CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

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Problem Set #6

Winter 2016

Issued: 17 Feb 16
Due: 24 Feb 16

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 3.4, problem 1**: Show that $\gamma(t) = (2\cos 2t, \sin 2t)$ is a periodic solution of the system

$$\dot{x} = -4y + x\left(1 - \frac{x^2}{4} - y^2\right)$$
$$\dot{y} = x + y\left(1 - \frac{x^2}{4} - y^2\right)$$

that lies on the ellipse $(x/2)^2 + y^2 = 1$ (i.e., $\gamma(t)$ represents a cycle Γ of this system). Then use the corollary to Theorem 2 in Section 3.4 to show that Γ is a stable limit cycle.

2. **Perko, Section 3.4, problem 3a**: Solve the linear system

$$\dot{x} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} x$$

and show that at any point $(x_0, 0)$ on the x-axis, the Poincare map for the focus at the origin is given by $P(x_0) = x_0 \exp(2\pi a\,/\,|b|)$. For $d(x) = P(x) - x$, compute $d'(0)$ and show that $d(-x) = -d(x)$.

3. **Perko, Section 3.5, problem 1**: Show that the nonlinear system

$$\dot{x} = -y + xz^2$$
$$\dot{y} = x + yz^2$$
$$\dot{z} = -z(x^2 + y^2)$$

has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0)$. Find the linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for the autonomous system that satisfies $\Phi(0) = I$, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

4. **Perko, Section 3.5, problem 5a**: Let $\Phi(t)$ be the fundamental matrix for $\dot{x} = A(t)x$ satisfying $\Phi(0) = I$. Use Liouville's theorem, which states that

$$\det \Phi(t) = \exp \int_0^t \text{trace} A(s) ds,$$

to show that if $m_j = e^{\lambda_j T}$, $j = 1, \ldots, n$ are the characteristic multipliers of $\gamma(t)$ then

$$\sum_{j=1}^{n} m_j = \text{trace}\Phi(T)$$

and

$$\prod_{j=1}^{n} m_j = \exp \int_0^T \mathrm{trace} A(t)\, dt.$$

* Hint: recall that the determinant of a matrix is equal to the product of its eigenvalues, and the trace of a matrix is equal to the sum of the eigenvalues.

5. **Perko, Section 3.9, problem 4a**: Show that the limit cycle of the van der Pol equation

$$\begin{aligned} \dot{x} &= y + x - x^3/3 \\ \dot{y} &= -x \end{aligned}$$

must cross the vertical lines $x = \pm 1$.

* Hint: you can use the fact (shown in Perko, Section 3.8) that a limit cycle exists for the van der Pol equation and that it is unique.