# Optimal Control

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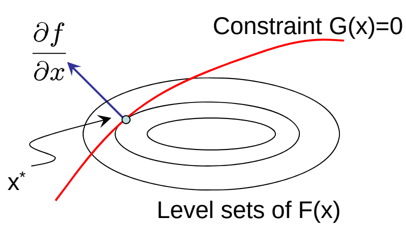
Ae 240

## Constrained Function optimization

Given $F : \mathbb{R}^n \to \mathbb{R}$ and $G_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1 \dots k$, then find $x^* \in \mathbb{R}^n$ such that $G_i(x^*) = 0\, \forall i$ and $F(x^*) \geq F(x)$ for all $x$ satisfying $G_i(x) = 0\, \forall i$.




- Then at optimal solution, gradient of F(x) must be parallel to gradient of G(x):

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0$$

- More generally, define:

$$\tilde{F} = F + \lambda^T G$$

- Then a necessary condition is:

$$\frac{\partial \tilde{F}}{\partial x}(x^*) = 0$$

- The *Lagrange multipliers* $\lambda$ are the sensitivity of the cost to a change in G

# Optimal Control of Systems

Given a system:

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$$

with $x(0) = x_0$. Then find

$$u = \text{argmin}_{u\in\Omega} \left( \int_0^T L(x,u)\text{d}t + V(x(T),u(T)) \right)$$

- Easy to include additional constraints on control *u*, and on state (along trajectory or at final time)
- Final time *T* may or may not be free (I'll only derive for fixed *T*)
- Define $z = \begin{bmatrix} x \\ u \end{bmatrix}$, then this is a standard problem of minimizing J(z) subject to constraints G(z)=0

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# Solution approach

- Add Lagrange multiplier $\lambda$(t) for dynamic constraint
  - And additional multipliers for terminal constraints or state constraints
- Form augmented cost functional:

$$\begin{aligned}\tilde{J}(x,u,\lambda) &= J(x,u) + \int_0^T \lambda^T (f(x,u) - \dot{x})\,\text{d}t \\ &= \int_0^T \left( L(x,u) + \lambda^T (f(x,u) - \dot{x}) \right) \text{d}t + V(x(T)) \\ &= \int_0^T \left( H(x,u) - \lambda^T \dot{x} \right) \text{d}t + V(x(T))\end{aligned}$$

- where we introduce the Hamiltonian: $H \triangleq L + \lambda^T f$
- A necessary condition for optimality is that $\delta\tilde{J}$ vanishes for any perturbation in x, u, or $\lambda$ about optimum:

$$x(t) = x^*(t) + \delta x(t)$$
$$u(t) = u^*(t) + \delta u(t)$$
$$\lambda(t) = \lambda^*(t) + \delta\lambda(t)$$

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# Derivation…

$$\begin{aligned}\delta\tilde{J} &\triangleq \tilde{J} - \tilde{J}^* \\ &\simeq \int_0^T \left(\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T \delta\dot{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right) \mathrm{d}t + \frac{\partial V}{\partial x}\delta x(T)\end{aligned}$$

- Note that (integration by parts):

$$\int_0^T \lambda^T \delta\dot{x} = -\int_0^T \dot{\lambda}^T \delta x + \lambda^T(T)\delta x(T) - \lambda^T(0)\delta x(0)$$

- So:

$$\begin{aligned}\delta\tilde{J} = \int_0^T &\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right]\mathrm{d}t \\ &+ \left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)\delta x(T) + \lambda^T(0)\delta x(0)\end{aligned}$$

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# Pontryagin's Maximum Principle

- If ($x^*$,$u^*$) is optimal, then:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

$$H(x^*(t), u^*(t), \lambda^*(t)) \le H(x^*(t), u, \lambda^*(t)) \;\forall u \in \Omega$$

- If $\Omega = \mathbb{R}^m$ and $H$ differentiable then $\partial H/\partial u = 0$
- Can be more general and include terminal constraints
- Follows directly from:

$$\begin{aligned}\delta\tilde{J} = \int_0^T &\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right]\mathrm{d}t \\ &+ \left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)\delta x(T) + \lambda^T(0)\delta x(0)\end{aligned}$$

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# Interpretation of λ

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0 \qquad \leftarrow \quad \dot{x} = f(x,u)$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

