1 The Stable Manifold Theorem

\[ \dot{x} = f(x) \]  
\[ \dot{x} = Df(x_0)x \]

We assume that the equilibrium point \( x_0 \) is located at the origin.

1.1 Some Examples

1.1.1 Example 1

Consider the linear system

\[ \dot{x}_1 = -x_1 \]
\[ \dot{x}_2 = 2x_2 \]

Clearly we have \( x_1(t) = a_1 e^{-t} \) and \( x_2(t) = a_2 e^{2t} \), with stable subspace \( E^s = span\{(1, 0)\} \) and unstable subspace \( E^u = span\{(0, 1)\} \). So \( \lim_{t \to \infty} \phi_t(a) = 0 \) only if \( a \in \mathbb{R}^s \). Consider a small perturbation of this linear system:

\[ \dot{x}_1 = -x_1 \]
\[ \dot{x}_2 = 2x_2 - 5\epsilon x_3^1 \]

The solution is given by \( x_1(t) = a_1 e^{-t} \) and \( x_2(t) = a_2 e^{2t} + a_2^3 \epsilon \left( e^{-3t} - e^{-2t} \right) = (a_2 - \epsilon a_1^3) e^{2t} + \epsilon a_1^3 e^{-3t} \). Clearly \( \lim_{t \to \infty} \phi_t(a) = 0 \) only if \( a_2 = \epsilon a_1^3 \). Indeed we can show that the set

\[ S = \{ x \in \mathbb{R}^2 \mid x_2 = \epsilon x_1^3 \} \]

is invariant with respect to the flow. It easy to see that \( a_2 = \epsilon a_1^3 \) leads to

\[ \phi_t(S) = \left[ \begin{array}{c}
  a_1 e^{-t} \\
  (a_2 - \epsilon a_1^3) e^{2t} + \epsilon a_1^3 e^{-3t}
\end{array} \right] \in S \]

So \( S \) is an invariant set (curve), and the flow on this curve is stable. So it seems that \( S \) is some nonlinear analog of \( E^s \). Furthermore, notice that \( S \) is tangent to the stable subspace of the linear system, and as \( \epsilon \to 0 \), the curve \( S \) becomes \( E^s \).

1.1.2 Example 2 (Perko 2.7 Example 1)

Consider

\[ \dot{x}_1 = -x_1 \]
\[ \dot{x}_2 = -x_2 + x_1^2 \]
\[ \dot{x}_3 = x_3 + x_1^2 \]

which we can rewrite as

\[ \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ x_2^1 \\ x_1^2 \end{bmatrix} \]
The flow is given by

\[ \phi_t(S) = \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} + a_3^2 \left( e^{-t} + e^{-2t} \right) \\ a_3 e^t + a_4^2 \left( e^t - e^{-2t} \right) \end{bmatrix} \]

where \( a = (a_1, a_2, a_3) = x(0) \). Clearly \( \lim_{t \to \infty} \phi_t(a) = 0 \) only if \( a_3 = -a_1^2/3 \). So

\[ S = \{ a \in \mathbb{R}^3 | a_3 = -a_1^2/3 \} \]

and similarly

\[ U = \{ a \in \mathbb{R}^3 | a_1 = a_2 = 0 \}. \]

Again it seems that \( S \) is some nonlinear analog of \( E^s \) and \( U \) is some nonlinear analog of \( E^u \). Furthermore, notice that \( S \) is tangent to the stable subspace of the linear system. We call \( S \) the stable manifold, and \( U \) the unstable manifold.

We are going to see how we can compute \( S \) and \( U \) in general.

### 1.2 Manifolds and stable manifold theorem

But first here is a “working” definition of a k-dimensional differential manifold. For more precise definition, there is a small section in the book, and CDS202 deals with differentiable manifolds in great details.

In this class, by **k-dimensional differential manifold** (or manifold of class \( C^m \)) we mean any “smooth” (of order \( C^m \)) k-dimensional surface in an n-dimensional space.

For example \( S = \{ a \in \mathbb{R}^3 | a_3 = -a_1^2/3 \} \) is 2-dimensional differentiable manifold.

