1. **Perko, Section 2.9, problem 3** Use the Lyapunov function \( V(x) = x_1^2 + x_2^2 + x_3^2 \) to show that the origin is an asymptotically stable equilibrium point of the system

\[
\dot{x} = \begin{bmatrix}
-x_2 - x_1 x_2^2 + x_3^2 - x_1^3 \\
x_1 + x_3^2 - x_2^3 \\
-x_1 x_3 - x_3 x_1^2 - x_2 x_3^2 - x_3^3
\end{bmatrix}
\]

Show that the trajectories of the linearized system \( \dot{x} = Df(0)x \) for this problem lie on circles in planes parallel to the \( x_1, x_2 \) plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

2. Determine the stability of the system

\[
\begin{align*}
\dot{x} &= -y - x^3 \\
\dot{y} &= x^5
\end{align*}
\]

Hint: motivated by the first equation, try a Lyapunov function of the form

\[
V(x, y) = \alpha x^6 + \beta y^2
\]

Is the origin asymptotically stable? Is the origin globally asymptotically stable?

3. Definition: An equilibrium point is “exponentially stable” if \( \exists M, \alpha > 0 \) and \( \epsilon > 0 \) such that

\[
\|x(t)\| \leq Me^{-\alpha t}\|x(0)\|, \forall \|x(0)\| \leq \epsilon, t \geq 0.
\]

Let \( \dot{x} = f(x) \) be a dynamical system with an equilibrium point at \( x_e = 0 \). Show that if there is a function \( V(x) \) satisfying

\[
k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2, \quad \dot{V}(x) \leq -k_3\|x\|^2
\]

for positive constants \( k_1, k_2 \) and \( k_3 \), then the equilibrium point at the origin is exponentially stable. Compute \( M \) and \( \alpha \) in terms of \( k_1, k_2, \) and \( k_3 \).

4. **Perko, Section 2.12, problem 2** Use Theorem 1 [Center Manifold Theorem] to determine the qualitative behaviour near the non-hyperbolic critical point at the origin for the system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -y + \alpha x^2 + xy
\end{align*}
\]

for \( \alpha \neq 0 \) and for \( \alpha = 0 \); i.e., follow the procedure in Example 1 after diagonalizing the system as in Example 3.
5. Consider the following system in $\mathbb{R}^2$:

\[
\begin{align*}
\dot{x} &= -\frac{\alpha}{2}(x^2 + y^2) + \alpha(x + y) - \alpha \\
\dot{y} &= -\alpha xy + \alpha(x + y) - \alpha
\end{align*}
\]

Determine the stable, unstable, and centre manifold of the equilibrium point at $(x, y) = (1, 1)$, and determine the stability of this equilibrium point for $\alpha \neq 0$. For determining stability, note that near the equilibrium point there are two 1-dimensional invariant linear manifolds of the form $M = \{(a_1, a_2) \in \mathbb{R}^2 | a_2 = ka_1\}$; determine the flow on these invariant manifolds.