- Two-point boundary value problem: $\lambda$ is solved backwards in time
- $\lambda$ is the "co-state" (or adjoint variable)
- Recall that $H = L + \lambda^T f(x,u)$
  - If $L$ = 0 (no integrated cost, only a final cost), then: $\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T$
  - This is the *tangent linear adjoint* of the original system
    - I.e. the adjoint of the system linearized about the forward trajectory

$$\langle \mathcal{L}^*(x), y\rangle = \langle x, \mathcal{L}(y)\rangle$$

- $\lambda(t)$ is the sensitivity of the cost to a perturbation in the state x(t)
  - In the integral as $\lambda(t)\delta\dot{x}$
  - Recall δJ = … +λ(0)δx(0)

$$\delta\dot{x} = z\delta_D(t-\tau)$$

$$\Rightarrow \delta\tilde{J} = \int \cdots \lambda^T \delta\dot{x} \cdots = \lambda^T(\tau) z$$

$$x(\tau^+) = x(\tau^-) + z$$

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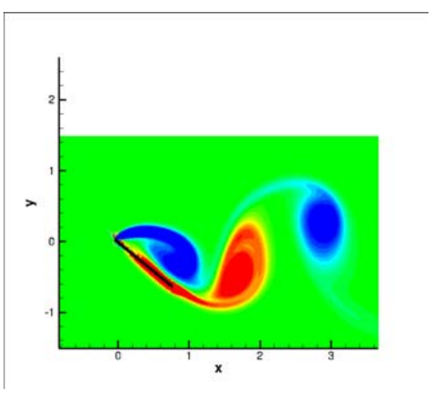
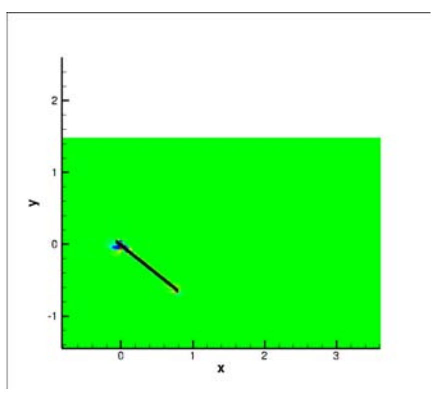
# 2D flat plate (Won Tae Joe)

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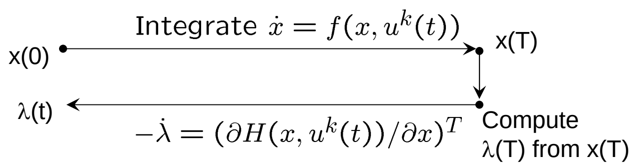
# General Nonlinear Optimization

- For a control input $u^k(t)$, compute the sensitivity:




- Given λ(t), note gradient of cost is $\frac{\delta \tilde{J}}{\delta u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}$
  - In a linear problem, can take one step to the minimum
  - In nonlinear problem, take step in gradient direction
  - Do line-search along gradient direction, since forward iterations are typically cheap compared to adjoint
- With new guess for control input $u^{k+1}(t)$, iterate… until converged

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# Receding-horizon implementation

- The above iteration gives a (locally) optimal control trajectory that minimizes the cost over a horizon [0,T] from state $x(0)=x_0$.
  - This may be sufficient to understand characteristics of system
- For real-time control:
  - Real horizon might be infinite, but finite computational power means can only compute over a finite horizon
  - In the presence of model uncertainty and disturbances, actual trajectory will not follow predicted trajectory
  - Approach: follow computed control trajectory over a (small) subset of the horizon $[0,t_k]$, measure new state, and recompute new optimum over interval $[t_k,T+t_k]$.
  - For some problems like Mars-landing, may have a compressing rather than receding horizon (i.e. fixed terminal time)
  - If terminal cost in formulation chosen to bound integrated cost over $[T,\infty]$ then cost is a Lyapunov function (hence stability guaranteed)

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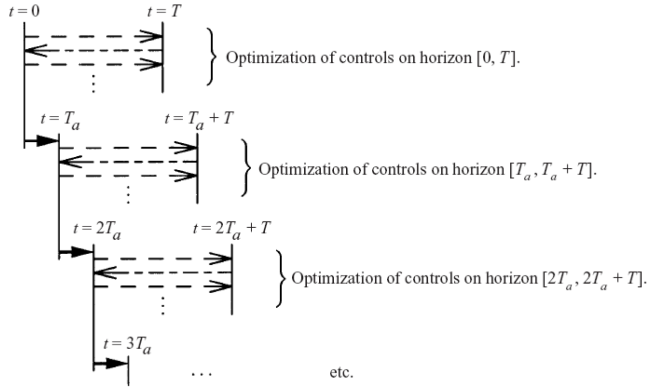
# Schematic…

- From Bewley, Moin and Temam (2001)
- Each finite-horizon control is computed through iteration with forward & adjoint solutions






# Linear system, Quadratic cost

$$\dot{x} = Ax + Bu \qquad J = \frac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right) \mathrm{d}t$$

- Apply PMP:

$$\dot{\lambda} = -A^T\lambda + Qx$$
$$Ru = -B^T\lambda$$

$B^T$ selects the part of the state that is influenced by u, so $B^T\lambda$ is sensitivity of aug. state cost to u

- Guess that λ(t)=P(t)x(t):

$$-\dot{P} = PA + A^T P + Q - PBR^{-1}B^T P \qquad P(T) = 0$$
$$u = -R^{-1}B^T P x$$

- $x^T Px$ has an interpretation as the "cost to go"
- Often see the infinite-horizon solution where dP/dt=0