**Theorem (The Stable Manifold Theorem):** Let \( E \) be an open subset of \( \mathbb{R}^n \) containing the origin, let \( f \in C^1(E) \), and let \( \phi_t \) be the flow of the non-linear system (1). Suppose that \( f(0) = 0 \) and that \( Df(0) \) has \( k \) eigenvalues with negative real part and \( n - k \) eigenvalues with positive real part. Then there exists a \( k \)-dimensional manifold \( S \) tangent to the stable subspace \( E^s \)of the linear system (2)at 0 such that for all \( t \geq 0 \), \( \phi_t(S) \subset S \) and for all \( x_0 \in S \),

\[ \lim_{t \to \infty} \phi_t(x_0) = 0; \]

and there exists an \( n - k \) differentiable manifold \( U \) tangent to the unstable subspace \( E^u \) of (2) at 0 such that for all \( t \leq 0 \), \( \phi_t(U) \subset U \) and for all \( x_0 \in U \),

\[ \lim_{t \to -\infty} \phi_t(x_0) = 0. \]

**Note:** As in the examples, since \( f \in C^1(E) \) and \( f(0) = 0 \), then system (1) can be written as

\[ \dot{x} = Ax + F(x) \]

where \( A = Df(0), F(x) = f(x) - Ax, F \in C^1(E), F(0) = 0 \) and \( DF(0) = 0 \).

Furthermore, we want to separate the stable and unstable parts of the matrix, i.e., choose a matrix \( C \) such that

\[ B = C^{-1}AC = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \]

where the eigenvalues of the \( k \times k \) matrix \( P \) have negative real part, and the eigenvalues of the \( (n-k) \times (n-k) \) matrix \( Q \) have positive real part. The transformed system \( (y = C^{-1}x) \) has the form

\[ \begin{align*}
\dot{y} &= By + C^{-1}F(Cy) \\
\dot{y} &= By + G(y)
\end{align*} \tag{3} \]
1.2.1 Calculating the stable manifold (Perko Method):

Perko shows that the solutions of the integral equation

\[
    u(t, a) = U(t)a + \int_0^t U(t-s)G(u(s, a))ds - \int_t^\infty V(t-s)G(u(s, a))ds
\]

satisfy (3) and \( \lim_{t \to \infty} u(t, a) = 0 \). Furthermore, it gives an iterative scheme for computing the solution:

\[
    u(t, a) = 0
\]

\[
    u^{(k+1)}(t, a) = U(t)a + \int_0^t U(t-s)G(u^{(k)}(s, a))ds - \int_t^\infty V(t-s)G(u^{(k)}(s, a))ds
\]

**Remark** Here is some intuition on why the particular integral equation is chosen. We basically want to remove the parts that blow up as \( t \to \infty \). In general, the solution of this system satisfies

\[
    u(t, a) = \left[ \begin{array}{cc}
    e^{P_t} & 0 \\
    0 & e^{Q_t}
    \end{array} \right] a + \int_0^t \left[ \begin{array}{cc}
    e^{P(t-s)} & 0 \\
    0 & e^{Q(t-s)}
    \end{array} \right] G(u(s, a))ds.
\]

Separate the convergent and non-convergent parts

\[
    u(t, a) = \left[ \begin{array}{cc}
    e^{P_t} & 0 \\
    0 & e^{Q_t}
    \end{array} \right] a + \int_0^t \left[ \begin{array}{cc}
    e^{P(t-s)} & 0 \\
    0 & e^{Q(t-s)}
    \end{array} \right] G(u(s, a))ds + \int_0^t \left[ \begin{array}{cc}
    0 & 0 \\
    0 & e^{Q(t-s)}
    \end{array} \right] G(u(s, a))ds
\]

Remove contributions that will cause it not to converge to the origin

\[
    u(t, a) = \left[ \begin{array}{cc}
    e^{P_t} & 0 \\
    0 & e^{Q_t}
    \end{array} \right] a + \int_0^t \left[ \begin{array}{cc}
    e^{P(t-s)} & 0 \\
    0 & e^{Q(t-s)}
    \end{array} \right] G(u(s, a))ds - \int_t^\infty \left[ \begin{array}{cc}
    0 & 0 \\
    0 & e^{Q(t-s)}
    \end{array} \right] G(u(s, a))ds
\]

