CDS140a
Nonlinear Systems: Local Theory
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3 Stability and Lyapunov Functions

3.1 Lyapunov Stability

Definition:

- An equilibrium point $x_0$of (1) is *stable* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$, we have $\phi_t(x) \in N_\epsilon(x_0)$.
- An equilibrium point $x_0$of (1) is *unstable* if it is not stable.
- An equilibrium point $x_0$of (1) is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have $\lim_{t\to\infty} \phi_t(x) = x_0$.

Remarks:

- The about limit being satisfied does not imply that $x_0$is stable (why?).
- From H-G theorem and Stable manifold theorem, it follows that hyperbolic equilibrium points are either asymptotically stable (sinks) or unstable (sources or saddles).
- If x_0 is stable then no eigenvalue of $Df(x_0)$ has positive real part (why?)
- $x_0$is stable but not asymptotically stable, then $x_0$is a non-hyperbolic equilibrium point

Example: Perko 2.9.2 (c) Determine stability of the equilibrium points of :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 + 4 \\ x_1 x_2 \end{bmatrix}$$

Equilibrium points are $(0, 2)$, $(1, 0)$.

$$\begin{aligned} Df(x) &= \begin{bmatrix} -4 & -2 \\ x_2 & x_1 \end{bmatrix} \\ Df(0,2) &= \begin{bmatrix} -4 & -2 \\ 2 & 0 \end{bmatrix} \\ Df(1,0) &= \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

What can we say in general about the stability of non-hyperbolic equilibrium points?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - x_1 x_2 \\ x_1 + x_1^2 \end{bmatrix}$$

3.2 Lyapunov Functions

Definition: Let $f \in C^1(E)$, $V \in C^1(E)$ and ϕ_t the flow of the differential equation 1. Then for $x \in E$ the derivative of the function $V(x)$ along the solution $\phi_t(x)$ is

$$\dot{V}(x) = \frac{d}{dt} V(\phi_t(x)) = \frac{\partial V(\phi_t)}{\partial \phi_t} \frac{d}{dt} \phi_t(x) = DV(x) f(x)$$

Theorem (Lyapunov's Direct Method): Let E be an open subset of $\mathbb{R}^n$ containing x_0. Suppose $f \in C^1(E)$ and that $f(x_0) = 0$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying

$V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then
(a) if $\dot{V}(x) \leq 0$ for all $x \in E$, x_0 is stable;
(b) if $\dot{V}(x) < 0$ for all $x \in E \backslash \{x_0\}$, x_0 is asymptotically stable;
(c) if $\dot{V}(x) > 0$ for all $x \in E \backslash \{x_0\}$, x_0 is unstable;

Proof: Without loss of generality, we assume that $x_0 = 0$.
Part (a)

We want to show that
"for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in N_\delta(0)$ and $t \geq 0$, we have $\phi_t(x) \in N_\epsilon(0)$"
Outline:

1. Construct a closed set (ball) $B_r \subset N_\epsilon$, such that $B_r \subset E$ (i.e., a technicality to make sure we remain in the domain)

2. Construct $\Omega_\beta = \{x \in B_r | V(x) \leq \beta\}$ (i.e., "a subset of the β sublevel set of V) such that Ω_β lies in the interior of B_r

   - Can show that condition (a) implies that $x \in \Omega_\beta \Rightarrow \phi_t(x) \in \Omega_\beta$

3. Construct $N_\delta(0) \subset \Omega_\beta$

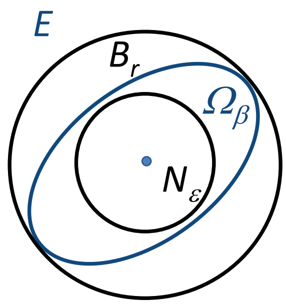

Figure 1: Sets used in the proof.

Then since $N_\delta(0) \subset \Omega_\beta \subset B_r \subset N_\epsilon(0)$, we have that $x \in N_\delta \subset \Omega_\beta \Rightarrow \phi_t(x) \in \Omega_\beta \Rightarrow \phi_t(x) \in N_\epsilon(0)$.

