Nonlinear Control

Controllability and observability more challenging than for linear system
- Local concept
- Linearization may not give same answer
  (e.g. parking a car:)

- Observability depends on control
  (doesn't decouple)

Control design.

1. Treat as nearly linear system
   design control around each operating point
   ⇒ Gain Scheduling
   - Appropriate for slowly varying
   e.g. \( \dot{x} = A(p)x + B(p)u \)
     \( p = \ldots \)
   - Time w Lyapunov argument

2. Feedback linearization:
   - Find input state transformation that renders system linear

\[
\begin{align*}
\dot{m} &= -mg \cos \theta + u_1 \sin \theta + u_2 \cos \theta \\
\dot{m} &= mg (1-\sin \theta) + u_1 \cos \theta - u_2 \sin \theta \\
\dot{\theta} &= ru_1 - \frac{1}{2}m \dot{x} \cos \theta
\end{align*}
\]

Choose
\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]
3. Lyapunov-based arguments

a) Backstepping: Recursive approach; reduce to low-order (simpler) problem & build up

\[ \dot{y} = \mathbf{f}(y) + \mathbf{g}(y) \, \mathbf{s} \]
\[ \dot{s} = u \]

i) Design \( s = \phi(y) \) to stabilize \( \dot{y} = \ldots \) subsystem

ii) Design \( u \) so that \( s - \phi(y) \) is stable

\[ \text{Simultaneously stable} \]

b) Sliding-mode: Robust

Design to

i) approach specified manifold in finite time

(\( s = \phi(y) \))

ii) Converge to origin along manifold (e.g. if \( s = \phi(y) \), then \( \dot{s} = \ldots \) stable)

4. Adaptive control

- E.g. model reference, update parameters of model
- Can design control assuming plant known
- Can estimate parameters (e.g. EKF) from output

- Relevant for linear but unknown system

\[ \dot{x} = A(p)x + B(p)u \]
\[ \dot{p} = \ldots \]
5. Bifurcation control

E.g. \[ \dot{x} = \mu x + x^3 + u \]

Subcritical pitchfork bifurcation

\[ U = -kx \]
Increase stability but still subcritical

Supercritical

6. Optimization and Receding Horizon Control

\[ \dot{x} = f(x, u), \text{ and maybe constraints} \]

\[ J = \int_0^T L(x, u) dt + V(x(T)) \]

Choose \( U = \text{argmin}(J) \)

\[ \begin{bmatrix} \text{Use Pontryagin's To write conditions} \\ - \text{Solve TP BVP} \\ - \text{Parameterize & discretize} \end{bmatrix} \]

RHC: Implement over \( t \in [t_0, t_0 + \delta], \delta < T, \)
and resolve over \( t_i = t_0 + \delta \text{ to } t_i + T \)

General, \( \delta \)T need to worry about solving optimization in fixed known time

Need conditions on \( V \) to guaranteed stability (with Lyapunov)
Reachability & Controllability

(Isidori, 1, 8, 2.1. - 2.2)

Consider
\[ \dot{x} = f(x) + g(x)u \quad (\text{Affine}) \]
\[ y = h(x) \quad x \in M \subset \mathbb{R}^n, \quad u \in K \subset \mathbb{R}^m \quad y \in \mathbb{R}^p \]

Definitions

- If \( \exists u(t): [0, T] \rightarrow K \) then \( q = x(T) \in M \) satisfying 1 with \( x(0) = p \) is reachable from \( p \) in Time \( T \).

- The set of such \( q \), \( Q(p, T) \) is the Time-T reachable set (for \( T = 0 \), \( Q = p \)).

- The system is controllable if \( \forall p, \exists M, \exists T \) such that \( q \in Q(p, T) \).

- The system is small-time locally controllable if \( p \) is an interior point of \( Q(0, T) \).

Example: "Planar drive vehicle" (roughly unicycle...)

\[ x_1 \quad x_2 \]

with \( u_1 = r_0 (\dot{x}_1 + \dot{x}_2) \quad [\text{Forward Force}] \]
\[ u_2 = r_0 (\dot{x}_1 - x_2) \quad [\text{Torque}] \]

so \[ \dot{x}_1 = u_1 \cos \Theta \]
\[ \dot{x}_2 = u_1 \sin \Theta \]
\[ \dot{\Theta} = u_2 \]

E.g. linearize about \( \Theta = 0 \):

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} u
\]

Linear system is not controllable

\[ \text{rank}( AB AB AB ) = 2 \]
(Aside: linear system)
\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{A(t-s)} B u(s) \, ds \]
\[ = B \phi_t + A B \phi_t + A^2 B \phi_t + \ldots \]

Controllable if \( \begin{bmatrix} B & A B & A^2 B & \ldots \end{bmatrix} \) is full-rank.

