Nonlinear Control

Controllability and observability more challenging than for linear system

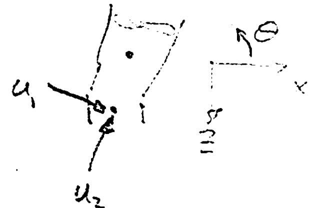
- Local concept
- Linearization may not give same answer
  (e.g. parking a car:  introduce Lie algebra for formal analysis
- Observability depends on control (doesn't decouple)

Control design.

1. Treat as nearly linear system, design control around each operating point
   $\Rightarrow$ Gain scheduling
   - Appropriate for slowly varying
     e.g. $\dot{x} = A(p)\,x + B(p)\,u$
     $\dot{p} = \ldots$
   - Prove w/ Lyapunov argument

• If linearization stable, nonlinear system is locally stable

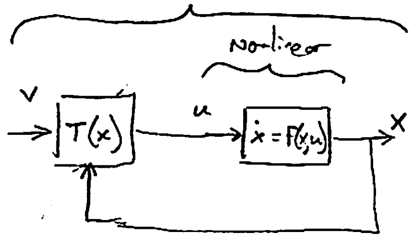
2. Feedback linearization:
   - Find input and state transformation that renders system linear



$$m\ddot{x} = -mg\cos\theta + u_1\sin\theta + u_2\cos\theta$$

$$m\ddot{z} = mg(1-\sin\theta) + u_1\cos\theta - u_2\sin\theta$$

$$I\ddot{\theta} = r u_1 - J_p\Omega\dot{x}\cos\theta$$



Choose $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

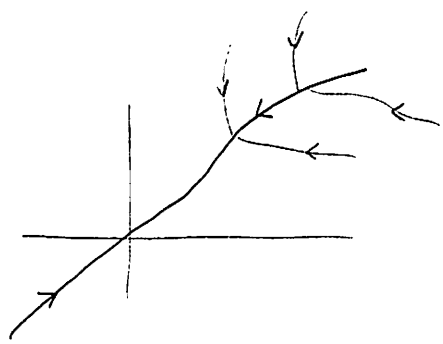
3. Lyapunov-based arguments

a) Backstepping: Recursive approach; reduce to low-order (simpler) problem & build up

E.g.
$$\dot{\eta} = f(\eta) + g(\eta)\,\xi$$
$$\dot{\xi} = u$$

i) Design $\xi = \phi(\eta)$ to stabilize $\dot{\eta} = \dots$ subsystem
ii) Design u so that $\xi - \phi(\eta)$ is stable
} Simultaneously stable

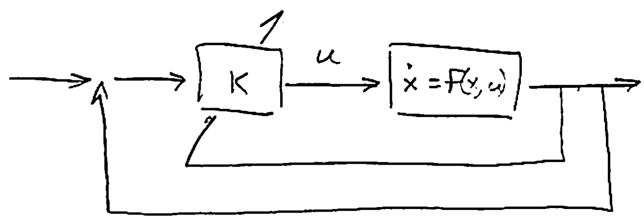
b) Sliding-mode: Robust



Design to
i) approach specified manifold in finite time (e.g. $\xi = \phi(\eta)$)

ii) Converge to origin along manifold (e.g. if $\xi = \phi(\eta)$, then $\dot{\eta} = \dots$ stable)

4. Adaptive control

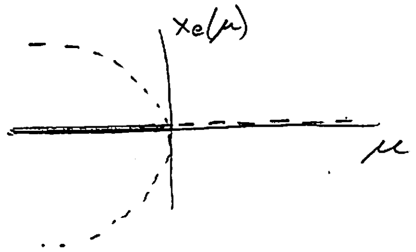


- E.g. model reference, update parameters of model
- Can design control assuming plant known
- Can estimate parameters (e.g. EKF) from output
} Combination not obviously stable...

