CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 140b
Problem Set #7

D. MacMartin and R. Murray
Spring 2014

Issued: 22 May 14
Due: 29 May 14

1. Complete the derivation sketched in class showing that for the driftless system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

with

$$u_1 = +1 \quad u_2 = 0 \quad \text{for} \quad 0 \le t < \epsilon \tag{1}$$
$$u_1 = 0 \quad u_2 = +1 \quad \text{for} \quad \epsilon \le t < 2\epsilon \tag{2}$$
$$u_1 = -1 \quad u_2 = 0 \quad \text{for} \quad 2\epsilon \le t < 3\epsilon \tag{3}$$
$$u_1 = 0 \quad u_2 = -1 \quad \text{for} \quad 3\epsilon \le t < 4\epsilon \tag{4}$$

then to second order in ϵ, $x(4\epsilon) = \epsilon^2[g_1, g_2]$.

2. From the definition of the Lie bracket, $[a, b] = -[b, a]$ and $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

   (a) Consider any two independent vector fields x and y, how many independent bilinear (e.g. $[x, y]$ and $[y, x]$ are clearly linearly dependent), trilinear (e.g. $[x, [x, y]]$, $[y, [y, x]]$, etc.), and quadrilinear terms (e.g. $[x, [x, [x, y]]]$, $[[x, y], [x, y]]$, etc.) are possible?

   (b) Let $V = \mathbb{R}^n$ be a Lie algebra (so $[a, b] \in V$ for any $a, b \in V$.) For any basis $\{v_1, \ldots, v_n\}$ of V, there exist constants C_{ij}^k, $i, j, k \in \{1, \ldots, n\}$ such that

$$[v_i, v_j] = \sum_{k=1}^{n} C_{ij}^k v_k, \quad \forall i, j \in \{1, \ldots, n\}$$

   Show that

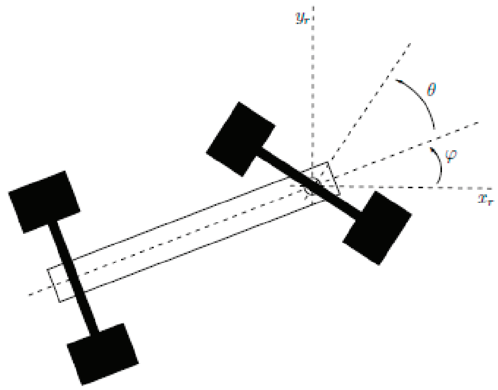
   i. $C_{ij}^k = -C_{ji}^k$ and

   ii. $\sum_{k=1}^n C_{ij}^k C_{kl}^m + C_{li}^k C_{kj}^m + C_{jl}^k C_{ki}^m = 0$

3. The dynamics of the car shown below are described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

with u_1 the driving input and u_2 the steering input. Show that this system is controllable.



4. Consider a nonlinear system that depends on slowly varying parameters $p(t)$,

$$\dot{x} = f(x, u, p)$$

with gain-scheduled control law

$$u = k(x, p)$$

that locally stabilizes the equilibrium $x = 0$ for any fixed values for the parameters p. Show that the system is also stable with time-varying p provided $\|\dot{p}\|$ is sufficiently small. (Note that, conversely, for any given μ such that $\|\dot{p}\| \leq \mu$ it is possible to choose $k(x, p)$ to guarantee stability, although practically this may require large u.)

Hint: You can choose a Lyapunov function of the form $V = x^T P(p) x$ for the linearized closed-loop system at any fixed p and show that this is also a Lyapunov function with time-varying p.

5. *Note that Homework 8 will ask you to design a controller for this system using feedback linearization, then backstepping, and then sliding-mode control.*
Show that the there is an input- and state-transformation that feedback-linearizes the system:

$$\dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - x_1^3$$
$$\dot{x}_2 = u$$

Note that you can find such a transformation if you can find an output $y = h(x)$ with relative degree 2; obvious candidates to try first are $y = x_1$ and $y = x_2$.