Lagrangian Coherent Structures (LCS)

CDS 140b - Spring 2012

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A time-dependent dynamical system

\[ \dot{x}(t; t_0, x_0) = v(x(t; t_0, x_0), t) \]

\[ x(t_0; t_0, x_0) = x_0 \]

- \( t \in I \subseteq \mathbb{R} \)
- \( x(t; t_0, x_0) \in D \subseteq \mathbb{R}^n \)
- Assume \( v \) is smooth (not necessary, but convenient)

We can define the flow map:

\[ \Phi_{t_0}^t : x_0 \mapsto \Phi_{t_0}^t(x_0) = x(t; t_0, x_0) \]

as the map which takes a point on the domain at time \( t_0 \) to its position at time \( t \)
Invariant manifolds in dynamical systems

- If \( \mathbf{v} \) is independent of \( t \), we can find invariant manifolds
- The stable manifolds of a fixed point \( \mathbf{x}_c \) are all the trajectories which asymptote to \( \mathbf{x}_c \) as \( t \rightarrow \infty \)
- The unstable manifolds are all trajectories which asymptote to \( \mathbf{x}_c \) as \( t \rightarrow -\infty \)
Invariant manifolds in dynamical systems

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• Two points on either side of a stable manifold will diverge in forward time
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- Barriers to transport.
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- Barriers to transport
- Separatrices between regions of different dynamics

Shawn Shadden
Time-dependent systems

Goal: Find a finite-time approximation to the stable and unstable manifolds, for time-dependent dynamical systems.

Lagrangian Coherent Structures (LCS)

- Finite-time: systems often defined in finite time-interval
- Structures that preserve the aforementioned properties, at least in some finite time interval

Autonomous system:
  - Find the fixed points $x_c$
  - Linearize about a fixed point
  - Find the stable and unstable subspaces
  - The stable and unstable manifolds are locally tangent

- Time dependent systems rarely have fixed points
- How do we find LCS? Use the fact that they are separatrices

Finite-Time Lyapunov Exponent (FTLE)
The Finite-Time Lyapunov Exponent (FTLE)

- Scalar denoted by $\sigma_t^T(x)$
- Measure of how much trajectories starting near a point $x$ at time $t$ diverge over the interval $[t, t+T]$
- Not an instantaneous separation rate. Measures the integrated separation between trajectories
- In general, the FTLE varies both in space and in time

Consider an arbitrary point $x \in D$ at $t_0$. When advected by the flow, after time $T$:

$$x \mapsto \phi_{t_0}^{t_0+T}(x)$$

Consider a point $y = x + \delta x(t_0)$ initially infinitesimally close to $x$. After a time $T$ the perturbation becomes:

$$\delta x(t_0 + T) = \phi_{t_0}^{t_0+t}(y) - \phi_{t_0}^{t_0+t}(x)$$

$$= \frac{d\phi_{t_0}^{t_0+t}(x)}{dx} \delta x(t_0) + O(||\delta x(t_0)||^2)$$
The Finite-Time Lyapunov Exponent (FTLE)

Since $\delta x(t_0)$ is infinitesimal, we assume $O(||\delta x(t_0)||^2)$ is negligible:

$$||\delta x(t_0 + T)|| = \sqrt{\left< \left( \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \delta x(t_0), \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \delta x(t_0) \right) \right>}$$

$$= \sqrt{\left< \delta x(t_0), \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \delta x(t_0) \right>}$$

Where $M^*$ denotes the adjoint of $M$.

A finite-time version of the Cauchy-Green deformation tensor:

$$\Delta = \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \frac{d\phi_{t_0}^{t_0+T}(x)}{dx}$$

$$||\delta x(t_0 + T)|| = \sqrt{\left< \delta x(t_0), \Delta \delta x(t_0) \right>}$$
The Finite-Time Lyapunov Exponent (FTLE)

To find the maximum deformation, choose $\delta x(t_0)$ in the direction of the max. eigenvalue of $\Delta$:

$$\max_{\delta x(t_0)} \| \delta x(t_0 + T) \| = \sqrt{\langle \delta x(t_0), \lambda_{max}(\Delta) \delta x(t_0) \rangle}$$

$$= \sqrt{\lambda_{max}(\Delta)} \| \delta x(t_0) \|$$

Define: $\sigma_{t_0}^T = \frac{1}{|T|} \ln \sqrt{\lambda_{max}(\Delta)}$

Then: $\max_{\delta x(t_0)} \| \delta x(t_0 + T) \| = e^{\sigma_{t_0}^T |T|} \| \delta x(t_0) \|$

$\sigma_{t_0}^T$ is the FTLE, and it gives a measure of how fast nearby trajectories separate.