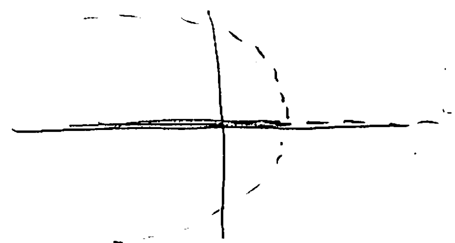
- Relevant for linear but unknown system
$$\dot{x} = A(p)x + B(p)u$$
$$\dot{p} = \dots$$

5. Bifurcation control

E.g. $\dot{x} = \mu x + x^3 + u$

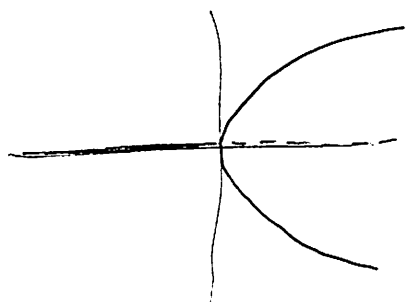


subcritical pitchfork bifurcation



$u = -kx$

Increase stability but still subcritical



$u = -kx^3$

supercritical

6. Optimization and Receding Horizon Control

$\dot{x} = f(x, u)$, and maybe constraints

$$J = \int_0^T L(x, u)\, dt + V(x(T))$$

Choose $u = \text{argmin}(J)$

[Use Pontryagin to write conditions
- Solve 2P BVP
- Parameterize & discretize]

RHC: Implement over
$t \in [t_0, t_0 + \delta]$, $\delta < T$,
and resolve over $t_1 = t_0 + \delta$ to $t_1 + T$

Need conditions on V to guarantee stability (with Lyapunov)

General, but need to worry about solving optimization in fixed known time.

Reachability & Controllability (Isidori 1.8, 2.1-2.2)

Consider $\dot{x} = f(x) + g(x)u$ ① (Affine)

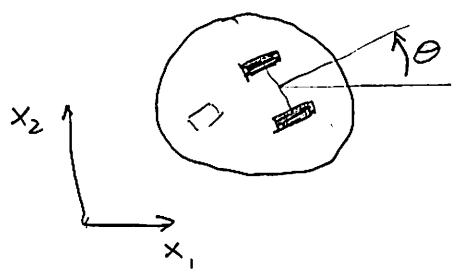
$y = h(x)$ $\qquad x \in M \subset \mathbb{R}^n,\ u \in K \subset \mathbb{R}^m$

$y \in \mathbb{R}^p$

Definitions

- If $\exists\, u(t) : [0,T] \to K$ then $q = x(T) \in M$ satisfying ① with $x(0) = p$ is reachable from p in time T
- The set of such q, $Q(p,T)$ is the time-T reachable set (for $T=0$, $Q = p$)
- The system is controllable if $\forall p, q \in M$, $\exists T$ such that $q \in Q(p,T)$
- The system is small time locally controllable if p is an interior point of $Q(p,T)$

Example: "Planar drive vehicle" (roughly unicycle, ...)



with $u_1 = r_w(\tau_1 + \tau_2)$ [Forward force]

$u_2 = r_w(\tau_1 - \tau_2)$ [Torque]

So
$$\dot{x}_1 = u_1 \cos\theta$$
$$\dot{x}_2 = u_1 \sin\theta$$
$$\dot{\theta} = u_2$$

Note a non-holonomic constraint of no motion sideways.

E.g. linearize about $\theta = 0$:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

Linear system is not controllable

$\text{rank}([B\ \ AB\ \ A^2B]) = 2$

(Aside :- linear system Then

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{At} B\, u(\tau)\, d\tau}$$

$$= B\phi_0 + AB\phi_1 + A^2B\phi_2 + \ldots$$

Controllable if $[B \;\; AB \;\; A^2B \;\ldots]$ is full-rank

From Cayley-Hamilton, $A^n = \sum_{i=0}^{n-1} \alpha_i A^i$ for some α_i)

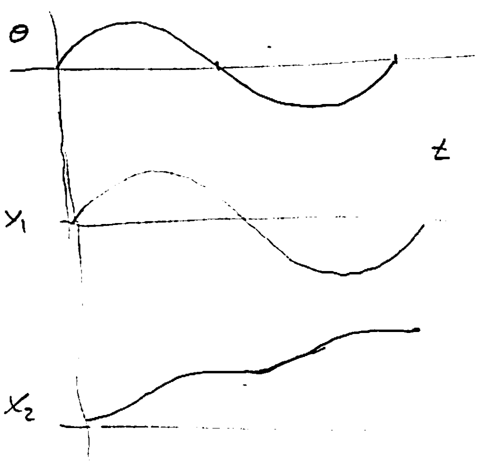
But nonlinear system is controllable :