Notice that last \( n-k \) components of \( a \) do not enter the computation, we can take them to be zero. Next we take the specific solution \( u(t, a) \)

\[
    u(t, a) = U(t)a + \int_0^t U(t-s)G(u(s, a))ds - \int_t^\infty V(t-s)G(u(s, a))ds
\]

and see what it implies for the initial conditions \( u(0, a) \). Notice that

\[
    u_j(0, a) = a_j, \quad j = 1, \ldots, k
\]

\[
    u_j(0, a) = -\left( \int_0^\infty V(-s)G(u(s, a))ds \right)_j, \quad j = k+1, \ldots, n
\]

So the last \( n-k \) components of the initial conditions satisfy

\[
    a_j = \psi_j(a_1, \ldots, a_k) := u_j(0, a_1, \ldots, a_k, 0, \ldots, 0), \quad j = k+1, \ldots, n.
\]

Therefore, the stable manifold is defined by

\[
    S = \{(y_1, \ldots, y_n) | y_j = \psi_j(y_1, \ldots, y_k), \quad j = k+1, \ldots, n\}.
\]
• The iterative scheme for calculating an approximation to $S$:
  
  - Calculate the approximate solution $u^{(m)}(t, a)$
  - For each $j = k + 1, \ldots, n$, $\psi_j(a_1, \ldots, a_k)$ is given by the $j$-th component of $u^{(m)}(0, a)$.

**Note:** Similarly can calculate $U$ by taking $t = -t$.

• Example:

$$\dot{x}_1 = -x_1 - x_2^2$$
$$\dot{x}_2 = x_2 + x_1^2$$

$$A = B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F(x) = G(x) = \begin{bmatrix} -x_2^2 \\ x_1^2 \end{bmatrix}$$

$$U = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 & e^t \\ 0 & 0 \end{bmatrix}, a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$$

Then

$$u^{(0)}(t, a) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u^{(1)}(t, a) = \begin{bmatrix} e^{-t}a_1 \\ 0 \end{bmatrix}$$

$$u^{(2)}(t, a) = \begin{bmatrix} e^{-t}a_1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{(t-s)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2s}a_2^2 \end{bmatrix} \, ds - \int_{-\infty}^0 \begin{bmatrix} 0 \\ 0 \\ e^{(t-s)} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2s}a_2^2 \end{bmatrix} \, ds = \begin{bmatrix} e^{-t}a_1 \\ -\frac{e^{-2t}}{3}a_2^2 \end{bmatrix}$$

$$u^{(3)}(t, a) = \begin{bmatrix} e^{-t}a_1 + \frac{1}{27}(e^{-4t} - e^{-t})a_1^3 \\ -\frac{e^{-2t}}{3}a_2^2 \end{bmatrix}$$

Next can show that $u^{(4)}(t, a) - u^{(3)}(t, a) = O(a_1^5)$ and therefore we can approximate by $\psi_2(a_1) = -\frac{1}{3}a_1^2 + O(a_1^5)$ and the stable manifold can be approximated by

$$S : x_2 = -\frac{1}{3}x_1^2 + O(x_1^3)$$

as $x_1 \to 0$. Similarly get

$$U : x_1 = -\frac{1}{3}x_2^2 + O(x_2^3)$$

1.2.2 **Note on invariant manifolds:**

Notice that if a manifold is specified by a constraint equation

$$y = h(x), \quad x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$$

and the dynamics given by

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

then condition

$$Dh(x)\dot{x} = \dot{y}$$

$$Dh(x)f(x, h(x)) = g(x, h(x))$$

suffices to show invariance. We’ll call this tangency condition. **Exercise:** Show that this is the case. If you’re going to use this in the homework this week, you should prove it first.
• Example:

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 2x_2 - 5\epsilon x_1^3
\end{align*}
\]

Show that the set

\[S = \{ x \in \mathbb{R}^2 | x_2 = \epsilon x_1^3 \}\]

is invariant. We have

\[3\epsilon x_1^2(-x_1) = 2\epsilon x_1^3 - 5\epsilon x_1^3.\]

