Stable (& Unstable) Manifolds

Idea:

- Initial conditions converge to origin
  - Locally: get subspace from linearization, can get successively better approx.

For hyperbolic equilibrium points,

Flow of non-linear system is topologically equivalent to the linearization (Hartman - Grobman)

- Flow along stable & unstable manifolds

For non-hyperbolic, divide into stable, unstable, and center

\[ \dot{x} = f(x) \Rightarrow \dot{x} = Ax + F(x) \]

Higher order

Change of variables to \( y = Cx \) so that

\[ y' = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} y + (4t) \] & eigenvalues of \( P \) have

- Real part < 0,
- \( \text{eig of } Q, \text{ real part } > 0 \)

For any eigenvalues of \( A = 0 \) (or Real part = 0), local behavior determined entirely by non-linear terms.

Questions

1. Prove stable manifold exists.
2. Tools to calculate it
3. Prove flow is topologically equivalent
4. Revisit stability
5. Behavior on center manifold

Today

- 1. Prove stable manifold exists.
- 2. Tools to calculate it

Thursday

- 3. Prove flow is topologically equivalent
- 4. Revisit stability

Next week

- 5. Behavior on center manifold
Stable manifold Theorem

Example

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 2x_2 - 5\varepsilon x_1^3
\end{align*}
\]

\[
\begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
-5\varepsilon x_1^3
\end{bmatrix}
\]

- From linearization, origin is saddle node
- Already diagonalized linearization
- For linear system \( E^s = \text{span} \{ [0] \} = \text{stable subspace} \)
  \( E^u = \text{span} \{ [0] \} = \text{unstable subspace} \)
- Can show that \( S = \{ x \in \mathbb{R}^2 \mid x_2 = \varepsilon x_1^3 \} \) is invariant with the flow

\[
\begin{bmatrix}
a \\
\varepsilon a^3
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 \\
-3\varepsilon a^2
\end{bmatrix}
\]

which is tangent to \( S \)

\( S \) can solve for flow, \( \Phi_t \left( \begin{bmatrix} a \\ \varepsilon a^3 \end{bmatrix} \right) =
\begin{bmatrix}
a e^{-t} \\
\frac{(\varepsilon a^3 - a^3) e^{2t} + \varepsilon a^3 e^{-3t}}{\varepsilon^3}
\end{bmatrix}
\)

\( E^s \)

\[
\begin{bmatrix}
a e^{-t} \\
\varepsilon^3 e^{-3t}
\end{bmatrix}
\] \( \in \mathbb{R}^2 \)

- \( S \) is the stable manifold

For this system, \( (\text{trajectories} \to 0 \text{ as } t \to \infty) \)
- \( S \) is tangent to \( E^s \) at the origin
- \( U = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \} \) is the unstable manifold

(trajectories \( \to 0 \text{ as } t \to -\infty \), and invariant with flow)

Definition (see Text for more formal or precise def'-)

A \( k \)-dimensional differentiable manifold is a "smooth" \( k \)-dimensional
surface in an \( n \)-dimensional space of order \( C^m \)

\( \text{manifold of class } C^m \)
Theorem: Let $E$ be an open subset of $\mathbb{R}^n$ containing the origin $0 \in E$, and let $\phi_t$ be the flow of $\dot{x} = F(x)$. Suppose $F(0) = 0$ and $DF(0)$ has $k$ eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a $k$-dimensional manifold $S$ tangent to the stable subspace $E^s$ of $\dot{x} = DF(0)x$ at the origin such that $\forall t \geq 0$, $\phi_t(0) \in S$ and $\forall x_0 \in S$, $\lim_{t \to \infty} \phi_t(x_0) = 0$.

And, $\exists$ $(n-k)$-dimensional differentiable manifold $U$ tangent to the unstable subspace $E^u$ of $\dot{x} = DF(0)x$, such that $\forall t \leq 0$, $\phi_t(0) \in U$ and $\forall x_0 \in U$, $\lim_{t \to -\infty} \phi_t(x_0) = 0$.

