being used. Of particular relevance here are the optimal control approaches of Miller et al. [2], and of
MacMartin and Hall [6]. Using Wiener–Hopf techniques to ensure causality, Miller et al. maximize the frequency weighted power dissipation associated with the control. The drawback to this approach is that it will allow power to be generated at some frequencies in order to achieve greater power dissipation at other frequencies, potentially destabilizing the structure. Since the driving point mobility of a structure is positive real, stability can be guaranteed by requiring that the compensator be positive real. Approximating the optimal compensator with a positive real form thus guarantees a stable closed loop, but is sub-optimal, because the positive real constraint is applied in a somewhat ad hoc manner. MacMartin and Hall [6] minimize the maximum value over frequency of the power flow into the structure, resulting in power dissipation at all frequencies, and a positive real compensator. This approach will be summarized in section 3, and parallels with the technique of reference [2] will be illustrated. Indeed, it is possible to solve the frequency weighted power dissipation problem of reference [2] in state space, using some of the results developed in reference [6].

Experiments were performed on a pinned–free beam in bending, a torque motor and collocated angular rate sensor at the pinned end of the beam being used. The experimental results verify the performance that can be achieved by modelling the structure with its de-reverberated mobility, and minimizing the worst case power flow. Several compensators are designed with this approach, and their performance is compared with that of velocity feedback. Previous experiments on the same structure have demonstrated the $H_2$ wave approach of Miller et al. [3].

2. MODELLING

In this section the de-reverberated mobility model for control design for uncertain modally dense systems [6] is reviewed. A modal model may not be useful in this case, since the detailed information it contains is often incorrect, and may also be unimportant. In the presence of significant uncertainty, the model information may be uncertain, but the local dynamics near an actuator can still be well modelled. The dynamics of the structure between an actuator and sensor which are separated by many wavelengths of the disturbance are, however, unknown due to the uncertain phase. Thus for broadband control, it is reasonable to require that feedback only be used between collocated sensors and actuators.

For some arbitrary structure, as shown in Figure 1, insight into the nature of the problem can be obtained from a wave perspective. Various disturbances are created at certain points in the structure and propagate through it. At any point in the structure, such as at an actuator, the disturbance will be scattered. In general, each of the resulting outgoing disturbances will eventually affect any global cost criterion. Thus from the perspective of the actuator, without a detailed and accurate description of how each wave propagates, its goal should be to minimize the energy of each of these outgoing disturbances. Since the scattering behaviour is a function of only the local dynamics, this goal can be achieved with only a local model of the structure.

One approach to obtaining such a model is through the use of waves. However, it may be difficult to obtain a useful wave description for many complicated structures. An alternative to the wave approach for obtaining a local model is to represent the structure by its de-reverberated driving point mobility [9]. The driving point mobility is the transfer function between two variables, the product of which is the power flow into the structure. The response at a point can be considered to be the sum of two parts: a direct field, due to the local dynamics; and a reverberant field, which is caused by energy reflected back
from other parts of the structure. The term "de-reverberated" implies that the "reverberant" part of the response has been removed before computing the mobility. It should be possible to model the direct field more easily and accurately than the reverberant field, as it depends on only a few parameters, while the reverberant field depends on the entire structure. For the same reason, it is the reverberant field that contains greater detail, and requires more degrees of freedom to model. Thus, by using the de-reverberated mobility, a lower order model can be used that is based only on the details of the structure which can be accurately modelled.

One method by which the de-reverberated mobility may be approximated is through the use of the cepstrum [9] of the impulse response. This procedure involves taking the inverse Fourier transform of the log of the complex spectrum, windowing this to remove the reverberant part, and transforming back to the frequency domain to yield the de-reverberated impulse response. A simpler approach is based on the observation that the effect of neglecting the reverberant field is to smooth out the transfer function. If no energy returns from beyond some closed surface surrounding the actuator, then this is equivalent to the structure beyond this surface either being infinite in extent, or having perfectly absorbing boundary conditions. This has been shown to be equivalent to replacing the log magnitude of the original transfer function with its mean [10, 11]. Thus another way to compute the de-reverberated mobility is simply to take a logarithmic average of the transfer function, with the phase being determined uniquely from the fact that the de-reverberated driving point mobility is positive real. In practice, this method should be adequate. Fitting the result with a rational polynomial gives a model that captures the essential dynamics of the system over a wide frequency range that encompasses many modes, with only a small number of poles and zeroes. Further details on determining the de-reverberated mobility can be found in references [6, 7].

This approach can be easily applied to arbitrarily complex structures, since all that is needed is the input/output behaviour at the driving point, which may be found from experimental data, calculated from some nominal model, or found analytically, perhaps even from a wave model. Indeed, for simple structures such as the beam in bending used in the experiment, the resulting model is equivalent to the local wave model of reference [2].
The de-reverberated mobility model is not intended to represent the structure accurately; it clearly fails in this regard. However, it is a useful model for the design of control systems for the structure. While the resonant and anti-resonant details of the full reverberant mobility are not explicitly modelled, the reverberant field is composed of waves, the behaviour of which is governed by the local dynamics of the controlled junction each time they pass through it. Thus if the local dynamics can be appropriately modified based on a local model, then the complete reverberant field can be controlled.