1.2.3 Calculating the stable manifold (Alternative Method - Taylor expansion):

Let

\[y = h(x) = ax^2 + bx^3 + cx^4 + \ldots\]

Since invariant manifold we have:

\[Dh(x)\dot{x} - \dot{y} = 0\]

we can match coefficients. For example

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 2x_2 - 5\epsilon x_1^3
\end{align*}
\]

we get

\[x_2 = h(x_1) = ax_1^2 + bx_1^3 + O(x_1^4)\]

we get

\[f(x_1, h(x_1)) = -x_1, g(x_1, h(x_1)) \approx 2(ax_1^3 + bx_1^3) - 5\epsilon x_1^3\]

\[Dh(x)f(x, h(x)) = g(x, h(x))\]

\[\downarrow\]

\[(2ax_1 + 3bx_1^2 + \cdots)(-x_1) = 2ax_1^2 + 2bx_1^3 - 5\epsilon x_1^3 + \]

Matching terms we get

\[-2a = 2a \Rightarrow a = 0, -3b = 2b - 5\epsilon \Rightarrow b = \epsilon.\]
1.2.4 Example

\[ \dot{x}_1 = -x_1 \]
\[ \dot{x}_2 = 2x_2 + x_1^2 \]

**Perko method:**

\[
A = B = \begin{bmatrix}
-1 & 0 \\ 0 & 2
\end{bmatrix}, \quad F(x) = G(x) = \begin{bmatrix}
0 \\ x_1^2
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
e^{-t} & 0 \\ 0 & 0
\end{bmatrix}, \quad V = \begin{bmatrix}
0 & 0 & e^{2t} \\ 0 & e^{2t}
\end{bmatrix}, \quad a = \begin{bmatrix}
a_1 \\ 0
\end{bmatrix}
\]

Then

\[ u^{(0)}(t,a) = \begin{bmatrix}
0 \\ 0
\end{bmatrix} \]
\[ u^{(1)}(t,a) = \begin{bmatrix}
e^{-t}a_1 \\ 0
\end{bmatrix} \]
\[ u^{(2)}(t,a) = \begin{bmatrix}
e^{-t}a_1 \\ 0
\end{bmatrix} + \int_0^t \begin{bmatrix}
e^{-(t-s)}a_1 \\ 0
\end{bmatrix} ds - \int_t^\infty \begin{bmatrix}
e^{-2s}a_1^2 \\ 0
\end{bmatrix} ds - \begin{bmatrix}
e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2
\end{bmatrix} \]
\[ u^{(3)}(t,a) = \begin{bmatrix}
e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2
\end{bmatrix} \]

So \[ u^{(m)}(t,a) = \begin{bmatrix}
e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2
\end{bmatrix}, m \geq 2 \Rightarrow u(t,a) = \begin{bmatrix}
e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2
\end{bmatrix} \]

and therefore we get \( \psi_2(a_1) = (u(0,a))_2 = -\frac{1}{4}a_1^2 \) and the stable manifold is given by

\[ S : \quad x_2 = -\frac{1}{4}x_1^2 \]

as \( x_1 \to 0 \). What is the unstable manifold?

**Taylor expansion:**

\[ x_2 = h(x_1) = ax_1^2 + bx_1^3 + \cdots \]
\[ Dh(x_1) = 2ax_1 + 3bx_1^2 + \cdots \]
\[ f(x_1,h(x_1)) = -x_1 \]
\[ g(x_1,h(x_1)) = 2(ax_1^2 + bx_1^3 + \cdots) + x_1^2 \]

then

\[ Dh(x)f(x,h(x)) = g(x,h(x)) \]

\[ (2ax_1 + 3bx_1^2 + \cdots)(-x_1) = 2ax_1^2 + x_1^2 + 2bx_1^3 + \cdots \]

\[-2a = 2a + 1 \Rightarrow a = -\frac{1}{4} \]
\[-3b = 2b \Rightarrow b = 0 \]

...
and so
\[ S : x_2 = -\frac{1}{4}a_1^2. \]

**Direct Solution:**
\[ \phi_t = \left[ -\frac{1}{4}a_1^2 \left( e^{-t}a_1 - e^{2t} - a_2 e^{2t} \right) \right] \]

1.2.5 Global Manifolds

- In the proof \( S \) and \( U \) are defined in a small neighborhood of the origin, and are referred to as the *local* stable and unstable manifolds of the origin.

