CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

D. MacMartin & J. Doyle — **Problem Set #8** — Issued: 29 Feb 16
Winter 2016 — Due: 9 Mar 16

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 4.4, Problem 1a** Show that for $a + b \neq 0$ the system

$$\dot{x} = \mu x - y + a(x^2 + y^2)x - b(x^2 + y^2)y + O(|x|^4)$$
$$\dot{y} = x + \mu y + a(x^2 + y^2)x + b(x^2 + y^2)y + O(|x|^4)$$

has a Hopf bifurcaton at the origin at the bifurcation value $\mu = 0$. Determine whether it is supercritical or subcritical.

2. **Perko, Section 4.4, Problem 2** Consider the C^1 system

$$\dot{x} = \mu x - y - x\sqrt{x^2 + y^2}$$
$$\dot{y} = x + \mu y - y\sqrt{x^2 + y^2}$$

   (a) Show that the vector field f defining this system belongs to $C^1(\mathbb{R}^2 \times \mathbb{R})$: i.e.. show that all of the first partial derivatives with respect to x, y and μ are continuous for all x, y and μ

   (b) Write this system in polar coordinates and show that for $\mu > 0$ there is a stable limit cycle around the origin and that for $\mu < 0$ there is no limit cycle around the origin. Sketch the phase portraits for these two cases.

   (c) Sketch the bifurcation diagram for the system, making sure that the size of the limit cycle as a function of μ is accurately represented.

3. The Moore-Greitzer equations model rotating stall and surge in gas turbine engines describe the dynamics of a compression system, such as those in gas turbine engines. The three-state "MG3" equations have the form:

$$\frac{d\psi}{dt} = \frac{1}{4B^2 l_c}\left(\phi - \Phi_T(\psi)\right),$$
$$\frac{d\phi}{dt} = \frac{1}{l_c}\left(\Psi_c(\phi) - \psi + \frac{J}{8}\frac{\partial^2 \Psi_c}{\partial \phi^2}\right),$$
$$\frac{dJ}{dt} = \frac{2}{\mu + m}\left(\frac{\partial \Psi_c}{\partial \phi} + \frac{J}{8}\frac{\partial^3 \Psi_c}{\partial \phi^3}\right) J,$$

where ψ represents the pressure rise across the compressor, ϕ represents the mass flow through the compressor and J represents the amplitude squared of the first modal flow perturbation (corresponding to a "rotating stall" disturbance). For the Caltech compressor rig, the parameters and characteristic curves are given by:

$$
\begin{aligned}
B &= 2, & \Phi_T(\psi) &= \gamma\sqrt{\psi}, \\
l_c &= 6, & \Psi_c(\phi) &= 1 + 1.5(\phi - 1) - 0.5(\phi - 1)^3, \\
\mu &= 1.25, & m &= 2.
\end{aligned}
$$

The parameter γ represents the throttle setting and typically varies between 1 and 2.

(a) Suppose that we restrict the dynamics of the system to the invariant set given by $J = 0$. Show that the system undergoes a subcritical Hopf bifurcation (this phenomena is called "surge").

(b) Compute the bifurcation diagram for the system showing the equilibrium value(s) for J as a function of γ. Make sure to capture the hysteresis loop.

(c) Suppose that we can modulate the throttle, so that $\gamma = \gamma_0 + u$. Analyze the performance of the system using the Liaw-Abed control law $u = kJ$. Show that if we choose k sufficiently large we can cause the bifurcation to stall to be supercritical.