Details:

1. Given *any* $\epsilon > 0$, choose $0 < r \leq \epsilon$, such that

$$B_r = \{x \in \mathbb{R}^n | \, |x| \leq r\} \subset E.$$

2. Let $\alpha = \min_{|x|=r} V(x)$ (i.e., the minimum of V in the boundary of B_r). Take $0 < \beta < \alpha$, $\Omega_\beta = \{x \in B_r | V(x) \leq \beta\}$. Then it can be easily shown that Ω_β lies in the interior of B_r (if a point a is in the boundary, then $V(a) \geq \alpha > \beta$). Notice that for $x = \phi_0(x) \in \Omega_\beta$, and for all t

$$\begin{aligned} V(\phi_t(x)) - V(\phi_0(x)) &= \int_0^t \frac{d}{ds} V(\phi_s(x)) ds \leq 0 \qquad \left(\text{since } \frac{d}{ds} V(\phi_s(x)) \leq 0\right) \\ &\Downarrow \\ V(\phi_t(x)) &\leq V(\phi_0(x)) \leq \beta \end{aligned}$$

and therefore $\phi_t(x) \in \Omega_\beta$ ($\phi_t$cannot exit B_r since it would mean going through the boundary of B_r).

3. Since V is continuous, $V(0) = 0$ then there exists a $\delta > 0$ such that $|x| < \delta \Rightarrow V(x) < \beta$. Therefore for

$$x \in N_\delta \Rightarrow x \in \Omega_\beta \Rightarrow \phi_t(x) \in \Omega_\beta \Rightarrow \phi_t(x) \in B_r \Rightarrow \phi_t(x) \in N_\epsilon(0).$$

So for any $\epsilon > 0$ we constructed a δ such that for all $x \in N_\delta(0)$ and $t \geq 0$, we have $\phi_t(x) \in N_\epsilon(0)$, and therefore the origin is stable.

Part (b) *Note:* Intuitively, condition $\dot{V}(x) < 0$, $x \neq 0$ s(i.e., $V(x)$ is strictly decreasing along the trajectories of 1) implies that as t increases, the trajectory moves into lower level sets of $V(x)$. We just need to show that it eventually goes to 0.

In part (a) we showed that the origin is stable. What we need to show is that "there exists a $\delta > 0$ such that for all $x \in N_\delta(0)$ we have $\lim_{t\to\infty} \phi_t(x) = 0$", i.e., "there exists a $\delta > 0$ such that for all $\epsilon > 0$, there exists a $T > 0$ such that for all $x \in N_\delta(0)$ and $t > T$, $|\phi_t(x)| < \epsilon$ (or $\phi_t(x) \in N_\epsilon(0)$)". But since we showed that for all $\epsilon > 0$ we can construct β such that $\Omega_\beta \subset N_\epsilon(0)$, i.e.,

$$\phi_t(x) \in \Omega_\beta \Rightarrow \phi_t(x) \in N_\epsilon(0).$$

Therefore, it is sufficient to show that for all $x \in N_\delta(0)$

$$\lim_{t\to\infty} V(\phi_t(x)) = 0$$

(why? because this means that for all $\beta > 0$ there exists a $T > 0$, such that for $t > T$, $|V(\phi_t(x))| < \beta$, i.e. $\phi_t(x) \in \Omega_\beta \subset N_\epsilon(0)$.)

Since V is a decreasing function along the trajectories (condition (b)) and bounded below, then

$$\lim_{t\to\infty} V(\phi_t(x)) = c \geq 0.$$

Assume $c > 0$. Let $\Omega_c = \{x \in B_r | V(x) \leq c\}$. By continuity of V and $V(0) = 0$, there exists a $d > 0$, such that $B_d = \{x \in \mathbb{R}^n | \, |x| \leq d\} \subset \Omega_c$. Since $\lim_{t\to\infty} V(\phi_t(x)) = c$, then $\phi_t(x)$ lies outside of B_d, i.e., $\phi_t(x)$ lies in the compact set $d \leq |x| \leq r$, $\dot{V}$ achieves its maximum in this set. Let $\alpha = -\max_{d\leq|x|\leq r} \dot{V}(x) > 0$. We have for $t > 0$

$$\begin{aligned} V(\phi_t(x)) &= V(\phi_0(x)) + \int_0^t \frac{d}{ds} V(\phi_s(x)) ds \\ &\leq V(\phi_0(x)) - \alpha t \\ &\Downarrow \text{ eventually} \\ V(\phi_t(x)) &< 0 \\ &\Downarrow \\ c &< 0 \end{aligned}$$

But we assumed $c > 0$, we have a contradiction.

Part (c) Reverse time (i.e., take $t = -t$) then one gets part (b).

