

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

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Problem Set #5

Issued: 8 Feb 16
Due: 17 Feb 16

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 3.1, problem 6, 7:** Two vector fields $f, g \in C^k(\mathbb{R}^n)$ are said to be C^k equivalent on \mathbb{R}^n if there is a homeomorphism $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $H, H^{-1} \in C^k(\mathbb{R}^n)$ that maps trajectories of $\dot{x} = f(x)$ to trajectories of $\dot{y} = g(y)$. If $\phi(t)$ and $\psi(t)$ are the dynamics systems defined by f and g respectively, then f and g are equivalent if and only if there exists a strictly increasing function $\tau(x, t)$ such that $\partial\tau/\partial t > 0$ and

$$H(\phi_t(x)) = \psi_{\tau(x,t)}(H(x)).$$

Show the following:

- (a) The equilibrium points of $\dot{x} = f(x)$ are mapped to equilibrium points of $\dot{y} = g(y)$.
(b) The eigenvalues of $Df(x_0)$ and the eigenvalues of $Dg(H(x_0))$ differ by the positive multiplicative constant $k_0 = \frac{\partial\tau}{\partial t}(x_0, 0)$.

* Hint: see the problem statements in Perko for some ideas if you get stuck

* This problem can be used to show that the stability of equilibrium points is the same for topologically equivalent systems, since the sign of the real part of the eigenvalues will not be changed.

2. **Perko, Section 3.2, problem 1:** Sketch the phase portrait for the system

$$\begin{aligned}\dot{x} &= x - x^3 \\ \dot{y} &= -y\end{aligned}$$

and answer the following questions:

- (a) Show that the interval $[-1, 1]$ on the x -axis is an attracting set for the system. Determine whether this set is an attractor or not and justify your answer.
(b) Are either of the intervals $(0, 1]$ or $[1, \infty)$ attractors?
(c) Are any of the infinite intervals $(0, \infty)$, $[0, \infty)$, $(-1, \infty)$, $[-1, \infty)$ or $(-\infty, \infty)$ on the x -axis attracting sets for this system?

3. **Perko, Section 3.2, problem 5:**

- (a) According to the corollary of Theorem 2 (in Section 3.2), every ω -limit set is an invariant set of the flow ϕ_t of $\dot{x} = f(x)$. Give an example to show that not every set invariant with respect to the flow ϕ_t is the α - or ω -limit set of a trajectory of $\dot{x} = f(x)$.
(b) Any stable limit cycle Γ is an attracting set and Γ is the ω -limit set of every trajectory in a neighborhood of Γ . Give an example to show that not every attracting set A is the ω -limit set of a trajectory in a neighborhood of A .

* Note: Perko's wording here might be a bit confusing. You need to find a dynamical system with attracting set A such that for any neighborhood around that set, you can find a trajectory (orbit) starting in that neighborhood whose ω -limit set is not equal to A .

(c) Is the cylinder in Example 3 of Section 3.2 an attractor for the system in that example?

4. **Perko, Section 3.3, problem 8:** Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)(4 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2)(4 - x^2 - y^2) \\ \dot{z} &= z.\end{aligned}$$

(a) Show that there are two periodic orbits Γ_1 and Γ_2 in the x, y plane and determine their stability.

(b) Show that there are two invariant cylinders for this system given by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(c) Describe $W^s(\Gamma_j)$ and $W^u(\Gamma_j)$, $j = 1, 2$, for the full system (in \mathbb{R}^3).