Non-normal growth, "Transient growth" & Lyapunov exponents

For a linear system $\dot{x} = Ax$, $x(0) = x_0$

$$x(t) = e^{At}x_0 \qquad \left(= \Phi(t,t_0)\,x(t_0)\right)$$

$$\max_{\|x_0\|\le\epsilon} \|x(t)\|_2 = \max_{\|x_0\|\le\epsilon} \left(x_0^T \Phi(t,t_0)^T \Phi(t,t_0) x_0\right)^{1/2} = \epsilon\,\bar{\sigma}\left(\Phi(t,t_0)\right). \quad \Rightarrow \max_{\|x_0\|\le\epsilon} \frac{\|x(t)\|_2}{\|x(t_0)\|_2} = \bar{\sigma}\left(\Phi(t,t_0)\right)$$

& worst-case "direction" is corresponding singular vector (what is the pattern giving 'optimal excitation') e.g. ENSO, THC, ...

[More generally: $\max x(t)^T X x(t) = \|x\|_X^2$
subject to $x_0^T E x_0 = 1 = \|x_0\|_E^2$.
Lagrange mult $\Rightarrow G^T X G x_0 = \gamma E x_0$; gen. eig. prob

IF A is normal $(A^TA = AA^T)$ (eigenvectors all orthogonal) $(A = V^{-1}\Lambda V,\ V^{-1} = V^T)$

Then $e^{A^Tt}e^{At} = V^{-1}e^{2Re(\Lambda)t}V$

$$\Rightarrow \frac{1}{\epsilon}\max_{\|x_0\|\le\epsilon} \|x(t)\| = e^{\left(\max_\lambda Re(\lambda)\right)t} \quad \text{or} \quad \boxed{\frac{1}{t}\log\left\{\frac{1}{\epsilon}\max_{\|x_0\|\le\epsilon}\|x\|\right\} = \max_\lambda Re(\lambda)}$$

(operator commutes with adjoint) $\|\Sigma^* x\| = \|\Sigma x\| \ \forall x$

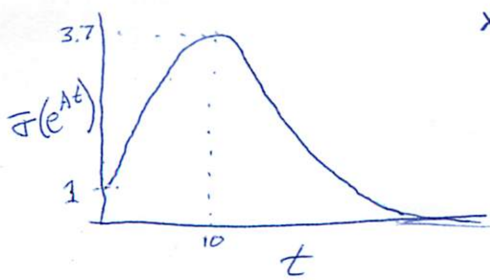
IF A is non-normal

Then this is not true

~~(although if eigenvectors unique, then $\lim_{t\to\infty}$ ... still holds)~~

E.g. $A = \begin{bmatrix} -.1 & 1 \\ 0 & -.1 \end{bmatrix}$ (non-trivial Jordan form, eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$)

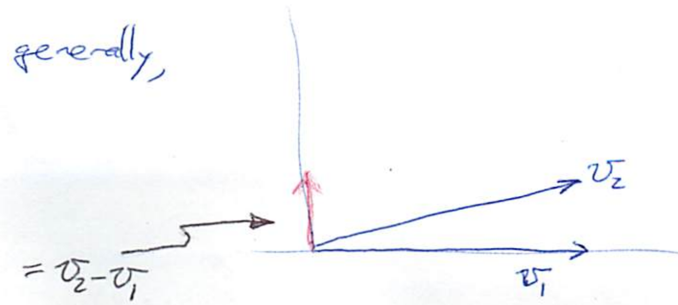
Then



$x_1(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{-.1t}$

(This is extreme case, but more generally,

...



For a nonlinear system $\dot{x} = F(x)$.

$$X(t) = \Phi(x_0; t, t_0)$$

Then analogous to linear system where for large t

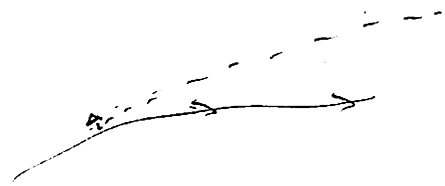
$$\frac{1}{t} \log \frac{|x(t)|}{|x(0)|} = \mathrm{Re}(\lambda_i)$$

(for $X(0)$ in direction of eigenvector v_i; with maximum value max...)

Define
$$M_{ij} = \frac{\partial \Phi^{(i)}(x_0; t, t_0)}{\partial x_0^{(j)}} = M_{ij}(x_0, t, t_0)$$

Then (maximum) Lyapunov exponent is (cf FTLE)

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \lim_{\epsilon \to 0} \log \frac{\| \delta x(t) \|}{\| \delta x_0 \|}$$



$$= \lim_{t \to \infty} \frac{1}{t} \log \bar{\sigma}(M)$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \sqrt{\lambda(M^T M)}$$

$$= \log \Lambda_i \quad \text{where } \Lambda_i \text{ are eigenvalues of } \lim_{t \to \infty} (M^T M)^{1/2t}$$

i.e. The average (over time) growth rate of perturbations about a trajectory.

. If MLE < 0 Then solution is asymptotically Lyapunov stable

(For discrete-time, $x_{k+1} = f(x_k)$,

Then: $\Lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \log \left| \frac{\partial F}{\partial x} \right|$)

MLE > 0 ⇒ unstable (chaotic if bounded)