Remarks:

- V satisfying the conditions of the theorem is called a Lyapunov function.
- The theorem allows to determine the stability of the equilibrium point without explicitly solving the differential equation. In a sense, since

  $$\dot{V}(x) = DV(x) f(x)$$

  the method converts a dynamics problem (i.e. determining the behavior of the trajectories over time), into a algebraic one (i.e., verifying inequalities of the form $F(x) > 0$, where F is some continuous function $F : \mathbb{R}^n \to \mathbb{R}$)
- One can think of the Lyapunov function as a generalization of the idea of the "energy" of a system. Then the method studies stability by looking at the rate of change of this "measure of energy".
- See [1] for a more detailed treatment of Lyapunov functions and nonlinear stability.
- The method does not show how to find a Lyapunov function V.
- **Definition:** The *region of attraction* (RoA) of an the equilibrium point at the origin for (1) is $\{x \in \mathbb{R}^n \mid \lim_{t\to\infty} \phi_t(x) = 0\}$.
  Ω_β are subsets of the RoA. This way we have a procedure for estimating the RoA (by maximizing β). More on this later.

Example 1 (Perko 9.5 (a) [2])

$$\begin{aligned}\dot{x} &= -x + y + xy \\ \dot{y} &= x - y - x^2 - y^3\end{aligned}$$

Lets try $V = x^2 + y^2$

$$\begin{aligned}\dot{V} &= 2x(-x + y + xy) + 2y(x - y - x^2 - y^3) \\ &= -2x^2 + 4xy - 2y^2 - 2y^4 \\ &= -2(x-y)^2 - 2y^4 \\ &< 0\end{aligned}$$

So the origin is asymptotically stable.

Example 2

- *Linear Harmonic Oscillator (spring mass)* $\ddot{x} + kx = 0$, $k > 0$. *Is the origin stable?*

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -kx\end{aligned}$$

  Energy $E(x,y) = PE + KE = \frac{1}{2}kx^2 + \frac{1}{2}y^2$. This is a good candidate for a Lyapunov function, i.e. $V(x,y) = E(x,y)$. Lets check:
  Let $D = \mathbb{R}^2$. First $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \in D \backslash (0,0)$.

$$\begin{aligned}\dot{V}(x,y) &= \frac{\partial}{\partial x}V\dot{x} + \frac{\partial}{\partial y}V\dot{y} \\ &= kxy - ykx \\ &= 0\end{aligned}$$

  So it is *stable*.

- *What happens if we add a "damping" term to the equation?*

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -kx - \epsilon y^3(1 + x^2)\end{aligned}$$

  First Jacobian $A = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$, $\lambda = \pm i\sqrt{k}$, so linear analysis not useful. Using same V we get

$$\begin{aligned}\dot{V}(x,y) &= \frac{\partial}{\partial x}V\dot{x} + \frac{\partial}{\partial y}V\dot{y} \\ &= kxy + y(-kx - \epsilon y^3(1 + x^2)) \\ &= -\epsilon y^4(1 + x^2)\end{aligned}$$

  So it is *stable* for $\epsilon > 0$. Can show, using LaSalle's Invariance Principle (coming up) that it is indeed *asymptotically stable* for $\epsilon > 0$ and *unstable* for $\epsilon < 0$.

3.3 Global Stability

Theorem (Barbashin-Krasovskii): Suppose $f \in C^1(\mathbb{R}^n)$ and that $f(0) = 0$. Suppose further that there exists a real valued function $V \in C^1(\mathbb{R}^n)$ satisfying $V(0) = 0$ and $V(x) > 0$ if $x \neq x_0$, and

$$|x| \to \infty \Rightarrow V(x) \to \infty$$
$$\dot{V}(x) < 0, \; \forall x \neq 0$$

then $x = 0$ is globally asymptotically stable.

Remark: The additional condition $|x| \to \infty \Rightarrow V(x) \to \infty$ guarantees that the level sets of $V(x)$ are bounded. Why?
For any $p \in \mathbb{R}^n$, let $c = V(p)$. Condition means that for any $c > 0$ there is $r > 0$ such that $|x| > r \Rightarrow V(x) > c$ (by definition; similar to definition of $\lim_{t\to\infty} f(t) = \infty$). $|x| > r \Rightarrow V(x) > c$ means that if $x \in \Omega_c$(i.e., $V(x) \le c$) then $x \in B_r$(i.e., $|x| \le r$), and therefore $\Omega_c \subset B_r$ and therefore bounded.