From Cayley-Hamilton, \( A^n = \sum_{i=0}^{n-1} \alpha_i A^i \) for some \( \alpha_i \).

But nonlinear system is controllable:

E.g., \( u_2 = \cos \omega t \Rightarrow \Theta(t) = \frac{1}{\omega} \sin \omega t \) (w.r.t. \( \Theta(0) = 0 \))

\[ \dot{u}_1 = \sin \omega t \]

where \( u_1 = u_1 - u_2 (x_1 \sin \theta - x_2 \cos \theta) \) - coordinate changes.

(Note, if nonlinear transformation where algebra is clean)

\[ \theta \]
\[ z \]
\[ x_1 \]
\[ x_2 \]

More
More general, for affine system

\[ \dot{x} = F(x) + g_1(x)u_1 + g_2(x)u_2 \]

Consider

- \( u_1 = 1, u_2 = 0 \) for \( 0 \leq t < \epsilon \)
- \( u_1 = 0, u_2 = 1 \) for \( \epsilon \leq t < 2\epsilon \)
- \( u_1 = -1, u_2 = 0 \) for \( 2\epsilon \leq t < 3\epsilon \)
- \( u_1 = 0, u_2 = -1 \) for \( 3\epsilon \leq t < 4\epsilon \)

Then

\[
\begin{align*}
\dot{x}(\epsilon) &= x_0 + \epsilon \left[ g_1(x_0) + \epsilon \frac{1}{2} \frac{\partial g_1}{\partial x} g_1(x_0) \right] + \text{h.o.t.} \\
\dot{x}(2\epsilon) &= x_0 + \epsilon \left[ g_1(x_0) + \epsilon \frac{1}{2} \frac{\partial g_1}{\partial x} g_1(x_0) \right] + \text{h.o.t.} \\
\dot{x}(3\epsilon) &= x_0 + \epsilon \left[ g_1(x_0) + \epsilon \frac{1}{2} \frac{\partial g_1}{\partial x} g_1(x_0) \right] + \text{h.o.t.} \\
\dot{x}(4\epsilon) &= x_0 + \epsilon \left[ \frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0) \right] + \text{h.o.t.}
\end{align*}
\]

Lie bracket \([g_1, g_2](x_0)\)

Additional possible direction that cyclic control can move the system

\[ \dot{x} = L_{q_1} q_2 - L_{q_2} q_1 \]

E.g., for

\[
\begin{bmatrix}
\cos x_3 \\
\sin x_3 \\
0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

Then

\[
\begin{align*}
[g_1, g_2] &= \frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0) \\
&= 0 - \begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}
\end{align*}
\]

which is orthogonal to \( g_1, g_2 \)

... The system is SLIC if the set of vector fields

\( g_i, [g_i, g_j], [g_i, [g_i, g_j]], \ldots \) spans the space.
With drift \((F(x) \neq 0)\), then consider
\[
\dot{x} = F(x) u_0 + g_1(x) u_1 + \ldots
\]
where \(u_0 = 1\).

Then similarly take space \(\mathcal{L}_i, [h_i, l_i], [h_i, [l_i, h_i]]\) etc.
For \(l_i \in \{F, g_i\}\).

This doesn't quite give controllability, since \(u_0\) is fixed.

Note for linear system
\[
\dot{x} = Ax + Bu = Ax + \sum_i b_i u_i
\]
Then \([b_i, b_i] = 0\)

and \([Ax, b_i] = -Ab_i\)
\([Ax, [Ax, b_i]] = A^2 b_i\)
Observability

Can we determine the state $x$ from time history of the output?

$$
\begin{align*}
\dot{x} &= F(x) + \sum q_i(x) u_i \\
y &= h(x)
\end{align*}
$$

For a linear system $\dot{x} = Ax + Bu$,

Then

$$
\begin{align*}
x(t) &= e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau
\end{align*}
$$

Effect of input known, can be subtracted. Not true for NL system.

Def' $: \text{Two states } x_1, x_2 \text{ are indistinguishable (} y(x_1) = y(x_2)) \text{ if for every admissible input } u \text{ then } y(t, x_1, u) \text{ with } x(0) = x_1 \text{ and } y(t, x_2, u) \text{ with } x(0) = x_2 \text{ are identical.} \text{ System observable if } x_1, x_2 \Rightarrow x_1 = x_2. \text{ (Or locally observable if ...)}$

Approach $: \text{Know } y(t) \Rightarrow \text{Know } y(t), \dot{y}(t), \ddot{y}(t), \ldots$

$$
\begin{align*}
y &= h(x) \\
\dot{y} &= L_fh + L_gh \ u \\
\ddot{y} &= \ldots
\end{align*}
$$

E.g. $: \text{If } u = 0 \ (\text{or } L_gh = 0, \ L_fh = 0, \ldots)$

Then form $D = \begin{bmatrix} h(x) \\ L_fh \\ L^2_fh \\
\vdots
\end{bmatrix}$; system observable if $D$ is invertible (locally observable if ...)

