8 Remark on Invariance of Manifolds

Let

$$\begin{array}{rcl} \dot{x} & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{array} \tag{12}$$

for $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$, $k+m=n$, and let a manifold S be defined by

$$S = \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^m \mid y = h(x)\}.$$

Proposition: If

$$g(x,h(x)) = Dh(x)f(x,h(x)) \tag{13}$$

then S is an invariant manifold of (12).

Proof: Let

$$(x,\ y) = (x,\ h(x))$$

be a point on S. The vector field at this point is given by

$$(f(x,h(x),\ g(x,h(x))$$

and a tangent vector to the manifold at this point is given by $\overrightarrow{\tau} = (1,\ Dh(x))$, and therefore a vector normal to the surface S is

$$\overrightarrow{n} = (-Dh(x),\ 1)$$

(since $\overrightarrow{\tau} \cdot \overrightarrow{n} = 0$). Since

$$\begin{array}{rcl} \overrightarrow{n} \cdot (f(x,h(x),\ g(x,h(x)) & = & (-Dh(x),\ 1) \cdot (f(x,h(x),\ g(x,h(x)) \\ & = & g(x,h(x)) - Dh(x)f(x,h(x)) \\ & = & 0, \end{array}$$

we have that the vector field at any point on S is tangent to S, and therefore S is invarant with respect to the flow of (12).