Example 3 (Strogatz 7.2.12 [3])

$$\begin{aligned} \dot{x} &= -x + 2y^3 - 2y^4 \\ \dot{y} &= -x - y + xy \end{aligned}$$

Lets try $V = x^{2m} + ay^{2n}$

$$\begin{aligned} \dot{V} &= 2mx^{2m-1}(-x + 2y^3 - 2y^4) + 2any^{2n-1}(-x - y + xy) \\ &= -2mx^{2m} + 4mx^{2m-1}y^3 - 4mx^{2m-1}y^4 - 2any^{2n-1}x + 2any^{2n}x - 2any^{2n} \\ &= -2mx^{2m} - 2any^{2n} + \left(4mx^{2m-1}y^3 - 4mx^{2m-1}y^4 - 2any^{2n-1}x + 2any^{2n}x\right) \\ \text{let m=1} \to \quad &= -2x^2 - 2any^{2n} + \left(4xy^3 - 4xy^4 - 2any^{2n-1}x + 2any^{2n}x\right) \\ \text{let n=2} \to \quad &= -2x^2 - 4ay^4 + \left(4xy^3 - 4xy^4 - 4ay^3x + 4ay^4x\right) \\ \text{let a=1} \to \quad &= -2x^2 - 4y^4 \end{aligned}$$

So $V = x^2 + y^4$ would work, and the origin is globally asymptotically stable.

LaSalle's Theorem (LaSalle's Invariance Principle): Let $\Omega \subset E$ be a compact set that is positively invariant with respect to the flow of (1). Suppose that there exists a real valued function $V \in C^1(E)$ such that $\dot{V}(x) \le 0$ in Ω. Let D_0 be the set of all points in Ω where $\dot{V}(x) = 0$, and M the largest invariant set in D_0. Then every solution starting in Ω approaches M as $t \to \infty$.

Remarks:

- The Theorem does not require that V is positive definite ($V(x) > 0,\ x \ne 0$).
- If $M = \{(0,0)\}$ then Ω is in the RoA of the origin.
- One can compute invariant subsets of the RoA of the origin by using the following Lemma:
  Lemma: If there exist a continuously differentiable real valued function V and $\beta > 0$, such that the level set $\Omega_{V,\beta} = \{x \in \mathbb{R}^n \mid V(x) \le \beta\}$ is bounded and

  $$\begin{aligned} &V(0) = 0,\ V(x) > 0,\ x \ne 0 \\ &\dot{V}(x) < 0,\ x \ne 0,\ x \in \Omega_{V,\beta} \end{aligned}$$

  then $\Omega_{V,\beta}$ is invariant and in the RoA of the origin.

Example 4 (Harmonic Oscillator) Let $\ddot{x} + \sin x = 0$.
a) Can you prove stability of the origin using linearization? Use an appropriate Lyapunov function to prove that the origin is a stable fixed point.
(b) Add a "damping term" $\ddot{x} + \epsilon\dot{x} + \sin x = 0$. Study the stability of the origin for $\epsilon > 0$.

Solution: (a)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x \end{aligned}$$

Lets look at fixed point $(0,0)$. Jacobian $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\lambda = \pm i$, so linear analysis not useful. Energy $E(x,y) = PE + KE = 1 - \cos x + \frac{1}{2}y^2$. This is a good candidate for a Lyapunov function, i.e., $V(x,y) = E(x,y)$. Lets check:
Let $E = (-\pi, \pi) \times \mathbb{R}$. First $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \in D \backslash (0,0)$.

$$\begin{aligned} \dot{V}(x,y) &= \frac{\partial}{\partial x} V \dot{x} + \frac{\partial}{\partial y} V \dot{y} \\ &= (\sin x) y - y \sin x \\ &= 0 \end{aligned}$$

So it is *stable*.

(b)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x - \epsilon y \end{aligned}$$

Using same$V$ we get

$$\begin{aligned} \dot{V}(x,y) &= \frac{\partial}{\partial x} V \dot{x} + \frac{\partial}{\partial y} V \dot{y} \\ &= (\sin x) y + y(-\sin x - \epsilon(1 - x^2) y) \\ &= -\epsilon y^2 \end{aligned}$$

Take then $\dot{V} \leq 0$ and $D_0 = \{(x,y) \in \mathbb{R}^2 \mid \dot{V}(x,y) = 0\} = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$. But since for $(x,y) \in D_0$ $(y = 0)$, $\dot{y} = -\sin x \neq 0$ for $x \neq k\pi$, $k \in \mathbb{Z}$, the largest invariant subset of D_0 is $M = \{(k\pi, 0),\ k = 0, \pm 1, \pm 2, \ldots\}$. I.e., the theorem states that the solutions will converge to one of the fixed points. If we choose $E = (-\pi, \pi) \times \mathbb{R}$, then we get $M = \{(0,0)\}$ and therefore the origin is asymptotically stable. Additionally, any solution starting in $\Omega_\beta = \{(x,y) \in E \mid V(x,y) \leq \beta\}$ will converge to 0. Ω_β is an estimate of the RoA of the origin.

References

[1] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3^{rd} edition, 2002.

[2] L. Perko. *Differential Equations and Dynamical Systems*. Springer, 3^{rd} edition, 2001.

[3] S. H. Strogatz. *Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering*. Westview Press, 2001.