Should be the Jacobian of this

Note for a linear system $h(x) = Cx$, $D = \begin{bmatrix} C \\ CA \\ CA^2 \\
\vdots
\end{bmatrix}$
Feedback linearization

When can we take a nonlinear system and transform it to a linear system?  (Not just locally?)

E.g. given \( \dot{x} = f(x) + g(x)u \quad x \in \mathbb{R}^n \)
\[ y = h(x) \]

Is there a transformation \( z = T(x) \), \( u = a(x) + b(x)\tilde{u} \) so that
\[ \dot{z} = \tilde{A}z + \tilde{B}\tilde{u} \quad \text{(Linear)} \]
\[ y = Cz \]

(IF a NL system is feedback linearizable it is essentially linear!)

Note \( T \) must be diffeomorphism (invertible and smooth)
- In practice only locally diffeomorphic

Define relative degree: how many times differentiate \( y \) before \( u \) shows up explicitly?

relative degree \( r \) if
\[ \frac{Lg}{Lf^k} h(x) = 0 \quad \forall x \text{ in nbhd of } x_0 \]
\[ \frac{Lg}{Lf^{k-1}} h(x) = 0 \quad \& \quad k < r - 1 \]

(For linear system, difference between degree of numerator & denominator)
\[ H(s) = \frac{C(sI-A)^{-1}B}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \]
and \( Lg/\text{Lf}^k h(x) \)

SISO

Suppose relative degree is \( n \)  (No zero dynamics)

Then
\[ y = h(x) \]
\[ y^{(1)} = L_fh(x) \]
\[ \vdots \]
\[ y^{(n)} = L_f^{n-1}h(x) \]
\[ y^{(n)} = L_fh(x) + Lg L_f^{n-1} h(x)u \]

\( \text{can't have } r > n \)
Then \[ z(t) = A(z) + B(u) \]

\[ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} h(x) \\ L_x h(x) \\ \vdots \\ L_x^{n-1} h(x) \end{bmatrix} \]

puts system in "normal form" ...

\[ \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= L_x h(x) + L_x L_x^{n-1} h(x) u \end{aligned} \]

or

\[ \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots \\ -K & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ u \end{bmatrix} \]

& e.g. \( u = -Kz \) (state feedback) can arbitrarily place poles of closed loop system \( A-BK \) 

\[ A-BK = \begin{bmatrix} \lambda (I - BU) \\ -K & -K & \cdots & -K \end{bmatrix} \]

\[ \lambda (I - BU) = \text{det} (\lambda I - (A-BK)) \]

If \( r < n \) need to:
1) Construct Transformation for \( z_{r+1} \) to \( z_n \) (and show that exists)
2) Understand Stability of the "zero dynamics"

In general, if can construct some output \( y = h(x) \) with relative degree \( n \),
Then \( x = F(x) + g(x) u \) is feedback linearizable.
(Can write as conditions involving \( g = [F, g], [F, [F, g]], \ldots \) need each of these involutive and constant dimensional)
Example:
Euler equations (rigid body rotation) with
Two inputs (e.g., momentum wheels)

\[ \dot{\omega}_1 = A_1 \omega_2 \omega_3 + u_1, \]
\[ \dot{\omega}_2 = A_2 \omega_1 \omega_3 + u_2, \]
\[ \dot{\omega}_3 = A_3 \dot{\omega}_1 \omega_1. \]

Single-link manipulator

\[ I \ddot{\theta}_1 + M g L \sin \theta_1 + k (\theta_1 - \theta_2) = 0, \]
\[ I \ddot{\theta}_2 = k (\theta_2 - \theta_2) = u. \]

or

\[ F(x) = \begin{bmatrix} x_2 \\ -a \sin x_1 - b (x_1 - x_3) \\ x_4 \\ c (x_1 - x_3) \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} \quad \text{for some constants } \ a, b, c, d > 0. \]

If \( y = x_1 \), choose

\[ z_1 = x_1 \quad ( = h(x)) \]
\[ z_2 = L \cdot h(x) = x_2 \]
\[ z_3 = L^2 \cdot h(x) = -a \sin x_1 - b (x_1 - x_3) \]
\[ z_4 = L^3 \cdot h(x) = a x_2 \cos x_1 - b (x_2 - x_4) \]

\[ \Rightarrow \quad \dot{z}_1 = \dot{z}_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = - (a \cos z_1 + b + c) z_3 + a (z_2 - c) \sin z_1 + b d \dot{u} \]

This transformation is valid globally since \( \mathbb{z} = T(x) \) is globally diffeomorphism (except \( x_1 = 0, \pm \pi/2 \)).