3. CONTROL DESIGN

In the previous section the modelling approach used was summarized, and in this section the design of the control system for this model is to be examined. All of the techniques that will be examined rely on an optimization of the power flow, maximizing in an appropriate sense the dissipation associated with the control system. For a lightly damped system, the power flow gives a measure of both the performance achieved, and the degree of stability.

Miller et al. [2] minimized the $H_2$ norm of the power flow, using a Wiener–Hopf procedure. The same problem can be solved in state space by using a Linear-Quadratic-Gaussian (LQG) algorithm. In either case, some assumptions are required about the power spectral density of the disturbance entering the junction. In the actual structure, this is related to the control action through the disturbance that previously departed the junction. With only a local model, however, it is assumed to be constant and independent of the control, and thus the resulting compensator may allow power to be added at some frequencies. This problem can be avoided by minimizing the power flow in an $H_\infty$ setting. For the open-loop system, the power removed by the controller at each frequency is zero, and the closed loop is guaranteed to be no worse.

Define $G(s)$ to be the de-reverberated driving point mobility, and assume some disturbance input $d$ to be additive at the output. Then the output $y$ is related to the input $u$ and the disturbance via

$$y(s) = G(s)u(s) + d(s).$$ \hspace{1cm} (1)

The disturbance $d$ in this equation can be thought of as originating from two sources; the original disturbance input to the real structure, and the reverberant field ignored in the modelling process.

The instantaneous power flow into the structure is the product of the input $u(t)$ and the output $y(t)$, since $G(s)$ is a mobility. The average power flow can be expressed as a time integral of the instantaneous power flow [12], and, by making use of Parseval’s theorem, this can be transformed into the frequency domain:

$$P_{\text{ave}} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)^T u(t) \, dt = \int_{0}^{\infty} \text{Tr} \left[ \Phi_{uu}(\omega) + \Phi_{yu}(\omega) \right] \frac{d\omega}{2\pi}.$$ \hspace{1cm} (2, 3)

The integrand in equation (3) represents the steady state, or average, power flow into the structure as a function of frequency. The average power flow at each frequency can then be defined as

$$P(\omega) = \text{Tr} \left[ \Phi_{uu}(\omega) + \Phi_{yu}(\omega) \right].$$ \hspace{1cm} (4)

The control law is assumed to be of the form

$$u(s) = -K(s)y(s).$$ \hspace{1cm} (5)

Solving for the control in terms of the disturbance from equation (1) gives

$$u = -(I + KG)^{-1}Kd = Hd,$$ \hspace{1cm} (6, 7)
where the explicit dependence on the Laplace transform variable has been dropped. From these equations, the equivalent feedback $K$ is related to the disturbance feedforward $H$ via

$$K = -H(I + GH)^{-1}. \tag{8}$$

Upon using equations (1), (6) and (7), equation (4) yields that the cross-spectral density of $u$ and $y$ is

$$\Phi_{uv} = (I + GH)\Phi_{dd}H^H \tag{9}$$

where $\Phi_{dd}$ is the power spectral density of the disturbance. The average power flow at each frequency is then

$$\mathbb{P}(\omega) = \text{Tr} \{ \Phi_{dd} [H^H(I + GH) + (I + GH)^H H] \}. \tag{10}$$

### 3.1. Unconstrained Optimum

The simplest optimization approach is to minimize the power flow at each value of the Laplace transform variable $s$. Equation (10) is valid only on the $j\omega$ axis, and must first be extended analytically to the remainder of the complex plane. The analytic continuation of the Hermitian operator is the *parahermitian conjugate* [13], denoted $(\cdot)^\sim$, and defined as

$$F^\sim(s) = F(-s)^T. \tag{11}$$

Since $F^\sim(j\omega) = F^H(j\omega)$, this notation will be used in place of the Hermitian operator throughout the rest of the paper. Optimizing the expected power flow at each point in the complex plane yields

$$K_{opt} = (G^\sim)^{-1}, \tag{12}$$

which is independent of the disturbance spectrum $\Phi_{dd}$.

This compensator extracts the maximum possible power from the structure at every frequency. This result is not new; it corresponds to the impedance matching condition found, for example, in reference [14]. The maximum energy dissipation is obtained if the impedance of the compensator is the complex conjugate of the impedance of the load, which in this case is the rest of the structure.

Unless the de-reverberated mobility is a constant, however, the compensator in equation (12) is non-causal, and cannot be implemented. If this compensator could be implemented, all the poles could be moved arbitrarily far into the left half-plane. Instead, the best causal compensator must be found.

### 3.2. Causal Optimum—$\mathcal{H}_\infty$ Approach

To guarantee dissipation at all frequencies, the worst case power dissipation will be minimized over the set of causal compensators: hence a minimax optimization of the power flow into the structure. This can be cast as an $\mathcal{H}_\infty$ minimization problem. First, however, the disturbance should be normalized to provide the same amount of power available to be dissipated at each frequency. This provides the designer with complete control over the relative importance of one frequency range to another, by removing any inherent frequency weighting from the problem.