(Aside: $S$ and $U$ may overlap, e.g. $\phi_t$)

Proof:

1. First note, can write $\dot{x} = F(x)$ as $\dot{x} = Ax + F(x)$ or $\dot{y} = By + G(y)$, where $B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, $P, Q \in \mathbb{R}^{k \times k}$, $G(y) = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-k} \end{bmatrix}$, $y(t) = e^{Bt}y(0) + \int_0^t e^{B(t-s)}G(y(s))\,ds$.

2. General form of solution satisfies $Q \in \mathbb{R}^{(n-k) \times (n-k)}$ positive.

3. Choose $\|y_0\|_\infty = 0 \Rightarrow y(t) \to 0$ as $t \to \infty$.

Work backwards: Find specific soln $u(t, 0) \to 0$. The flow $\phi_t$
General Idea:

Define $U = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$

and $a \in E^3 = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$

For the initial condition $y(0) = \begin{bmatrix} a_1 \\ a_2 \\ \phi \end{bmatrix}$, then define

$\psi(t,a) = y(t)$, and

$u(t,a) = U(t)y(0) + V(t)y(0) + \int_0^t U(t-s) G(u(s,a)) ds$ + $\int_0^t V(t-s) G(\psi(s,a)) ds - \int_0^t V(t-s) G(\phi(s,a)) ds$

is a solution to $\dot{y} = By + G(y)$.

These terms do not converge for $t \to \infty$.

5. If we choose $\psi_k$ so that these terms cancel, then $u(t,a) \to 0$.

Thus

$\psi_k(t,a) = -\int_0^t V(t-s) G(u(s,a)) ds$

and

$u(t,a) = U(t)a + \int_0^t U(t-s) G(u(s,a)) ds - \int_t^\infty V(t-s) G(u(s,a)) ds$

satisfies $\dot{y} = By + G(y)$ and $\lim_{t \to \infty} u(t,a) = 0$.

The stable manifold is $S = \{ (y_1, \ldots, y_n) | y_k = \psi_k(y_1, \ldots, y_n), i = k+1 \ldots n \}$.
Finding Stable Manifold

Perko: Find successive approximate solutions to integral equation
(Note proof requires showing these converge)

Alternate: Taylor series, add quadratic term, then...

\[ U^0(t,a) = 0 \]
\[ U^{n+1}(t,a) = U(t) a_t + \int_0^t U(t-s) G(U^s, a) ds - \int_0^\infty V(t-s) G(a(s), a) ds \]

(Aside: Full proof in Perko shows this converges)

Example (in Perko)

\[ \dot{x}_1 = -x_1 - x_2^2 \quad \text{so} \quad A = B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]
\[ \dot{x}_2 = x_2 + x_1^2 \quad \text{(already diagonalized)} \]

and \( G(x) = \begin{bmatrix} -x_2^2 \\ x_1^2 \end{bmatrix} \)

\[ a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \]

\[ U^0(t,a) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad U^{(1)}(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix} \implies \text{gives } E^3 \text{ for linear system} \]

\[ U^{(2)}(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-s)} 0 \\ 0 e^{-(t-s)} \end{bmatrix} \begin{bmatrix} 0 & -2 s a_1 \\ 0 & e^{2 s} a_1^2 \end{bmatrix} ds - \int_0^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{(t-s)} \end{bmatrix} \begin{bmatrix} 0 \\ e^{2 s} a_1^2 \end{bmatrix} ds \]

\[ = \begin{bmatrix} e^{-t} a_1 \\ -\frac{e^{-t} a_1^2}{3} \end{bmatrix} \]

This would give approximation

\[ y_2(a) = -\int_0^\infty e^{-s} e^{-2 s} a_1^2 ds = -\frac{1}{3} a_1^2 \]

\[ = U^{(2)}(0, a_1) \]

So \( S \approx \{ x \in \mathbb{R}^2 \mid x_2 = -\frac{x_1^2}{3} \} \quad \text{or} \quad X_2 = -\frac{x_1}{3} + O(x_1^3) \)

Similarly, get \( U \) as the stable manifold of \( \dot{x}_1 = x_1 + x_2 \)

(Remember to choose \( y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \), and also...)

\( \dot{x}_2 = -x_2 - x_1^2 \)
Before looking at Taylor expansion...