**Definition:** Let \( \phi_t \) be the flow of (1). The *global stable* and *unstable manifolds* of (1) at 0 are defined by
\[ W^s(0) = \cup_{t \leq 0} \phi_t(S) \]
and
\[ W^u(0) = \cup_{t \geq 0} \phi_t(S) \]
respectively.

The global stable and unstable manifold \( W^s(0) \) and \( W^u(0) \) are unique and invariant with respect to the flow. Furthermore, for all \( x \in W^s(0) \), \( \lim_{t \to \infty} \phi_t(x) = 0 \) and for all \( x \in W^u(0) \), \( \lim_{t \to -\infty} \phi_t(x) = 0 \).

**Corollary:** Under the hypothesis of the Stable Manifold theorem, if \( \Re(\lambda_j) < -\alpha < 0 < \beta < \Re(\lambda_m) \) for \( j = 1, \ldots, k \) and \( m = k + 1, \ldots, n \) then given \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x_0 \in N_{\delta}(0) \cap S \) then
\[ |\phi_t(x_0)| \leq \epsilon e^{-\alpha t} \]
for all \( t \geq 0 \) and if \( x_0 \in N_{\delta}(0) \cap U \) then
\[ |\phi_t(x_0)| \leq \epsilon e^{\beta t} \]
for all \( t \leq 0 \).

This shows that solutions starting in \( S \) sufficiently near the origin, approach the origin exponentially fast as \( t \to \infty \).

1.3 Center Manifold Theorem

**Theorem (The Center Manifold Theorem)** Let \( f \in C^r(E) \) where \( E \) is an open subset of \( \mathbb{R}^n \) containing the origin and \( r \geq 1 \). Suppose that \( f(0) = 0 \) and that \( Df(0) \) has \( k \) eigenvalues with negative real part, \( j \) eigenvalues with positive real part, and \( m = n - k - j \) eigenvalues with zero real part. Then there exists an \( m \)-dimensional center manifold \( W^c(0) \) of class \( C^r \) tangent to the center subspace \( E^c \) of (2) at 0, there exists an \( k \)-dimensional center manifold \( W^s(0) \) of class \( C^r \) tangent to the stable subspace \( E^s \) of (2) at 0, and there exists an \( j \)-dimensional center manifold \( W^u(0) \) of class \( C^r \) tangent to the unstable subspace \( E^u \) of (2) at 0; furthermore, \( W^c(0) \), \( W^s(0) \) and \( W^u(0) \) are invariant under the flow \( \phi_t \) of (1).

2 The Hartman-Grobman Theorem

**Definition:**
- Let \( X \) be a metric space (such as \( \mathbb{R}^n \)) and let \( A \) and \( B \) be subsets of \( X \). A *homeomorphism* of \( A \) onto \( B \) is a continuous one-to-one map of \( A \) onto \( B \), \( h : A \to B \), such that \( h^{-1} : B \to A \) is continuous.
- The sets \( A \) and \( B \) are called *homeomorphic* or *topologically equivalent* if there is a homeomorphism of \( A \) onto \( B \).
• Two autonomous systems of differential equations such as (1) and (2) are said to be \textit{topologically equivalent} in a neighborhood of the origin, or to have the \textit{same qualitative structure near the origin} if there is a homeomorphism \( H \) mapping an open set \( U \) containing the origin onto a set \( V \) containing the origin, which maps trajectories of (1) in \( U \) onto trajectories of (2) in \( V \) and preserves their orientation by time.

**Theorem (The Hartman-Grobman Theorem)** Let \( f \in C^1(E) \) where \( E \) is an open subset of \( \mathbb{R}^n \) containing the origin, and \( \phi \), the flow of (1). Suppose that \( f(0) = 0 \) and that \( Df(0) \) has no eigenvalues with zero real part. Then there is a homeomorphism \( H \) of an open set \( U \) containing the origin onto a set \( V \) containing the origin such that for each \( x_0 \in U \), there is an open interval \( I_0 \subset \mathbb{R} \) containing zero such that for all \( x_0 \in U \) and \( t \in I_0 \)

\[
H \circ \phi_t(x_0) = e^{At}H(x_0);
\]
i.e., (1) and (2) are topologically equivalent in a neighborhood of the origin.

