CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

D. MacMartin & J. Doyle
Winter 2016

Problem Set #3

Issued: 25 Jan 16
Due: 3 Feb 16

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 2.7, problem 2, 3**

   (a) Find the first three successive approximations $u^{(1)}(t,a)$, $u^{(2)}(t,a)$, and $u^{(3)}(t,a)$ for

   $$\dot{x}_1 = -x_1,$$
   $$\dot{x}_2 = x_2 + x_1^2$$

   and use $u^{(3)}(t,a)$ to approximate S near the origin. Also approximate the unstable manifold U near the origin for this system. Note that $u^{(2)}(t,a) = u^{(3)}(t,a)$ and therefore $u^{(j+1)}(t,a) = u^{(j)}(t,a)$ for $j \geq 2$. Thus $u(t,a) = u^{(2)}(t,a)$ which gives the exact function defining S.

   (b) Sketch S, U, E^s and E^u.

2. Consider a dynamical system with $x = (u,v) \in \mathbb{R}^n$. For the case $n = 2$, prove that if

   $$\dot{u} = f(u,v), \qquad u \in \mathbb{R}^k$$
   $$\dot{v} = g(u,v), \qquad v \in \mathbb{R}^{n-k}$$

   then the manifold $S = \{(u,v) \in \mathbb{R}^k \times \mathbb{R}^{n-k} | v = h(u)\}$ is an invariant manifold of the system if

   $$g(u,h(u)) = Dh(u)f(u,h(u))$$

   Use this result to compute the stable manifold for the system of problem 1 above using the Taylor series for $h(x)$ to define $S = \{(x_1,x_2)|x_2 = h(x)\}$ and matching coefficients to solve for $h(x)$.

   - Note: The result holds for $\mathbb{R}^n$, but you only need to consider the case $n = 2$ and $k = 1$ (although you are free to prove the more general case if you prefer).
   - Hint: One way to show S is an invariant manifold in $\mathbb{R}^2$ is to show that the normal vector (orthogonal to the tangent to S at $(x,h(x))$) is orthogonal to the vector field (f,g) at that point. (It is sufficient to prove the result for $\mathbb{R}^2$.)

3. **Perko, Section 2.6, problem 3** Show that the continuous map $H : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

   $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = H(x) = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + x_1^2/3 \end{bmatrix}$$

has a continuous inverse $H^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ and that the nonlinear system

$$\frac{dx}{dt} = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$$

is transformed into a linear system using this transformation, i.e., if $y = H(x)$, show that $\dot{y} = Ax$ for an appropriate A. Use this transformation to compute and sketch the stable and unstable manifolds for the nonlinear system.

* Note: since the nonlinear system can be transformed into a linear one, it follows that $S = H^{-1}(E^s)$ and $U = H^{-1}(E^u)$.

4. **Perko, Section 2.8, Problem 1** Solve the system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= -y_2 + z^2 \\ \dot{z} &= z \end{aligned}$$

and show that the successive approximations $\Phi^{(k)} \to \Phi$ and $\Psi^{(k)} \to \Psi$ as $k \to \infty$ for all $x = (y_1, y_2, z) \in \mathbb{R}^3$. Define $H_0 = (\Phi, \Psi)^T$ and use this homeomorphism to find

$$H(x) = \int_0^1 e^{-As} H_0\big(\phi_s(x)\big) ds,$$

where A is the linearization of the nonlinear dynamics at the origin and $\phi_t(x)$ is the flow of the full system. Use the homemorphism H to find the stable and unstable manifolds

$$W^s(0) = H^{-1}(E^s) \quad \text{and} \quad W^u(0) = H^{-1}(E^u)$$

for this system.

- Note: this problem involves some simple but somewhat tedious computations. If you know how to use Mathematica or a similar program, you may wish to carry out the computations for the successive approximations using that software. However, make sure to show the results at each step of the calculation.
- Hint: You should find

$$H(y_1, y_2, z) = \begin{bmatrix} y_1 \\ y_2 - z^2/3 \\ z \end{bmatrix}, \qquad \begin{aligned} W^s(0) &= \{x \in \mathbb{R} | z = 0\}, \\ W^u(0) &= \{x \in \mathbb{R} | y_1 = 0, y_2 = z^2/3\}. \end{aligned}$$