CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 140b
Problem Set #3

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1. From the definition of the Lie bracket, $[a, b] = -[b, a]$ and $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

   (a) Consider any two independent vector fields x and y, how many independent bilinear (e.g. $[x, y]$ and $[y, x]$ are clearly linearly dependent), trilinear (e.g. $[x, [x, y]]$, $[y, [y, x]]$, etc.), and quadrilinear terms (e.g. $[x, [x, [x, y]]]$, $[[x, y], [x, y]]$, etc.) are possible?

   (b) Let $V = \mathbb{R}^n$ be a Lie algebra (so $[a, b] \in V$ for any $a, b \in V$.) For any basis $\{v_1, \ldots, v_n\}$ of V, there exist constants C_{ij}^k, $i, j, k \in \{1, \ldots, n\}$ such that

   $$[v_i, v_j] = \sum_{k=1}^{n} C_{ij}^k v_k, \quad \forall i, j \in \{1, \ldots, n\}$$

   Show that

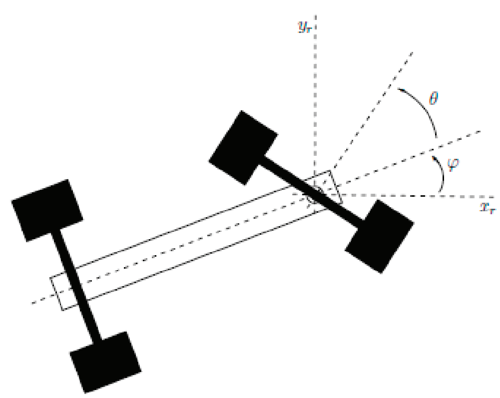
   i. $C_{ij}^k = -C_{ji}^k$ and

   ii. $\sum_{k=1}^{n} C_{ij}^k C_{kl}^m + C_{li}^k C_{kj}^m + C_{jl}^k C_{ki}^m = 0$

2. The dynamics of the car shown below are described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

   with u_1 the driving input and u_2 the steering input. Show that this system is controllable.



3. Consider a nonlinear system that depends on slowly varying parameters $p(t)$,

$$\dot{x} = f(x, u, p)$$

with gain-scheduled control law

$$u = k(x, p)$$

that locally stabilizes the equilibrium $x = 0$ for any fixed values for the parameters p. Show that the system is also stable with time-varying p provided $\|\dot{p}\|$ is sufficiently small. (Note that, conversely, for any given μ such that $\|\dot{p}\| \leq \mu$ it is possible to choose $k(x, p)$ to guarantee stability, although practically this may require large u.)

Hint: You can choose a Lyapunov function of the form $V = x^T P(\mu)x$ for the linearized closed-loop system at any fixed μ and show that this is also a Lyapunov function with time-varying μ.