Note that the FTLE can be defined equivalently as:

$$\sigma_{t_0}^T = \frac{1}{|T|} \ln \left\| \frac{d\phi_{t_0+T}(x)}{dx} \right\|_2$$
Computing the FTLE

- Compute FTLE field at time $t_0$ on a grid
- Advect particles on the grid by the flow map for a time $T$
- At each point on the grid:
  - Compute the deformation tensor at $t_0$:
    $$ \frac{d\phi_{t_0}^{t_0 + t}(x)}{dx} $$
  - Find the maximum singular value if the deformation tensor.
  - Compute $\sigma_{t_0}^T$

$T$ can be positive or negative (integrate trajectories backwards in time)
The FTLE field

Example: pendulum

- High values of the positive-time FTLE indicate regions of high particle separation, which coincide with the location of the stable manifolds.
- High values of the negative-time FTLE indicate regions of high particle attraction, which coincide with the location of the unstable manifolds.
- Combining both FTLE fields, we recover the location of the heteroclinic orbit, which acts as a separatrix between the two dynamically distinct regions.
Lagrangian Coherent Structures are ridges in the FTLE

- Define an LCS as a locally-maximizing curve (ridge) in the FTLE
- A curve is a ridge when:
  - A step off the curve in any direction is a step down
  - The direction of steepest descent is perpendicular to the ridge
- Define the Hessian of the FTLE:
  \[
  \sum = \frac{d^2 \sigma_{t_0}^T(x)}{dx^2}
  \]
  - A curve \( c(s) \) is a ridge if it satisfies:
    - \( \nabla \sigma_{t_0}^T(c(s)) \cdot c'(s) = 0 \)
    - \( \lambda_{min} < 0 \)

where \( \lambda_{min} \) is the smallest eigenvalue of \( \sum \)
The time-dependent double-gyre

Consider a double-gyre defined by the Stokes streamfunction:

\[ \psi(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y) \]

where:

\[ f(x, y) = a(t)x^2 + b(t)x \]

\[ a(t) = \epsilon \sin(\omega t) \]

\[ b(t) = 1 - 2\epsilon \sin(\omega t) \]

The velocity field:

\[ u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x} \]

\[ A = 0.1 \]

\[ \omega = \frac{2\pi}{10} \]

\[ \epsilon = 0.25 \]
The time-dependent double-gyre

When \( \epsilon = 0 \):

\[
\psi(x, y, t) = A \sin(\pi x) \sin(\pi y)
\]

The velocity field:

The forward-time FTLE:
The time-dependent double-gyre

When $\epsilon \neq 0$:

The forward-time FTLE:

The backward-time FTLE:
Properties of LCS

- Trajectories that start on an LCS remain on the LCS over the time interval on which it is defined
Properties of LCS

• Trajectories that start on an LCS remain on the LCS over the time interval on which it is defined
• Particles on either side of a repelling LCS diverge away from each other exponentially in forward time
• Particles on either side of an attracting LCS diverge away from each other exponentially in backward time
• Barriers to transport. Flux across a well-defined LCS is, at worst, a slow leak (See Shadden et al 2005 for expression and proof)
• Separatrices between dynamically different regions
Applications of LCS

- ‘Skeleton’ of the flow

Figure 7.3: Repelling FTLE for the Global Ocean. The LCS reveal boundaries to mesoscale eddies that are responsible for lateral mixing. Regions of intense activity include the Pacific Equatorial jets, the Atlantic Gulf Stream, the Cape Cauldron, and the Antarctic Circumpolar Current.
Applications of LCS

- ‘Skeleton’ of the flow
- Transport and mixing
Applications of LCS

- ‘Skeleton’ of the flow
- Transport and mixing
- Structure identification

Combine forward and backward LCS to find the boundaries of a vortex ring.

Shadden et al 2006.
Applications of LCS

- ‘Skeleton’ of the flow
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- Structure identification
- Capture areas
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- ‘Skeleton’ of the flow
- Transport and mixing
- Structure identification
- Capture areas
- Flow control
Advantages of LCS

• The FTLE is an integral quantity, so it is robust to noise in the velocity data (Haller 2002). Useful for experimental results.
• Unambiguous: not sensitive to threshold value.

• Frame invariant.
Disadvantages of LCS

- Requires time-resolved velocity data.
- FTLE field returns a wealth of information about particle trajectories, but it is sometimes difficult to interpret, and not useful in every application.
- For every time step $t$ you want the the FTLE for:
  - Initialize particles on a rectangular grid at time $t$
  - Advect trajectories forward for the integration time $T$ using a Runge-Kutta scheme
  - Find the final particle positions at time $t+T$
  - Compute the FTLE field

EXTREMELY EXPENSIVE!
The speed issue

- Brunton and Rowley *Chaos* 2010:
  Store trajectory information to reduce computational time

Conventional method

“Unidirectional” method

- Lipinski and Mohseni *Chaos* 2010:
  - Initially compute FTLE field everywhere
  - Identify ridges on the first frame
  - Thereafter, only compute FTLE in the neighborhood of the ridges
  - Identify and track ridges at each time step
Resources and references

Introduction to LCS

• Shawn Shadden’s LCS tutorial:
  http://mmae.iit.edu/shadden/LCS-tutorial/contents.html

• Phil Du Toit’s CDS thesis:
  http://thesis.library.caltech.edu/5293/

Software

• Jeff Peng’s LCS Matlab kit:
  http://dabiri.caltech.edu/software.html

• Francois Lekien’s MANGEN:
  http://mmae.iit.edu/shadden/LCS-tutorial/mangen.html

• Phil Du Toit’s Newman:
  (No longer available online. Contact me at ofarrell@)

Journal articles

Project suggestions

• FTLE computation and testing in classical flows
• LCS computation using open source software
• Hyperbolicity in LCS
  • Mathur et al 2007

In the press

• “The skeleton of water,” The Economist, 12 November 2009