\textbf{Remark (\textit{o-invariant manifold})}

If \( x = F(x,y), \, x \in \mathbb{R}^k \)
\[ y = g(x,y), \, y \in \mathbb{R}^m \quad (m = n-k) \]

and
\[ S = \{ (x,y) \in \mathbb{R}^k \times \mathbb{R}^m \mid y = h(x) \} \]

manifold defined by constraint \( h(x) \)

Then if \( g(x, h(x)) = Dh(x)F(x, h(x)) \) then \( S \) is an invariant manifold of the system

\textbf{Proof:} Target vector to \( S \) at \((x, y) = (x, h(x))\) is \( \vec{\tau} = (1, Dh(x)) \)

normal \( \vec{n} = (-Dh(x), 1) \)

\[ (\vec{\tau} \cdot \vec{n} = 0) \]

The vector field at this point is \((F(x, h(x)), g(x, h(x)))\)

So \( \vec{n} \cdot (F(x, h(x)), g(x, h(x)) ) = -Dh(x) F(x, h(x)) + g(x, h(x)) = 0 \)

Since vector field at any point on \( S \) is target to \( S \)

\( S \) is invariant wrt flow of the system

\textbf{Taylor expansion for finding} \( S \):

\[ S = \{ (x,y) \in \mathbb{R}^k \times \mathbb{R}^m \mid y = h(x) \} \]

write \( y = h(x) \)

\[ = ax^2 + bx^3 + \ldots \]

Since \( S \) is an invariant manifold

\[ \text{then} \quad Dh(x) \dot{x} = \dot{y} \quad \text{or} \quad Dh(x) F(x, h(x)) = g(x, h(x)) \]
Example: \[ x_1 = -x_1 - x_2 \leftarrow "F(x, y)" \]
\[ x_2 = x_2 + x_1 \leftarrow "g(x, y)" \]

Write \( h(x) = a x_1^2 + b x_1^3 + O(x_4) \)

So \((2ax_1 + 3bx_1^2)(-x_1 - x_2) = x_1^2 + (ax_1^2 + bx_1^3) \uparrow = ax_1^2 + bx_1^3 \)

or \(-2ax_1 - 2bx_1^3 - 3bx_1^2 - 3bx_1^2 - 3bx_1^3 = x_1^2 (1+a) + bx_1^3 \)

\[ x_1^2 : -2a = 1 + a \Rightarrow a = -\frac{1}{3} \]
\[ x_1^3 : -2ab - 3b = b \Rightarrow b = 0 \]

e tc.

So \( S = \{ (x_1, x_2) : x_2 = -\frac{1}{3} x_1^2 \} \) which is the same as before.

Global Manifolds

Note that the proof only finds \( S, U \) locally.

The \underline{global} stable and unstable manifolds of \((1)\) will flow \( \phi_t \) at the origin \( x = 0 \) are defined by

\[ W^s(0) = \bigcup_{t \leq 0} \phi_t(S) \]
\[ W^u(0) = \bigcup_{t \geq 0} \phi_t(0) \]

These are unique and invariant with flow, and
\[ \forall x \in W^s(0), \lim_{t \to 0^+} \phi_t(x) = 0, \forall x \in W^u(0), \lim_{t \to 0^-} \phi_t(x) = 0 \]

Also note exponential convergence/divergence near enough to origin.
Hartman–Grobman Thm

Let \( F \in C^1(E) \), where \( E \) is an open subset of \( \mathbb{R}^n \) containing the origin, let \( \phi_t \) be the flow of \( \dot{x} = F(x) \).

If \( F(0) = 0 \) and \( A = DF(0) \) has no eigenvalues with zero real part (hyperbolic equlibrium) then there exists a homeomorphism \( H : U \to V \), for open sets \( U, V \) containing the origin, such that \( H(x_0) \in U \) and \( \text{CR} \) containing zero so if \( x_0 \in U, t \in \mathbb{T} \),

\[
H_0 \phi_t(x_0) = e^{At} H(x_0)
\]

Translation: \( H \) maps trajectories of \( \dot{x} = F(x) \) onto trajectories of the linearization \( \dot{x} = DF(0)x \).

where

\[
H \text{ is a continuous, one to one, invertible map; } \quad \text{Definition of } H : U \to V \text{ and } H^{-1} : V \to U \text{ is also continuous } \quad \text{homeomorphism on a metric space.}
\]

i.e. close to equilibrium point \( x_0 \)

\[
\dot{x} = f(x) \text{ behaves like } \dot{x} = DF(0)(x - x_0)
\]

Remarks:

1. \( A \) & \( B \) are called homeomorphic or topologically equivalent if there is a homeomorphism of \( A \) onto \( B \).

