CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 140b
Problem Set #2

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1. A planar pendulum (in the x-z plane) of mass m and length ℓ hangs from a support point that moves according to $x = a\cos(\omega t)$. Find the Lagrangian, the Hamiltonian, and write the first-order equations of motion for the pendulum.

2. **Perko, Section 3.3, problem 5.** Show that

$$\begin{aligned}\dot{x} &= y + y(x^2 + y^2)\\ \dot{y} &= x - x(x^2 + y^2)\end{aligned}$$

is a Hamiltonian system with $4H(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2)$. Show that $dH/dt = 0$ along solution curves of this system and therefore that solution curves of this system are given by

$$(x^2 + y^2)^2 - 2(x^2 - y^2) = C$$

Show that the origin is a saddle for this system and that $(\pm 1, 0)$ are centers for this system. (Note the symmetry with respect to the x-axis.) Sketch the two homoclinic orbits corresponding to $C = 0$ and sketch the phase portrait for this system, noting the occurrence of a compound separatrix cycle.

3. **Perko, Section 3.6, problem 4.** Consider the Hamiltonian system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x\\ \dot{z} &= w\\ \dot{w} &= -z + z^2\end{aligned}$$

with Hamiltonian $H(x, y, z, w) = (x^2 + y^2 + z^2 + w^2)/2 - z^3/3$

(a) Show that this system has two periodic orbits for $H > 1/6$, given by

$$\begin{aligned}\Gamma_0 : &\quad \gamma_0(t) = (k\cos t, -k\sin t, 0, 0)\\ \Gamma_1 : &\quad \gamma_1(t) = (\sqrt{k^2 - 1/3}\cos t, -\sqrt{k^2 - 1/3}\sin t, 1, 0)\end{aligned}$$

which lie on the surface $S : x^2 + y^2 + w^2 + z^2 - 2z^3/3 = k^2$ for $k^2 > 1/3$.

(b) For the projective transformation (from the point $(k, 0, 0, 0) \in S$ onto $\mathbb{R}^3$)

$$Y = \frac{y}{k - x} \quad Z = \frac{z}{k - x} \quad W = \frac{w}{k - x}$$

show that Γ_0 gets mapped onto the Y-axis and Γ_1 gets mapped onto the ellipse $(Z - 3k)^2 + 3Y^2 = 9k^2$. (Note for the latter that one can express x and y in terms of Y and Z, and substitute into the expression for S to obtain a quadratic relationship between Z and Y for solutions lying on S. I did this quickly and do not guarantee that the specific ellipse equation is correct!)

(c) Show that Γ_0 has four zero characteristic exponents and that Γ_1 has characteristic exponents $\lambda_1 = \lambda_2 = 0$, $\lambda_{3,4} = \pm 1$.

You may use the observations that (i) the fundamental matrix solution for a block-diagonal matrix is itself block-diagonal (that is, you will get 2×2 blocks when you linearize the system about the respective equilibrium points) and (ii) the fundamental matrix solution for a a 2×2 system with purely imaginary eigenvalues will be of the form $\Phi(t) = R_t e^{Bt}$ with $B = 0$, i.e., with characteristic exponents equal to zero.

4. Consider the stability of the Lagrange points in the Sun-Planet system (with some simplifying steps, including restricting motion to a plane). With the mass of the sun as $1 - \mu$ and planet as μ, then in the rotating coordinate system with origin at the centre of mass, the Hamiltonian as a function of normalized planar position (x, y) and momenta (p_x, p_y) is given by

$$H = \frac{(p_x + y)^2 + (p_y - x)^2}{2} + \Omega(x, y)$$

(a) Show that the equilibrium points are given by the critical points of the function Ω.

(b) To explore the linearized dynamics it is sufficient to retain only quadratic terms in H (why?). For the collinear Lagrange points this leads to

$$H = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - ax^2 + by^2$$

for $a > 0$ and $b > 0$. E.g., for the L1 point in the Sun-Jupiter system then a=9.892, b=3.446. With these numerical values, describe the linearized dynamics about this Lagrange point; are periodic orbits stable?

It is not necessary to know the form of the potential function to answer the above. However, if you are curious, Ω is given by

$$\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1 - \mu)}{2}$$

with $r_1 = \sqrt{(x + \mu)^2 + y^2}$ and $r_2 = \sqrt{(x - 1 + \mu)^2 + y^2}$. Solving for the critical points of Ω leads to 5 equilibrium solutions for the Lagrange points L1 through L5; points labeled L1, L2, and L3 are collinear with $y = 0$. If you are interested, the collinear solutions are not too difficult to solve for numerically for some value of μ (e.g., $\mu = 0.0009537$ for Jupiter).