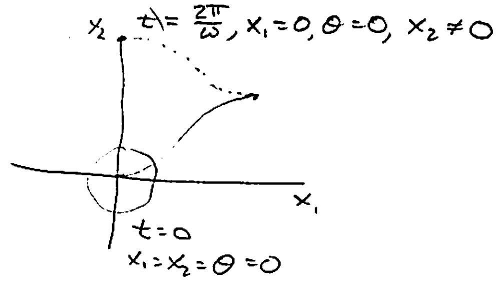
E.g. $u_2 = \cos\omega t \Rightarrow \theta(t) = \frac{1}{\omega}\sin\omega t$ (with $\theta(0) = 0$)

$u_1 = \sin\omega t$

~~where $v_1 = u_1 - u_2(x_1 \sin\theta - x_2\cos\theta)$ ← co-ordinate change~~

(Note, $\exists$ nonlinear transformation where algebra is clean)





More

More general, for affine system

$$\dot{x} = F(x) + g_1(x)u_1 + g_2(x)u_2$$

Consider
$u_1 = 1,\ u_2 = 0 \qquad 0 \le t < \epsilon$ & $F(x) = 0$ for simplicity (Driftless)
$u_1 = 0,\ u_2 = 1 \qquad \epsilon \le t < 2\epsilon$
$u_1 = -1,\ u_2 = 0 \qquad 2\epsilon \le t < 3\epsilon$
$u_1 = 0,\ u_2 = -1, \qquad 3\epsilon \le t < 4\epsilon$

Then
$$x(\epsilon) = x_0 + \epsilon g_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial g_1}{\partial x} g_1(x_0) + \text{h.o.t.}$$
$$x(2\epsilon) = x_0 + \cdots + \epsilon g_2\left(x_0 + \epsilon g_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial g_1}{\partial x} g_1(x_0)\right) + \epsilon^2 \frac{1}{2}\frac{\partial g_2}{\partial x} g_2\left(\cdots\right) + \cdots$$
$$\approx x(0) + \epsilon g_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial g_1}{\partial x} g_1(x_0) + \epsilon g_2(x_0) + \epsilon^2 \frac{\partial g_2}{\partial x} g_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial g_2}{\partial x} g_2(x_0)$$
$$\vdots$$
$$x(4\epsilon) \approx x_0 + \epsilon^2 \underbrace{\left[\frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0)\right]}_{\text{Lie bracket } [g_1, g_2](x_0)}$$

→ Additional possible direction that cyclic control can move the system

$$= \underbrace{L_{g_1} g_2}_{\substack{\text{derivative of } g_2 \text{ along} \\ \text{trajectory determined} \\ \text{by } g_1}} - L_{g_2} g_1$$

E.g. for
$$\dot{x} = \underbrace{\begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g_2(x)} u_2$$

Then
$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0)$$
$$= 0 - \begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}$$

which is orthogonal to g_1, g_2

... The system is STLC if the set of vector fields

$g_i,\ [g_i, g_j],\ [g_i, [g_j, g_k]], \ldots$ spans the space

With drift ($F(x) \neq 0$), Then

consider $\dot{x} = F(x) u_0 + g_1(x) u_1 + \ldots$

where $u_0 = 1$

Then similarly Take space l_i, $[l_i, l_j]$, $[l_i, [l_j, l_k]]$ etc.
for $l_i \in \{F, g_i\}$

This doesn't quite give controllability, since u_0 is fixed

Note for linear system
$$\dot{x} = Ax + Bu = Ax + \sum b_i u_i$$

Then $[b_i, b_j] = 0$

and $[Ax, b_i] = -Ab_i$
$$[Ax, [Ax, b_i]] = A^2 b_i$$

Feedback linearization

When can we take a nonlinear system and transform it to a linear system? (NOT just locally?)