**Example:** The systems

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-x_1 \\
x_2 + x_1^2
\end{bmatrix}
\text{ and }
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

are topologically equivalent. Let \( x_0 = (a_1, a_2) \)

\[
H(x) = \begin{bmatrix}
-x_1 \\
x_2 + \frac{1}{3}x_1^2
\end{bmatrix}
\]

Then

\[
e^{At}H(x_0) = \begin{bmatrix}
e^{-t} & 0 \\
0 & e^t
\end{bmatrix} \begin{bmatrix}
-a_1 & -a_1 e^{-t} \\
a_2 + \frac{1}{3}a_1^2 & (a_2 + \frac{1}{3}a_1^2) e^t
\end{bmatrix} = \begin{bmatrix}
-a_1 e^{-t} & -a_1 e^{-t} \\
a_2 + \frac{1}{3}a_1^2 & (a_2 + \frac{1}{3}a_1^2) e^t
\end{bmatrix}
\]

\[
H \circ \phi_t(x_0) = H\left( \begin{bmatrix}
-a_1 e^{-t} \\
a_2 + \frac{1}{3}a_1^2
\end{bmatrix} \right) = \begin{bmatrix}
-a_1 e^{-t} \\
a_2 + \frac{1}{3}a_1^2
\end{bmatrix} \begin{bmatrix}
-e^t - \frac{1}{3}a_1^2 e^{-2t} + \frac{1}{3}a_1^2 e^{-2t} \\
\end{bmatrix} = \begin{bmatrix}
-a_1 e^{-t} \\
a_2 + \frac{1}{3}a_1^2
\end{bmatrix}
\]

**Remarks:**

• Perko gives an outline of the proof and gives a method using successive approximations for calculating \( H \).

• However, computationally not very useful since to compute \( H \) by this method requires solving for the flow \( \phi_t \) first.

• Conceptually, it is extremely useful since knowing that such \( H \) exists (without needing to compute it), allows us to determine the qualitative behavior of nonlinear systems near a hyperbolic equilibrium point by simply looking at the linearization (without solving it).

### 3 Stability and Lyapunov Functions

**Definition:**

• An equilibrium point \( x_0 \) of (1) is \textit{stable} if for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x \in N_\delta(x_0) \) and \( t \geq 0 \), we have \( \phi_t(x) \in N_\epsilon(x_0) \).

• An equilibrium point \( x_0 \) of (1) is \textit{unstable} if it is not stable.

• An equilibrium point \( x_0 \) of (1) is \textit{asymptotically stable} if it is stable and if there exists a \( \delta > 0 \) such that for all \( x \in N_\delta(x_0) \) we have \( \lim_{t \to \infty} \phi_t(x) = x_0 \).

**Remarks:**

• The about limit being satisfied does not imply that \( x_0 \) is stable (why?).
• From H-G theorem and Stable manifold theorem, it follows that hyperbolic equilibrium points are either asymptotically stable (sinks) or unstable (sources or saddles).

• If \( x_0 \) is stable then no eigenvalue of \( Df(x_0) \) has positive real part (why?)

• \( x_0 \) is stable but not asymptotically stable, then \( x_0 \) is a non-hyperbolic equilibrium point

**Example: Perko 2.9.2 (c)** Determine stability of the equilibrium points of:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-4x_1 - 2x_2 + 4 \\
x_1 x_2
\end{bmatrix}
\]

Equilibrium points are \((0, 2), (1, 0)\).

\[
Df(x) = \begin{bmatrix}
-4 & -2 \\
x_2 & x_1
\end{bmatrix}
\]

\[
Df(0, 2) = \begin{bmatrix}
-4 & -2 \\
2 & 0
\end{bmatrix}
\]

\[
Df(1, 0) = \begin{bmatrix}
-4 & -2 \\
0 & 1
\end{bmatrix}
\]

What can we say in general about the stability of non-hyperbolic equilibrium points?

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-x_2 - x_1 x_2 \\
x_1 + x_1^2
\end{bmatrix}
\]