2. Two autonomous systems of differential equations are said to be topologically equivalent if there is a homeomorphism \( H \) mapping open set \( U \) containing origin onto set \( V \) containing origin which maps trajectories of one system in \( U \) onto trajectories of the other system in \( V \).

3. Hartman–Grobman is most useful conceptually rather than any value it knowing \( H \) directly.
4. Actually follows from Stable Manifold Thm, but close enough to equilibrium, convergence/divergence is exponential:

\[ \text{Re}(\lambda_j) < -\alpha < 0 < \beta < \text{Re}(\lambda_m) \text{ for } j = 1, \ldots, k, m = k + 1, \ldots, n \]

Then \( \forall \epsilon > 0 \exists \delta > 0 \exists x_0 \in N_\delta(0) \land S \Rightarrow \| \phi_t(x_0) \| < \epsilon e^{-\alpha t} \forall t > 0 \)

& \( x_0 \in N_\delta(0) \land U \Rightarrow \| \phi_t(x_0) \| < \epsilon e^{\beta t} \forall t > 0 \)

Example:

1. \( \frac{dx_1}{dt} = (-x_1) \quad \text{&} \quad \frac{dx_i}{dt} = (-1, 0) (x_i) \)

are topologically equivalent.

\[
\begin{bmatrix}
  e^{-t} & 0 \\
  0 & e^t
\end{bmatrix}, \quad \phi_t(x_0) = \begin{cases}
  a_1 e^{-t} \\
  (a_2 + \frac{1}{3} a_1^2) e^t - \frac{1}{3} a_1^2 e^{-2t}
\end{cases}
\]

\[ a_0 = \frac{a_1}{a_2} \]

The homeomorphism

\[ H(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 + \frac{1}{3} x_1^2 \end{bmatrix}, \quad H^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 - \frac{1}{3} x_1^2 \end{bmatrix} \]

satisfies

\[ e^{At} H(x_0) = \begin{bmatrix} a_1 e^{-t} \\
  (a_2 + \frac{1}{3} a_1^2) e^t - \frac{1}{3} a_1^2 e^{-2t}
\end{bmatrix} \]

\[ H \circ \phi_t(x_0) = \begin{cases}
  a_1 e^{-t} \\
  (a_2 + \frac{1}{3} a_1^2) e^t - \frac{1}{3} a_1^2 e^{-2t} + \frac{1}{3} a_1^2 e^{-2t}
\end{cases} \]

\[ \Rightarrow \text{So not only is there a "correct" unstable manifold, tangent to unstable subspace of linearization, but the flow can be (locally) mapped to the linearization.} \]
Lyapunov Functions

Idea: Find an energy-like function. If it is always decreasing, then the system must converge to the zero "energy" state.

Note:
For $F \in C(E)$ with flow $\phi$, and function $V(x) \in C(E)$, then

$\dot{V}(x) = \frac{dV}{dt} = D(V(x)) \cdot F(x)$

Thus:
Let $E$ be an open subset of $\mathbb{R}^n$ containing $x_0$.
Suppose $F \in C(E)$, $F(x) = 0$, and $\exists V \in C^1(E)$ (real-valued) with $V(x_0) = 0$, $V(x) > 0$ if $x \neq x_0$.

Then:
1) if $\dot{V}(x) \leq 0 \ \forall x \in E$, $x_0$ is stable.
2) if $\dot{V}(x) < 0 \ \forall x \in E \setminus \{x_0\}$, $x_0$ is asymptotically stable.
3) if $\dot{V}(x) > 0 \ \forall x \in E \setminus \{x_0\}$, $x_0$ is unstable.

(Otherwise, no information $\Rightarrow$ Try a different function $V$.)