With the optimal non-causal compensator from equation (12), the closed loop power flow into the structure is

$$\mathbb{P} = -\text{Tr} [\Phi_{dd} (G + G^\sim)^{-1}]. \tag{13}$$

Introduce a scaled disturbance $w$ related to the original disturbance $d$ via

$$d = G_0 w. \tag{14}$$
Then if the input $w$ has unit magnitude at a certain frequency, the optimal non-causal compensator will dissipate unit power at this frequency, provided that the transfer function $G_0$ is the co-spectral factor of $G + G^-$, given by

$$G_0G_0^- = G + G^-.$$  \hspace{1cm} (15)

The block diagram for the resulting system is shown in Figure 2, and the system (equation (1)) becomes

$$y(s) = G(s)u(s) + G_0(s)w(s).$$ \hspace{1cm} (16)

Now, consider the problem of finding a causal compensator that will minimize the worst case power flow in equation (4). This quantity represents the power flow into the structure, which will be negative for any energy-absorbing (and hence guaranteed stabilizing) controller. In order to cast this as an $\mathcal{H}_\infty$ optimization, the performance index must be positive definite. Since the best causal compensator can dissipate no more power than the non-causal optimum, positive definiteness will be assured if the disturbance power $\Phi_{ww}(\omega)$ is added to the cost. Also note that $\Phi_{ww}(\omega) = F(\omega)$ is the same as $u^-\omega y(\omega)$ for $w^-\omega w(\omega) = F(\omega)$. Thus the cost at each frequency is

$$\text{Cost}(\omega) = w^-w + u^-y + y^-u = |G_0^-u + w|^2 = |G_1G_0^-u + G_1w|^2,$$ \hspace{1cm} (17, 18a, b)

where

$$G_1(s) = \Delta(G_0^-)(s)/\Delta(G_0(s))$$ \hspace{1cm} (19)

and $\Delta(\cdot)$ is the characteristic polynomial of the transfer function $(\cdot)$. The inner function $G_1$ does not change the cost, and is included in order to represent the cost in terms of stable transfer functions.

From equation (18), the relevant output that should be minimized is

$$z = G_1G_0^-u + G_1w.$$ \hspace{1cm} (20)

Upon combining this with the system equation (16), the result can be written as a standard $\mathcal{H}_\infty$ problem [15]:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_1I & G_1G_0^- \\ G_0 & G \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$ \hspace{1cm} (21)

The compensator from $y$ to $u$ that minimizes the $\mathcal{H}_\infty$ norm of the transfer function from $w$ to $z$ will minimize the maximum power flow into the structure.

In general, it may be desirable to weight some frequency ranges more heavily than others, while still requiring that power be removed at all frequencies. This could be because there is a known disturbance source in a certain range, because structural modes are less well damped within this range, or because the performance requirements put more emphasis on this range. Similarly, there will usually be some frequency beyond which performance is not required, and the weighting can also be chosen to reflect this.

![Figure 2. System block diagram.](image-url)
The manner in which the weighting is introduced into the problem must be such that if power is added to the structure somewhere, the resulting cost will be worse than the open-loop cost. Hence, rather than weighting the sum of the disturbance input power and the power input by the control, as in equation (17), define the cost to be the sum of the disturbance power and some frequency weighted control power, as

$$\text{Cost}(\omega) = w^\top w + W_1^\top (u^\top y + y^\top u) W_1.$$  \hspace{1cm} (22)

The only constraint on the weighting function $W_1$ is that its magnitude be less than or equal to unity at all frequencies. If this is the case, then the right side of equation (22) can again be factored as $\text{Cost}(\omega) = |z|^2$ and therefore written as a standard $\mathcal{H}_\infty$ problem. The performance variable $z$ is given by

$$z = \begin{bmatrix} W_1 (G_0^\top u + w) \\ W_2 w \end{bmatrix},$$  \hspace{1cm} (23)

where $W_2$ is defined by the relationship

$$|W_1|^2 + |W_2|^2 = 1.$$  \hspace{1cm} (24)

Note that as desired, the open-loop cost is unity everywhere, and the cost is greater than unity at any frequency where power is added to the structure. Thus as before, a closed-loop cost of less than unity guarantees closed-loop stability.

One of the properties of $\mathcal{H}_\infty$ compensators is that at the optimum, the closed-loop transfer function being minimized is a constant function of frequency, equal to some number $\gamma$ [15]. From this, and equation (22), the fraction of the incoming power that is dissipated by the compensator can be related to $\gamma$ and the weighting function as

$$\mathcal{P}(\omega) = (1 - \gamma^2)/|W_1|^2.$$  \hspace{1cm} (25)

This provides some insight into how to select $W_1$. Where $W_1$ is small, a greater amount of control effort is required to reduce the cost than before, and thus there is more power removed. Hence, in order to emphasize some frequency range more heavily, the weighting function $W_1$ should be chosen to be smaller within that region.

The cost in equation (17) or (22) can also be modified to include a penalty on the control effort, $pu^\top u$. The problem (21) is modified to include an additional output in the vector $z$, corresponding to $\sqrt{p} u$. Similarly, it is straightforward to modify the problem (21) to include sensor noise. An additional disturbance input is included in the vector $w$, which affects only the sensor output $y$.

