CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

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Winter 2016

Problem Set #7

Issued: 22 Feb 16
Due: 31 Feb 16

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 4.1, Problem 1**:

   (a) Consider the two vector fields

   $$f(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} -x_2 + \mu x_1 \\ x_1 + \mu x_2 \end{bmatrix}.$$

   Show that

   $$\|f - g\|_1 = |\mu|(\max_{x \in K} \|x\| + 1)$$

   where $K \subset \mathbb{R}^2$ is a compact set containing the origin in its interior.

   (b) Show that for $\mu \neq 0$ the systems

   $$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x}_1 &= -x_2 + \mu x_1 \\ \dot{x}_2 &= x_1 + \mu x_2 \end{aligned}$$

   are not topologically equivalent.

2. Consider the dynamical system

   $$m\ddot{q} + b\dot{q} + kq = u(t), \qquad u(t) = \begin{cases} 0 & t = 0, \\ 1 & t > 0, \end{cases} \qquad q(0) = \dot{q}(0) = 0,$$

   which describes the "step response" of a mass-spring-damper system.

   (a) Derive the differential equations for the sensitivities of $q(t) \in \mathbb{R}$ to the parameters b and k. Write out explicit systems of ODEs for computing these, including any initial conditions. (You don't have to actually solve the differential equations explicitly, though it is not so hard to do so.)

   (b) Compute the sensitivities and the relative (normlized) sensitivies of the equilibrium value of q_e to the parameters b and k. You should give explicit formulas in terms of the relevant parameters and initial conditions.

   (c) Sketch the plots of the relative sensitivities $S_{q,b}$ and $S_{q,k}$ as a function of time for the nominal parameter values $m = 1$, $b = 2$, $k = 1$.

3. **Perko, Section 4.2, Problem 4**: Consider the planar system

$$\begin{aligned}\dot{x} &= \mu x - x^2\\ \dot{y} &= -y.\end{aligned}$$

Verify that the system satisfies the conditions for a transcritical bifurcation (equation (3) in Section 4.2) and determine the dimensions of the various stable, unstable and center manifolds that occur.

* Hint: try computing the linearization around the various equilibrium points.

4. **Perko, Section 4.2, Problem 7**: Consider the two dimensional system

$$\begin{aligned}\dot{x} &= -x^4 + 5\mu x^2 - 4\mu^2\\ \dot{y} &= -y.\end{aligned}$$

Determine the critical points and the bifurcation diagram for this system. Draw phase portraits for the various values of μ and draw the bifurcation diagram.

* Hint: Start by computing the linearization