CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

D. MacMartin & J. Doyle — **Problem Set #1** — Issued: 4 Jan 16
Winter 2016 — Due: 13 Jan 16

1. **(Based on Perko, Section 1.3, Exercise 5, 6)**

   (a) For each matrix below, find the eigenvalues and eigenvectors of A and e^A:

   (i) $\begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

   (remember to show the steps required for these [simple!] computations, don't just plug in values from MATLAB or Mathematica; see notes at the bottom of the page).

   (b) For any arbitrary $A \in \mathbb{R}^{n\times n}$, show that if x is the eigenvector of A corresponding to the eigenvalue λ, then x is also an eigenvector of e^A corresponding to the eigenvalue e^λ.

2. **(Based on Perko, Section 1.4, Exercise 6)** Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that leaves a subspace $E \subset \mathbb{R}^n$ invariant (i.e., for all $x \in E$, $Ax \in E$). Show that if $x(t)$ is the solution of the initial value problem

$$\dot{x} = Ax \qquad x(0) = x_0 \tag{1}$$

   with $x_0 \in E$, then $x(t) \in E$ for all $t \in \mathbb{R}$.

3. **Perko, Section 1.6, Exercise 2:** Solve the initial value problem (1) with

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{2}$$

   Determine the stable and unstable subspaces and sketch the (3D) phase portrait. (Hint: see Figure 1 in Section 1.6 for an example of a 3D phase portrait.)

4. **Perko, Section 1.9, problem 5, parts (c), (d2)**: Let A be an $n \times n$ nonsingular matrix and let $x(t)$ be the solution of the initial value problem (1) with $x(0) = x_0$. Show that

   (c) If $x_0 \in E^c$, $x_0 \neq 0$ and A is semisimple, then there are positive constants m and M such that for all $t \in \mathbb{R}$, $m \leq |x(t)| \leq M$

   * Note: in the book, Perko defines $\sim$ to mean "set subtraction". So $E \sim \{0\}$ in the book is the set E minus the point 0.

   (d2) If $E^s \neq \{0\}$, $E^u \neq \{0\}$, and x_0 has nonzero components in both E^s and E^u, then $\lim_{t\to\pm\infty} |x(t)| = \infty$;

5. Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) Show that the unforced system ($u = 0$) is stable but not asymptotically stable.

(b) Given $x(0) = x_0$ and $u(t) = \cos(\omega t)$, solve for the output $y(t)$. Show that when $\omega = 1$ the output is unbounded.

Notes:

- The problems are transcribed above in case you don't have access to Perko. However, in the case of discrepancy, you should use Perko (third edition) as the definitive source of the problem statement.
- There are a number of problems that can be solved using MATLAB, Mathematica or a similar program. If you just give the answer with no explanation (or say "via MATLAB"), the TA will take off points. Instead, you should show how the solutions can be worked out by hand, along the lines of what is done in the text book. It is fine to check everything with MATLAB or your favorite software tool.
- For numerical calculations, it is OK to use MATLAB to invert a matrix. But you should not use it to compute the matrix exponential and just put down the answer. Instead, show how to get the matrix exponential into a form in which the calculation can be done by hand and then carry out the computation.
- For phase portraits, you should generate the diagram by hand and make sure to label any important features. Describe why the portrait looks as it does based on the relevant properties of the dynamical system (eg, eigenvalues of the A matrix).
- For the final exam, you will not be allowed to use MATLAB, Mathematica or similar programs, so make sure you understand what you are computing and drawing!