The final result of this approach is a positive real compensator, which is guaranteed stabilizing for any positive real plant. However, if there are any time delays, actuator or sensor dynamics, or if the actuator and sensor are not truly collocated and dual, then the structure will not be positive real at all frequencies. Stability can still be guaranteed if the complementary sensitivity is bounded above by the inverse of the difference of the true structure from positivity, as noted by Slater [18].

The calculation of the optimal compensator for the $\mathcal{H}_\infty$ problem is most easily performed in state space [16]. The algorithms for computing $G_0$, $G_1 G_0^\top$ and $W_2$ are given in references [6] and [7]. Each of these problems is related to a spectral factorization, the solution to which can be found from a Riccati or Lyapunov equation [15].

3.3. CAUSAL OPTIMUM—$\mathcal{H}_2$ APPROACH

For structures for which the local wave model of Miller et al. [2] can be identified, this model can be represented in the form of equation (1). The $\mathcal{H}_2$ power flow minimization
in reference [2] was constrained to be causal by using a Wiener–Hopf [17] approach. A similar Wiener–Hopf solution can be found in the current framework.

First, introduce the notation

$$\Phi = \Phi_R \Phi_L, \quad \Phi = \Phi_+ + \Phi_-$$  \hspace{1cm} (26, 27)

for the right half-plane analytic and left half-plane analytic factors of $\Phi$, and the positive and negative time parts of $\Phi$ respectively. Both of these types of spectral factorizations can be solved in state space with the solution to a Riccati or Lyapunov equation [15].

With disturbance feedforward $u = H d$, the frequency weighted power flow being minimized is given by equation (10) as

$$J = \int_{-\infty}^{\infty} \{\Phi_{dd} [H^{-1}(I + GH)^* + (I + GH)^{-1} H]\} \, d\omega. \hspace{1cm} (28)$$

The first order variation in $J$ with respect to $H$ is

$$\delta J = 2 \int_{-\infty}^{\infty} \delta H^{-1} ((G + G^*) H + I) \Phi_{dd} \, d\omega. \hspace{1cm} (29)$$

This should be zero for all admissible variations $\delta H^{-1}$. To ensure causality, $\delta H$ must be right half-plane analytic (RHP), and then equation (29) is zero provided that

$$((G + G^*) H + I) \Phi_{dd} = a_L$$  \hspace{1cm} (30)

for some arbitrary left half-plane analytic (LHP) function $a_L$. Solving for the RHPA compensator that satisfies equation (30) yields the optimal disturbance feedforward compensator as

$$H = -(G + G^*)_R^{-1} [(G + G^*)_L^{-1} (\Phi_{dd})_R] + (\Phi_{dd})_R^{-1},$$  \hspace{1cm} (31)

from which the feedback law $u = -K y$ can be determined via equation (8). Note that the quantity $(G + G^*)_R$ in this equation is precisely $G_0$ from equation (15).

Miller et al. [2] specified the power spectral density of the incoming wave modes $w_i$, while this solution requires the power spectral density of the disturbance $d$. The disturbance $d$ is the disturbance in the generalized velocity caused by both the incoming and outgoing wave modes. Thus

$$d = (j \omega) T (Y_{ui} + Y_{uo} S) w_i.$$  \hspace{1cm} (32)

$Y_{ui}$ and $Y_{uo}$ are partitions of the transformation matrix from wave mode variables to physical variables, with $Y_{ui}$ relating the displacement vector $u$ to the incoming wave mode vector $w_i$, and $Y_{uo}$ relating $u$ to the outgoing wave modes $w_0$. $S$ is the scattering matrix of the junction relating outgoing wave modes to incoming wave modes. The matrix $T$ is present to select the appropriate elements of the displacement vector $u$ corresponding to each element of the disturbance $d$, and the factor of $j \omega$ is required since $d$ is a velocity and $u$ is a displacement. The power spectral density of $d$ can be easily related to that of the incoming wave modes $w_i$ from this equation.

The Wiener–Hopf optimization problem is also equivalent to a standard LQG problem [17]. By using the results of the previous subsection, the $H_2$ problem can be solved more easily with this approach. The cost $J$ is proportional to the $H_2$ norm of the power flow,

$$J = \int_{-\infty}^{\infty} (u^* y + y^* u) \, d\omega,$$  \hspace{1cm} (33)
and, as in equation (17), the addition of the constant \( w \rightarrow w \) to the integrand does not change the problem, so

\[
J = \int_{-\infty}^{\infty} (z - z) \ d\omega = \| z \|_2^2,
\]

with \( z \) given by equation (20). Hence, the \( \mathcal{H}_2 \) optimal compensator is that which minimizes the \( \mathcal{H}_2 \) norm of the transfer function from \( w \) to \( z \) in the standard problem (21). This is very similar to the \( \mathcal{H}_\infty \) approach; the norm used in the optimization has changed, and the deterministic (but unknown) finite power noise \( w \) has been replaced by a stochastic process, but the set-up is otherwise identical.

