Backstepping

- For a certain class of systems, \( x \in \mathbb{R}^{n+1} \)
can reduce to control design for system of dimension \( n \)
( & recursively to simpler until solvable.)

- Start with system of form

\[
\begin{align*}
\dot{y} &= F(y) + g(y) s \\
\dot{s} &= u \\
x &= \begin{bmatrix} y \\ s \end{bmatrix} \in \mathbb{R}^{n+1}
\end{align*}
\]

(Aside: ) \( \dot{s} = F_s(s) + g_s(s) u \), then choose \( u = F_s(s) + g_s(s) u \)
requires \( F_s, g_s \) smooth, \( g_s \) invertible
ii) for \( u \in \mathbb{R}^p \), use initial state transformation to
so that \( \dot{x} = F_s(x) + g_s(x) u \)
where \( g_s(x) = \begin{bmatrix} 0 \\ g_s \end{bmatrix} \)

Not always possible

- Assume we know how to choose \( s = \phi(y) \) to stabilize \( 0 \)
about equilibrium at \( y = 0 \), \( A \) and \( \phi(0) = 0 \)
with Lyapunov function \( V(y) \)

\[
\frac{dV}{dy} \left[ F(y) + g(y) \phi(y) \right] \leq -W(y) \quad \text{where } W(y) > 0
\]

\( \forall y \in D \)

- Define \( z = s - \phi(y) \) (so stable if \( z = 0 \))

\[
\begin{align*}
\dot{z} &= (F(y) + g(y) \phi(y)) + g(y) \dot{s} \\
\dot{s} &= u - \phi
\end{align*}
\]

and \( \dot{\phi} = \frac{\partial \phi}{\partial y} (F(y) + g(y) s) \)

- Define Lyapunov function for augmented system

\[
V_a(y, z) = V(y) + \frac{1}{2} z^2
\]
\[ V_0 = \frac{\partial V}{\partial \eta} (F(\eta) + g(\eta) \phi(\eta)) + \frac{\partial V}{\partial \eta} g(\eta) z + \frac{\partial V}{\partial \eta} (u - \phi) \]

\[ \text{e.g.} \]

\[ \text{choose } u = \phi - \frac{\partial V}{\partial \eta} g(\eta) - K z \]

\[ \text{For this to be negative definite} \]

\[ \Rightarrow \text{Then } [\begin{bmatrix} z \\ 0 \end{bmatrix}] = [\begin{bmatrix} 0 \\ 0 \end{bmatrix}] \text{ is a stable equilibrium point} \]

\[ u \rightarrow [s] \rightarrow g(\eta) \rightarrow [s] \rightarrow \phi \rightarrow [F + s \phi] \]

\[ \text{Key: non-linear term only depends on } \eta \]

\[ u \rightarrow [s] \rightarrow g \rightarrow [s] \rightarrow \phi \rightarrow [F + s \phi] \]

\[ \text{Trivial example: } \dot{x} = 5 \]

\[ \text{e.g. stable if } z = -x \text{ and } V = \frac{1}{2} x^2 \]

\[ \ddot{x} = u \]

\[ \dot{x} = -x \]

\[ \text{So } \phi(x) = -x \]

\[ \dot{\phi} = \frac{\partial \phi}{\partial x} [F(x) + g(x) s] = -s \]

\[ z = s - \phi = s + x \]

\[ z = s - \phi = s + x \]

\[ \text{So } u = -x - x - K (s + x) = -(K+1)(s + x) \]

\[ \frac{d}{dt} [\begin{bmatrix} s \\ 0 \end{bmatrix}] = \begin{bmatrix} 0 & 1 \\ -K & -K+1 \end{bmatrix} \]
Sliding Mode Control

Goal: Robustness

\[
\dot{y} = F(y, \xi) + S_y(y, \xi) \quad \text{\textit{uncertainty}}
\]

\[
\dot{\xi} = F_\xi(y, \xi) + G_a(y, \xi) \left[ u + S_y(y, \xi, u) \right]
\]

(Note \( \dot{x} = F_c(x) + S_1(x) + G(x) \left[ u + E_2(x, u) \right] \))

Find diffeomorphism \( T(x) \)

So \( \frac{d}{dt} G(x) = \begin{bmatrix} 0 \\ G_a(x) \end{bmatrix} \), define \( \begin{bmatrix} \xi \\ \xi \end{bmatrix} = T(x) \quad \xi \in \mathbb{R}^n \quad \xi \in \mathbb{R}^p \)

1) Find \( \xi = \phi(\xi) \) w.r.t. \( \phi(0) = 0 \) so

\[ \dot{\xi} = F(\xi, \phi(\xi)) + S_y(\xi, \phi(\xi)) \]

is stable at origin

2) Define \( \zeta = \xi - \phi(\xi) \) as the sliding manifold

\[
\ddot{\zeta} = F_\xi(y, \xi) + G_a(y, \xi) \left[ u + S_y(y, \xi, u) \right] - \frac{\partial \phi}{\partial y} \left[ F(y, \xi) + S_y(y, \xi) \right]
\]

So \( u = G_a^{-1}(y, \xi) \left[ -F_\xi(y, \xi) + \frac{\partial \phi}{\partial y} F(y, \xi) + U \right] \)

Then

\[
\dot{\xi} = U + \underbrace{G_a S_y - \frac{\partial \phi}{\partial y} S_y}_{A(y, \xi, U)} \Delta(y, \xi, U)
\]

\( \Rightarrow \) With no uncertainty & \( U = 0 \), once on manifold stay there \( \forall t \)
3) Design $U$ to force $z \to 0$ despite uncertainty:

Need bound

$$\|A(q, s, \nu)\|_{\infty} \leq \rho(q, s) + k \|U\|_{\infty} \quad \forall (q, s, \nu) \in \mathcal{D}$$

where $\rho(q, s) > 0$ and $k \in [0, 1)$ are known.

This says the known part of control effect is "bigger" than the unknown part.

For each element of vector $z_i$:

$$z_i = U_i + A_i(q, s, \nu) \quad 1 \leq i \leq p$$

Use Lyapunov Function $V_i = \frac{1}{2} z_i^2$

$$V_i = z_i \dot{z}_i = z_i U_i + z_i A_i(q, s, \nu)$$

$$\leq z_i U_i + |z_i| (\rho(q, s) + k \|U\|_{\infty})$$

So choose $U_i$ large enough:

$$V_i = -\frac{\beta(q, s)}{1-k} \text{sgn}(z_i)$$

where $\beta(q, s) \geq \rho(q, s) + b \quad \forall q, s \in \mathcal{D}$

Then

$$\dot{V}_i \leq -\frac{\beta(q, s)}{1-k} |z_i|$$

$$+ \frac{\beta(q, s)}{1-k} |z_i|$$

$$= -\beta(q, s) |z_i| + \beta(q, s) |z_i|$$

$$\leq -b |z_i|$$

Note, design really started from here and worked backward.

So

i) $\lim_{t \to \infty} z_i(t) = 0 \Rightarrow z_i(t) = 0 \quad \forall t > t_0$

ii) $|z_i(t)| \leq |z_i(0)| - bt \quad \forall t > 0 \Rightarrow \text{Trajectories approach manifold } z = 0$ in finite time.
Phase portraits look like

(in Theory)

In practice, use of signum gives rise to chattering.

Use Sat F

related to size of region
That converges to
(\& prove \( v \) negative on
boundaries of region)