Question: How do solns depend on θ (or µ)?
- Family of solns x(θ) w/ smooth dependence on θ
- Infinitesimal δθ \rightarrow complete change in soln

\[ x = F(x, θ) \quad \text{for some parameter } θ \]
\[ y = H(x, θ) \]

Equilibrium: \[ F(x_e, θ) = 0 \]
\[ \Rightarrow \frac{δF}{δx} \frac{dx_e}{δθ} + \frac{δF}{δθ} = 0 \]
\[ \frac{dx_e}{δθ} = -\left( \frac{δF}{δx} \right)^{-1} \frac{δF}{δθ} \]

(Provided non-singular)

\[ \frac{δx_e}{δθ} = \frac{1}{x_e} \frac{dx_e}{δθ} + \frac{δh}{δθ} \]
\[ = \frac{1}{x_e} \left( -\frac{δF}{δx} \right) \frac{δF}{δθ} + \frac{δh}{δθ} \]

... compute numerically

Or look at relative changes:

\[ \frac{δy_0}{δθ} = \frac{δx_e/xe}{δθ} = \frac{θ}{x_e} \frac{dx_e}{δθ} \]

(or multivariable)

\[ \frac{δy_{θ_1, θ_2}}{δθ} \frac{θ_1}{θ_2} = \text{array} \]

\[ \frac{δy_0}{δθ} \text{ [θ, θ_1]} \]
\[ C' \text{ norm} \quad \|F\| = \sup_{x \in K} \|F(x)\| + \sup_{x \in K} \|\frac{\partial F(x)}{\partial x}\| \quad (\text{Eulerian}) \]

\( \text{Defn: } F \in C'(E) \text{ is structurally stable if } \exists \delta > 0 \text{ s.t. } \forall \epsilon \in C(E) \]

\[ \|F - \epsilon\| < \delta \implies F \text{ and } \epsilon \text{ are topologically equivalent} \quad (\text{with positive time-scaling}) \]

Then let \( F \in C'(E) \) be hyperbolic at pt \( x_0 \). Then \( \exists \delta > 0 \text{ s.t.} \]

\[ \|F - \epsilon\| < \delta \implies \exists \delta < \delta \text{ s.t.} \]

i) \( \delta(x) = 0 \), \( x_0 \) is hyperbolic
ii) \# +ve & -ve eig unchanged (but not Torbo for)

Remarks

1. If \( x_0 \) is a hyperbolic eq pt. Then any \( C' \text{ - nearby} \) dynamical system has the same \# of +ve and -ve eigenvalues.

(\# of \( F \text{ is structurally stable on } K \) containing a hyperbolic eq pt.)

Then all nearby vector fields must have \# of +ve & -ve eig.

\( \text{No eig go evn though } \Re(\lambda) = 0 \)

- Given \( F(x, \theta) \) \& \( F(x, 0) \) is structurally stable then \( F(x, \theta) \) will not change stability type for \( \theta \) small enough.

2. Can show that for a linear system \( x = Ax \) then

\( \text{struct stab } \iff x = 0 \) is hyperbolic

3. Lots of interesting behaviour for non-local behaviour. (Use \( \mu = 1 \rightarrow 0 \))

Example:

\[ x_1 = -x_1 + \mu x_2^3 \quad \text{Q1: Is this system stable for } \mu = 0? \]
\[ x_2 = -x_2 + \mu x_1^3 \quad \text{A1: Yes, } \lambda = -1, 1 \]

Q2: For any \( \mu \neq 0 \), is there an eq pt \& is it stable? (Hard to solve in general)

... if this \( \delta \)-s ball - some distance of linearization ...

Check distance: let \( K = B_\epsilon(0) \)

\[ \|F - \epsilon\| = \sup_{x \in K} \left( \mu \sqrt{x_1^2 + x_2^2} \right) + \sup_{x \in K} \left( 3 \mu x_2^3 x_1 \right) \]

For any \( x \) \& \( \delta \), \( \exists \delta > 0 \text{ s.t. } \|F - \epsilon\| < \delta \)

And since \( \epsilon \) is struct. stable \( \exists \epsilon > 0 \text{ s.t. } \)

\( x_0 \) is a stable eq pt. for \( F(x, \epsilon) \).
1. Stable Node Bifurcation
   \[ x = x - x^2 \]

2. Transcritical Bifurcation
   \[ x = x(x - x^2) \]

3. Pitchfork Bifurcation
   \[ x = x(x - x^2) \]

Theorem (Sotomayor):
If \( a, b, c > 0 \) and \( \frac{\partial^2 F}{\partial x^2} (s, a, b, c) = 0 \) and \( \frac{\partial^2 F}{\partial x \partial y} (s, a, b, c) = 0 \), then \( (s, a, b, c) \) is a saddle-node bifurcation point.