The power spectral density of \( w = G_0 d \) can be related to that of \( d \), and therefore to that of the incoming wave modes \( w_i \), by equation (32). This PSD can also be used to introduce frequency weighting into the problem; more importance is attached to a certain frequency range by increasing the power available to be dissipated in that range.

As was noted in reference [2], the \( \mathcal{H}_2 \) approach suffers from the fact that it does not guarantee a stabilizing compensator. That the state space LQG method presented here yields the same results as the Wiener–Hopf approach in reference [2] will be demonstrated in the next section.

4. EXAMPLE

The approach described in the previous sections can be demonstrated in the design of compensators for a pinned–free Bernoulli–Euler beam with a moment actuator at the pinned end. This structure is chosen to represent that of the experiment described in the next section; as a result the beam properties used in this example will be those of the experiment, given in Table 1. This example also allows a comparison to be made of the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) compensators, and demonstrates that the LQG-based \( \mathcal{H}_2 \) method presented here is equivalent to the Wiener–Hopf method of reference [2].

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<th>Beam dimensions and properties</th>
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The de-reverberated mobility can be found analytically as the transfer function of the “infinitely extended” system. By using a wave approach, the transfer function of a semi-infinite pinned Bernoulli–Euler beam between collocated tip moment and slope rate can be expressed as

\[
G(s) = \sqrt{s} / \sqrt{2(\rho A)^{1/4}(EI)^{3/4}}.
\]

With no causality constraint, the compensator that dissipates the maximum possible power at all frequencies is, from equation (12),

\[
K(s) = \sqrt{2} (\rho A)^{1/4}(EI)^{3/4} / \sqrt{s}.
\]

This is the ideal compensator for the structure, but cannot be implemented. Instead, a number of other compensators can be designed.
Velocity feedback is the simplest of these, and was chosen to compare the optimal designs with a similar existing design approach. To achieve maximum power dissipation at a frequency $\omega_0$, the gain should be as close as possible to the unconstrained compensator at this frequency, so

$$K_1(s) = \sqrt{2}(\rho A)^{1/4}(EI)^{3/4}/\sqrt{\omega_0}.$$  \hspace{1cm} (37)

The second compensator is the $H_\infty$-optimal compensator with unity weighting at all frequencies, given by

$$K_2(s) = \sqrt{2}(\rho A)^{1/4}(EI)^{3/4}/\sqrt{s}.$$ \hspace{1cm} (38)

The analytic derivation of this compensator is shown in the Appendix. The magnitude is the same as that of the unconstrained optimal compensator, but the phase is $-45^\circ$, rather than $+45^\circ$. This compensator was also derived and implemented by Miller and Hall [3], as the fixed form optimal compensator.

In order to test further the $H_\infty$ approach, a weighting function was selected to emphasize a narrow frequency band near 35 rad/s. This corresponds approximately to the frequency of the seventh mode of this beam. The minimum value of $W_1$ in this region was approximately 0.65, and the weighting increased to near unity a factor of $\sqrt{2}$ above and below this frequency. An analytic solution for the compensator in this case would be difficult. However, the plant in equation (35) can be approximated adequately over a wide frequency range with a finite number of alternating poles and zeroes on the real axis, with equal logarithmic spacing. State space methods can then be used to obtain an approximate compensator. For this example, equation (35) was approximated by nine poles and nine zeroes on the negative real axis, between $3.5 \times 10^{-3}$ and $3.5 \times 10^5$ rad/s. The transfer function of this approximation matches the assumed de-reverberated mobility to within 2 degrees of phase and 0.25 dB magnitude for three decades above and below the centre frequency of the weighting function.

The optimal compensator from slope rate to moment was found to be well approximated by the product of the unweighted optimum in equation (38), and a two-pole, two-zero network. This network provided the phase lead that is required so that at the centre of the weighted region, the phase approaches the unconstrained optimal phase of $45^\circ$ (from equation (36)), allowing the compensator to dissipate more power. The optimal poles and zeroes of this network are symmetric about the centre frequency of the weighting function $W_1$, at 35 rad/s. The two free parameters of this network were optimized to minimize the $H_\infty$ norm of the cost. This results in the compensator from slope rate to moment being

$$K_3(s) = 2 \cdot 62 \frac{\sqrt{2}(\rho A)^{1/4}(EI)^{3/4}}{\sqrt{s}} \frac{(s^2 + 38 \cdot 5s + 466)}{(s^2 + 100s + 3210)}. \hspace{1cm} (39)$$

For comparison with these designs, the same model can be used to compute an $H_2$ optimal solution, as the solution of an LQG problem. For comparison with the results of Miller et al. [3], the power spectral density $\Phi_{w_w}$ is chosen to be the same as in reference [3]:

$$\Phi_{w_w} = \frac{a^2}{(s - \omega_n^2)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{a^2}{(s + \omega_n^2)^2}. \hspace{1cm} (40)$$

By using equation (32) and the definitions for $Y_{ww}$, $Y_{wo}$ and $S$ for the pinned beam from reference [3], the tip slope rate can be related to the incoming wave modes by

$$d = 2j\omega(\rho A/ EI)^{1/4}[j\sqrt{\omega} \ \ \sqrt{\omega}]w.$$ \hspace{1cm} (41)

The normalized disturbance $w$ is related to $d$ by equation (32). Using these equations
gives the power spectral density of $w$ as
\[ \Phi_{ww} = b^2 \omega^{5/2}/(s - \omega_n^2)^2(s + \omega_n^2)^2, \]  
for some constant $b$. This is proportional to $\omega^{-5}$ at low frequencies, and to $\omega^{-1.5}$ at high frequencies, with a maximum at $\omega_n$, selected to be 40 rad/s. Thus one would expect the most damping near this frequency, and a much sharper drop in damping for lower frequencies than for higher frequencies.