# Examples

- Note adjoint solution alone is useful
  - Weather forecasting, to understand what measurements are most useful to make, and to propagate uncertainty (i.e. sensitivity to initial condition errors)
  - Understanding what perturbations are most likely to lead to an El Niño event
- Theory
  - Use PMP to prove that bang-bang control is time-optimal, for example
- For control
  - Real-time implementation of full non-linear limited to relatively simple systems (e.g. chemical plants, Mars entry/descent/landing (EDL),…)
  - Used to understand limits of performance, and characteristics…
    - 2D separation over airfoil (Won Tae)
    - El Niño control (me)
    - Turbulent boundary layers (e.g. Bewley et al)
    - Jet noise (Freund, Colonius)

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# El Niño Dynamics (chaotic system)

- Forward model is Fortran-77 legacy code (Cane & Zebiak)
- Adjoint model obtained from automatic differentiation (Adifor)
- Compare linear feedback to optimal control

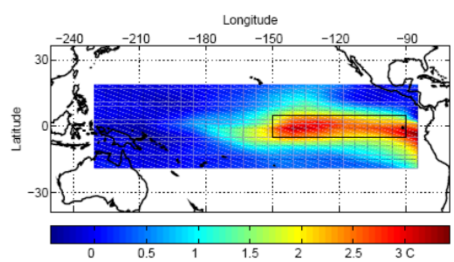


Fig. 2. SST anomaly predicted by model at the peak of an El Niño event (at ~19 yrs in the uncontrolled time-history in Fig. 4(a)). The Niño-3 region is shown boxed. For control, solar insolation is dynamically varied over the eastern half of the Niño-3 region.

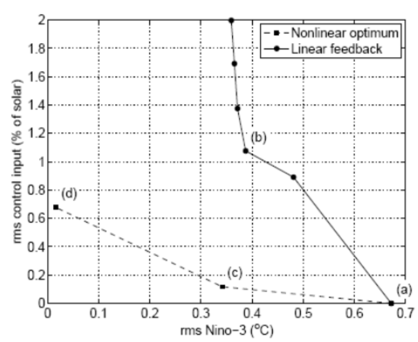


Fig. 3. Performance vs. control effort, comparing SISO linear feedback with the optimal nonlinear control, for a 25-yr simulation.

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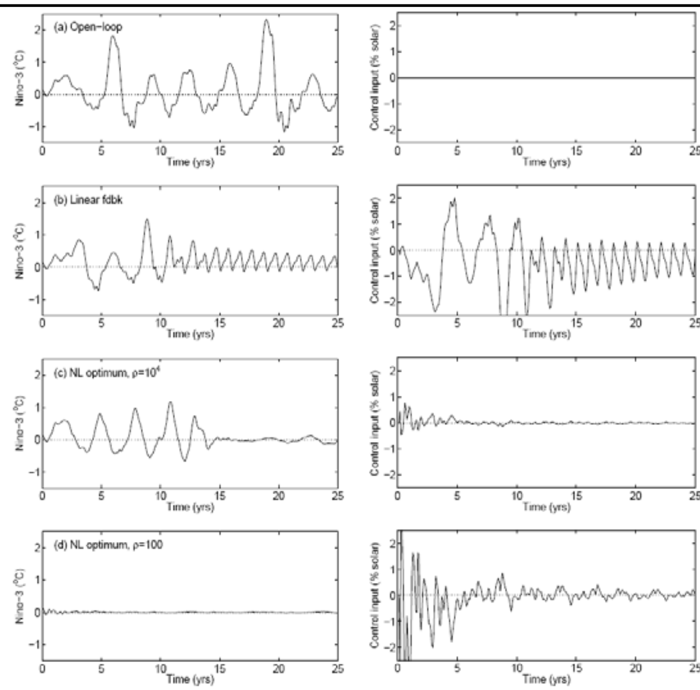
Linear feedback:

Optimum, high control weight

Optimum, low control weight




;. 4. Time history of Niño-3 index (left) and control input (right) for different control cases. From top to bottom, cases are (a) uncontrolled, (b) linea

# Summary

- Necessary conditions to solve a general non-linear optimal control problem are straightforward to write down
  - Only characterizes extrema; problem is not in general convex!
- The solution involves the adjoint
  - In general, a Lagrange multiplier is the sensitivity of the cost to a change in the constraint
  - Specifically for this dynamic constraint, the Lagrange multiplier is the sensitivity of the cost to a change in the state at each time

$$\delta\dot{x} = z\delta(t-\tau)$$

$$\Rightarrow \delta\tilde{J} = \int \cdots \lambda^T \delta\dot{x} \cdots = \lambda^T(\tau) z$$

$$x(\tau^+) = x(\tau^-) + z$$

- The solution in general is iterative
  - Requires solving 2PBVP
- For linear system, quadratic cost, then a closed-form solution exists
  - Called LQR (Linear Quadratic Regulator)

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