Let \( A \cdot x = 0 \) and \( \lambda \mathbf{A} = 0 \) define left eigenvectors \( \lambda \).

Assume \( K \) of remaining eigenvalues are stable and \( n - 1 \) are unstable.

Define
\[
\theta = C \quad \text{if} \quad \text{stable,} \quad \text{and} \quad \lambda \mathbf{A} = 0 \]
\[
\varphi = D \quad \text{if} \quad \text{unstable,} \quad \text{and} \quad \lambda \mathbf{A} = 0 \]

Then:
(i) \( \alpha = 0, \beta > 0 \)
   \[
   a = 0, b > 0, c > 0, d > 0
   \]
(ii) \( \alpha = 0, b = 0 \)
   \[
   a = 0, b > 0, c > 0, d > 0
   \]
Notes on Bifurcations

1. Bifurcation when any eigenvalue reaches $\text{Re}(\lambda) = 0$
   - One real eigenvalue
   - Pair of complex eigenvalue (Hopf)
   - Simultaneous crossing of more than one (higher order)
   - Note: Time constants get long as you approach bifurcation

2. Restrict attention to dimension of centre manifold...

3. For one real eigenvalue, potential function picture may be useful:

   a) Saddle-node
      \[ \dot{x} = \mu - x^2 \]
      \[ V(x) \text{ is cubic} \]

      \[ \mu > 0 \quad \rightarrow \quad m = 0 \quad \rightarrow \quad \mu < 0 \]

   b) Transcritical
      \[ \dot{x} = \mu x - x^2 \]
      \[ V(x) \text{ cubic} \]
      \[ m \text{ changes quadratic} \]

      \[ m > 0 \quad \rightarrow \quad m = 0 \quad \rightarrow \quad m < 0 \]

   c) Pitchfork
      \[ \dot{x} = \mu x - x^3 \]

      \[ m > 0 \quad \rightarrow \quad m = 0 \quad \rightarrow \quad m < 0 \]
Bifurcations

Consider a $C^1$ dynamical system on $E \subset \mathbb{R}^n$ given by

$$\Sigma: \dot{x} = F(x, \mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^p, \quad (t = 1 \text{ for new})$$

**Defn:** The system $\Sigma$ undergoes a bifurcation at $\mu = \mu_0$ if $F(x, \mu)$ is not struct. stable at $\mu = \mu_0$.

$\Rightarrow$ As $\mu$ goes from $\mu < \mu_0$ to $\mu > \mu_0$ we might get a discrete change in solution.

**Examples:**

1. $\dot{x}_1 = x_2$
   $\dot{x}_2 = -kx_1 - mx_2$

   ![Diagram](image)

   No bifurcation
   (transformation to undist.)

2. $\dot{x}_1 = x_2$
   $\dot{x}_2 = m x_2 + x_1 - x_3$

   ![Diagram](image)

   $\mu < 0$ $\Rightarrow$ No equilibria $\Rightarrow$ continuity preserves limit

   ![Diagram](image)

   $\mu > 0$

   This is not a bifurcation for $x = 0 \Rightarrow$ always a saddle.

   $\mu = 0$

   This is bifurcation
   For $x_i = \pm 1$, $E = B_{\epsilon}(X_i)$

   This is also a global bifurcation.

**Remark:** Lots of specialized results for $\mathbb{R}^2$ - we will focus on $\mathbb{R}^n$, but can often restrict to $\mathbb{R}^2$ [explore a centre manifold bifurcation]

**Common bifurcations:**

Visualize using bifurcation diagram

$$x = F(x, \mu)$$

$$y = h(x)$$

![Diagram](image)

Dashed & solid = US S.
Let \( n = 1 \), saddle node: \( y^* = F(y; x) = 0 \) \( \Rightarrow \) \( y = \frac{\sqrt{F}}{\gamma} \).

From 0 \( \Rightarrow \) 0 we can use implicit function theorem. To write \( y = m(x) \), choose \( \mu_0 = 0 \).

To compute change in eigenvalues:
\[
\frac{\partial \lambda}{\partial \mu} = -\frac{\partial F}{\partial x} \frac{\partial x}{\partial \mu}
\]

For \( n > 1 \), choose projections as relevant eigenvalues (transpose).

Example: MG3 (Here-Greffe).

\( \text{State } \emptyset, y, J \)