The $H_\infty$, weighted $H_\infty$, unweighted $H_\infty$ and unconstrained optimal compensators are plotted in Figure 3, as is the velocity feedback compensator. Note that at the frequency weighted most heavily, the $H_\infty$ solution achieves the magnitude of the unconstrained optimal compensator, but not quite the phase, while the $H_2$ optimal solution achieves the phase, but not the magnitude. The $H_2$ compensator calculated by Miller et al. in reference [3] by a wave approach and Wiener–Hopf methods is also plotted for comparison with the $H_2$ compensator calculated here with the LQG method. The agreement indicates that the two approaches are equivalent; the discrepancy at high frequencies is due to the inclusion of a small penalty on control effort in the LQG solution.

![Figure 3](image)

Figure 3. Comparison of compensators. (a) Unconstrained optimum; (b) velocity feedback; (c) unweighted $H_\infty$ design; (d) weighted $H_\infty$ design; (e) LQG $H_2$; (f) Wiener–Hopf $H_2$ design.

The power input to the structure by these compensators is plotted in Figure 4, expressed as a fraction of the available incoming power at each frequency. Thus a value of $-1$ indicates that the maximum possible power is being dissipated, a value of zero means that the compensator does nothing at this frequency, and values larger than zero indicate that power is being added to the structure, which could lead to instabilities. Note that the $H_2$ solution adds power to the structure at certain frequencies, while the $H_\infty$ solutions do not. Furthermore, as expected from the weighting function chosen, the $H_2$ solution treats higher frequencies with more importance than lower, while the $H_\infty$ solution treats both equally.

Both of the $H_\infty$ compensators dissipate some power, and thus provide some damping at all frequencies, whereas velocity feedback is ineffective at sufficiently high and low frequencies. The weighted $H_\infty$ design will also provide better narrow-band damping than velocity feedback, and only slightly worse broadband damping than the unweighted $H_\infty$ design.
Figure 4. Power absorption for unconstrained optimum (a), velocity feedback (b), unweighted $\mathcal{H}_\infty$ design (c), weighted $\mathcal{H}_\infty$ design (d), and $\mathcal{H}_2$ design (e).

5. EXPERIMENT

The $\mathcal{H}_\infty$ and velocity feedback compensators designed in the previous section were implemented on a brass beam suspended in the Space Engineering Research Center laboratory at M.I.T. Previous experiments with this beam include collocated rate feedback and $\mathcal{H}_2$ optimal wave control [3].

5.1. SET-UP

The set-up is shown schematically in Figure 5. The beam is suspended horizontally in the laboratory, with actuation and sensing such that the bending vibration can be controlled. One end is effectively pinned, while the other is free. The properties and dimensions of the beam are summarized in Table 1. The open-loop damping of the first 17 modes, up to a frequency of approximately 30 Hz, averages about 0.3%. A small
amount of passive damping was found to be required in order to maintain closed-loop stability of modes of the system above 1000 Hz. A constrained layer of foam rubber was added to a short section of the beam, which does not appreciably increase the open-loop damping in any of the modes below about 300 Hz, corresponding to about the 55th bending mode of the beam.

Control is applied through a torque motor at the pinned end, and sensing is provided by a linear accelerometer mounted a short distance from this end. The member connecting the sensor to the tip is assumed to be rigid, so that the sensor provides a rotational acceleration measurement collocated with the moment actuator. In practice, this assumption is not quite valid, although it is reasonable in the frequency range of interest.

In addition to the control actuator and sensor, a proof-mass shaker and data acquisition accelerometer were mounted at the free end of the beam. The shaker was mounted to provide a force collocated with the acceleration measurement. The closed-loop transfer function between these two was used as an indication of the performance achieved. Above 30 Hz, this transfer function is dominated by the inertia of the shaker. The poles of the beam are almost cancelled by zeroes, and are thus almost unobservable. As a result, closed-loop transfer functions at the free end were not taken above this frequency.

The signal from the accelerometer at the controlled end was fed through a signal amplifier into an analog computer which contained the compensator program. The output of this was fed through a power amplifier into the moment actuator. The accelerometer signal from the uncontrolled end was fed into a Signology SP-20 Signal Processing Peripheral to record and analyze the response data, and to obtain frequency domain information. An oscilloscope was used to monitor the accelerometer signal so that any instabilities could be quickly identified, and their frequencies determined. Detailed information on the characteristics of the sensors and actuators can be found in reference [3].