E.g. given $\dot{x} = f(x) + g(x)u$ $\quad x \in \mathbb{R}^n$

$y = h(x)$

Is there a transformation $z = T(x)$, $u = a(x) + b(x)v$

so that

$$\left.\begin{aligned} \dot{z} &= Az + Bv \\ y &= Cz \end{aligned}\right\} \text{Linear}$$

(If a NL system is feedback linearizable it is essentially linear!)

Note T must be diffeomorphism (invertible and smooth)
- In practice only locally diffeomorphic

Define relative degree: how many times differentiate y before u shows up explicitly?

relative degree r if $L_g L_f^{\,k} h(x) = 0$ $\quad \forall x$ in nbhd of x_0

$L_g L_f^{\,r-1} h(x) = 0$ $\quad$ & $k < r-1$

(For linear system, difference between degree of numerator & denominator

$$H(s) = C(sI - A)^{-1}B = \frac{b_{n-r}s^{n-r} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0},$$

and $L_g L_f^{\,k} h = CA^k B x$

SISO

Suppose relative degree is n (NO zero dynamics)

← can't have $r > n$

Then $y = h(x)$

$y^{(1)} = L_f h(x)$

$\vdots$

$y^{(n-1)} = L_f^{\,n-1} h(x)$

$y^{(n)} = L_f^{\,n} h(x) + L_g L_f^{\,n-1} h(x) u$

Then $z = T(x) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_F h(x) \\ \vdots \\ L_F^{n-1} h(x) \end{bmatrix}$

puts system in "normal form" ...

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\vdots$$
$$\dot{z}_n = \underbrace{L_F^n h(x) + L_g L_F^{n-1} h(x) u}_{v}$$

& $u = \frac{1}{L_g L_F^{n-1} h(x)} \left(-L_F^n h(x) + v \right)$

or

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & \ddots & \\ & 0 & & \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

[& e.g. $v = -Kz$ (state feedback) can arbitrarily place poles of closed-loop system]

$A - BK = \begin{bmatrix} 0 & 1 & \cdots & \\ & & \ddots & \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix}$, $\lambda(A-BK)$
$= \det(\lambda I - (A - BK))$
$= \cdots$

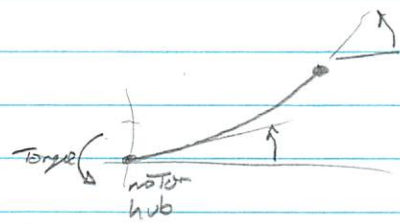
If $r < n$ need to:

1) Construct transformation for z_{r+1} to z_n (and show that exists)
2) Understand stability of the "zero dynamics"

In general, if can construct some output $y = h(x)$ with relative degree n, then $\dot{x} = F(x) + g(x) u$ is feedback linearizable.
(Can write as conditions involving $\{g,\ [F,g],\ [F,[F,g]], \ldots\}$ need each of these involutive and constant dimensional)

Example:
Euler equations (rigid body rotation) with
Two inputs (e.g. momentum wheels)

$$\dot{\omega}_1 = A_1 \omega_2 \omega_3 + u_1$$
$$\dot{\omega}_2 = A_2 \omega_1 \omega_3 + u_2$$
$$\dot{\omega}_3 = A_3 \omega_2 \omega_1$$



Single-link manipulator

$$I \ddot{q}_1 + MgL \sin q_1 + k(q_1 - q_2) = 0$$
$$J \ddot{q}_2 - k(q_1 - q_2) = u$$

or
$$F(x) = \begin{bmatrix} x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$
for some constants $a, b, c, d > 0$

If $y = x_1$, choose

$$z_1 = x_1 \qquad (= h(x))$$
$$z_2 = L_f h(x) = x_2$$
$$z_3 = L_f^2 h(x) = -a \sin x_1 - b(x_1 - x_3)$$
$$z_4 = L_f^3 h(x) = a x_2 \cos x_1 - b(x_2 - x_4)$$

$$\Rightarrow \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = \underbrace{-(a \cos z_1 + b + c) z_3 + a(z_2^2 - c)\sin z_1 + bdu}_{= v}$$

This transformation is
valid globally since $z = T(x)$ is globally diffeomorphic
(except $x_1 = 0, \pm \pi/2$