5.2. COMPENSATOR IMPLEMENTATION

A detailed model of the beam is not necessary for the experiment; it is sufficient to examine the transfer function from the control actuator to the control sensor. This transfer function is shown in Figure 6. The de-reverberated mobility could be calculated from this transfer function through cepstral analysis, or by averaging. Alternatively, it can be approximated by the theoretical de-reverberated mobility for a pinned end of a beam, given by equation (35). This transfer function is also plotted in Figure 6, and closely approximates the logarithmic average of the measured transfer function in the region of interest. The presence of a rotational inertia at the tip, corresponding to the inertia of that part of the actuator armature and sensor that is fixed to the beam, introduces a roll-off into the transfer function at high frequencies. However, the effect of this inertia was at a sufficiently high frequency so that for the control design, it was assumed to be zero and not modelled. As a result, the previously designed compensators can be used here.

The velocity feedback, unweighted $\mathcal{H}_\infty$, and the weighted $\mathcal{H}_\infty$ compensator designs are all positive real, and thus guaranteed to be stable for any positive real structure. The transfer function from the actuator to the sensor of this beam, however, was not positive real at high frequencies. This is due to the non-collocatedness of the sensor and actuator, the additional dynamics of the sensor and actuator, and any time delays in the system. The system can still be guaranteed to be stable if the complementary sensitivity is bounded above by the inverse of this difference from positivity [18]. So, to be stable, the complementary sensitivity, and therefore the compensator, must roll off at high frequency. Ideally, the compensator design procedure would result in this behaviour automatically. Since this is not the case for either the $\mathcal{H}_\infty$ approach or for rate feedback, the additional roll-off required must be added in an ad hoc manner. Low-pass filters were therefore added to
all three designs, with poles at 500 rad/s. However, this was not sufficient to implement the velocity feedback compensator at its optimal gain without destabilizing high-frequency modes. Indeed, a second low-pass filter was necessary to achieve stability at even 40% of the optimal gain, at which level data was taken. This implemented velocity feedback compensator provides its maximum damping at about 35 Hz.

The available measurement in the experiment was proportional to angular acceleration, and thus a further integration was necessary to obtain angular rate. This integrator was rolled off at d.c. to prevent saturation and drift problems. The second order dynamics were chosen to have a natural frequency of 1 rad/s, and a damping ratio of 0.7071. Finally, a high-pass filter was included to remove the d.c. offset of the accelerometer.

The low-pass and high-pass filters and integrator dynamics are combined into the filter:

$$F(s) = \frac{s}{s^2 + 1.41s + 1} \left( \frac{500}{s + 500} \right) \left( \frac{s}{s + 1} \right).$$  \hspace{3cm} (43)

The implemented compensators between moment and angular acceleration are then

$$K_1(s) = 1.64 \left( \frac{500}{s + 500} \right) F(s), \quad K_2(s) = 24.2 \frac{1}{\sqrt{s}} F(s),$$  \hspace{3cm} (44, 45)

$$K_3(s) = 63.4 \frac{1}{\sqrt{s}} \left( \frac{s^2 + 3.85s + 466}{s^2 + 100s + 3210} \right) F(s).$$  \hspace{3cm} (46)

The circuit used to implement the half integrator $\frac{1}{\sqrt{s}}$ is based on that presented in reference [5]. The approximation is excellent up to about 700 Hz, well above the region of interest. The measured compensator for the weighted $H_\infty$ design is compared with the desired compensator in Figure 7. Good agreement is obtained, except at low frequencies where the dynamics of the integrator and the high-pass filter have a noticeable effect, and at frequencies higher than those shown, where the low-pass filter was added. Similar agreement exists between the measured and desired compensators for the other two cases.

5.3. RESULTS

The closed-loop transfer functions from force at the free end to collocated velocity with the three compensators are compared with the open-loop response in Figure 8. Note
that the spikes present in the data at 16·4, 19·8 and 24·5 Hz correspond to torsional modes of the beam, which are excited by the shaker but are uncontrolled by the moment actuator. The corresponding predicted responses appear in Figure 9. These were calculated from the compensator transfer function by using the phase closure approach of reference [12]. Reasonable agreement is obtained between this prediction and the actual transfer function, although the achieved performance is noticeably better than that predicted. Similar experimental and predicted transfer functions with $\mathcal{H}_2$ optimal compensators can be found in reference [3].

These results confirm the expected advantages of each technique. The unweighted $\mathcal{H}_\infty$ design achieves damping in a broadband region. This is obtained by sacrificing some of the narrow-band damping achieved by velocity feedback. The lowest modes present in the frequency range plotted are damped more heavily by the unweighted $\mathcal{H}_\infty$ compensator than by velocity feedback; one would expect that this would also be true of modes at a sufficiently high frequency. Unfortunately, the hardware limitations noted earlier prevented the free end transfer function from being determined above 30 Hz, where this effect would be apparent. The weighted $\mathcal{H}_\infty$ design achieves excellent narrow-band damping in the desired frequency range, while maintaining some damping everywhere. This is a result of the exact match in magnitude, and close match in phase, with the unconstrained optimal compensator that absorbs all of the incoming power at each frequency. In fact, the modes near 6 Hz can be virtually eliminated if the phase of the compensator is boosted still closer to the unconstrained non-causal optimum, at the expense of performance at other frequencies.

6. CONCLUSIONS

The de-reverberated driving point mobility is a simple but useful model for control design of uncertain, modally dense structures. For simple structures, such as the beam used in this experiment, this is equivalent to a local wave model, but the approach is capable of modelling much more general structures, as it can be determined directly from an experimental transfer function.
The compensator that dissipates the most power possible at every frequency is in general non-causal, and cannot be implemented. Two approaches were examined for obtaining a causal compensator that dissipates power. The $\mathcal{H}_\infty$-optimal solution can be found by using either Wiener-Hopf or LQG techniques. However, this compensator may allow power to be generated at certain frequencies. Another approach is to find the $\mathcal{H}_\infty$-optimal solution which minimizes the maximum power flow into the structure. This compensator dissipates power at all frequencies, and is therefore guaranteed to be stabilizing.

Experimental results demonstrate that the damping that can be achieved with the $\mathcal{H}_\infty$ approach is much greater than that achievable with rate feedback. With no frequency weighting, good broadband damping can be obtained. With a frequency weighting, excellent narrow-band performance can be achieved while some broadband damping is maintained. At the frequency where the best performance is obtained, the compensator closely matches the unconstrained (non-causal) optimum in both magnitude and phase.

One difficulty with the approach is that it does not automatically enforce roll-off that is necessary to deal with high-frequency sensor and actuator dynamics, or non-collocatedness. The required roll-off must be added on an ad hoc basis. After this was done, this approach to modelling and control design successfully added significant damping to many

Figure 8. Experimental open- (dotted) and closed-loop (solid) transfer functions as obtained by using (a) velocity feedback, (b) unweighted $\mathcal{H}_\infty$ design, and (c) weighted $\mathcal{H}_\infty$ design.
Figure 9. Predicted open- (dotted) and closed-loop (solid) transfer functions as obtained by using (a) velocity feedback, (b) unweighted $H_\infty$ design, and (c) weighted $H_\infty$ design.

modes of a laboratory structure, without the large effort in system identification, off-line computation, and compensator complexity that would be required of many control design techniques.

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APPENDIX: BEAM $\mathcal{H}_\infty$ COMPENSATOR

For a pinned beam with de-reverberated mobility of the form $G(s) = \beta \sqrt{s}$, the compensator that minimizes the maximum power flow into the structure can be found analytically. From equation (20) the problem is to find a stable, causal compensator that minimizes the $\mathcal{H}_\infty$ norm of the transfer function from $w$ to $G_1 G_0^- u + G_1 w$ or, equivalently, to $G_0^- u + w$. From the definition of $G_0$ (equation (15)),

$$G_0 G_0^- = \beta \sqrt{s} + \beta \sqrt{-s} = \beta \sqrt{2} \sqrt{s} \sqrt{-s},$$

or

$$G_0(s) = \sqrt{\beta \sqrt{2} s}.$$  \hfill (47, 48)

Since $d = G_0 w$, then, from equation (7),

$$u = H G_0 w.$$  \hfill (49)

The compensator $K$ from $y$ to $u$ will be stable and causal provided that $H$ is also stable and causal. Thus the problem is to find $H$ to minimize

$$\| G_0^- H G_0 + 1 \|_\infty.$$  \hfill (50)
The solution to this, in the notation of Francis [15], is
\[ \gamma = \min_H \| G_0 H G_0 + 1 \|_\infty = \min_H \| \beta \sqrt{2} \sqrt{s} \sqrt{s} H + 1 \|_\infty = \min_H \| \beta \sqrt{2s} H + \sqrt{s} \sqrt{-s} \|_\infty. \] (52-54)

Equation (54) is of the form
\[ \gamma = \min_H \| R - X \|_\infty, \] (55)

where
\[ R = \sqrt{s} \sqrt{-s} = (1+j)/\sqrt{2}, \quad X = -\beta \sqrt{2s} H. \] (56-58)

The problem now is to find \( X \in \mathcal{H}_\infty \) to minimize \( \| R - X \|_\infty \). There are three possible options for the behaviour of \( X(s) \) at the origin. Either \( X \) has a pole at zero, in which case \( \| R - X \|_\infty \) is infinite, \( X \) has a zero, in which case \( \| R - X \|_\infty \approx 1 \), or \( X \) is a constant, with either 0° or 180° phase. In the last case, the smallest value \( |R(0) - X(0)| \) can have is \( 1/\sqrt{2} \), for \( X(0) = 1/\sqrt{2} \). Thus there cannot exist \( X(s) \) for which \( \| R - X \|_\infty < 1/\sqrt{2} \). Since the solution \( X(s) = 1/\sqrt{2} \) results in \( \| R - X \|_\infty = 1/\sqrt{2} \), this must be an optimal solution. From equation (58),
\[ H = -1/2\beta \sqrt{s}, \] (59)

and from equations (6) and (7), the compensator from the output \( y \) to \( u \) is given by
\[ K = 1/\beta \sqrt{s}. \] (60)