CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Winter 2010 | | Due: 19 Jan 10

Note: Please put the number of hours that you spent on this homework set (including reading) on the back of the first page of your homework.

1. (OBC, 2.8) Consider the control system transfer function

$$H(s) = \frac{s+b}{s(s+a)}, \qquad a, b > 0$$

with state space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} b & 1 \end{bmatrix} x$$

and performance criterion

$$V = \int_0^\infty (x_1^2 + u^2) dt.$$

(a) Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

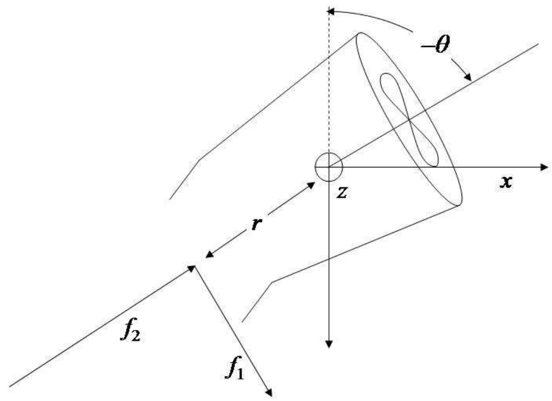
with $p_{12} = p_{21}$ and $P > 0$ (positive definite). Write the steady state Riccati equation as a system of four explicit equations in terms of the elements of P and the constants a and b.

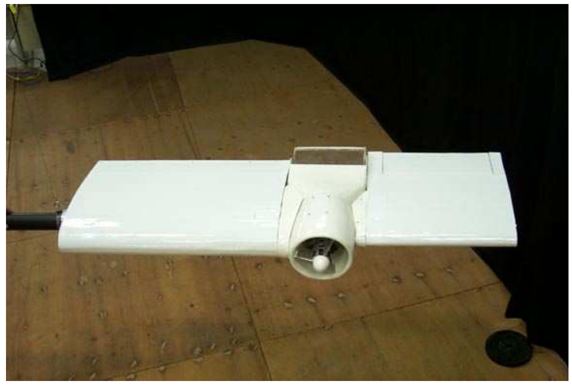
(b) Find the gains for the optimal controller assuming the full state is available for feedback.

(c) Find the closed loop natural frequency and damping ratio.

2. For the ducted fan described below, design an LQR controller assuming that the full state is available for feedback. The purpose of this exercise is to explore how different weighting matrices affect the performance of the control.

The ducted fan experiment is representative of a small model aircraft powered by a vectored thrust engine, as shown below. Note that the center of mass is aligned with the pivot point.





In hover, the thrust balances the gravitational force, so $f_2 = m\gamma$, and the control input is the side force $u = f_1$ produced by vectoring the thrust. The linearized dynamics about hover are described by

$$m\ddot{x} = -m\gamma\theta + u$$
$$J\ddot{\theta} = ru$$

where J is the moment of inertia, m is the inertial mass in the horizontal direction, θ the angle of the fan with respect to vertical, x the horizontal displacement of the centre of mass, u the net side force produced by vectoring the thrust (f_1 in the figure), and r the distance from the centre of mass to the point where the thrust is applied. Because of the counter-weight, the effective gravitational force is reduced ($\gamma \ll 9.81\,\mathrm{m/s^2}$).

For parameters, use

$$J = 0.13 \text{ kg m}^2 \qquad r = 38.5 \text{ cm}$$
$$m\gamma = 7 \text{ kg m/sec}^2 \qquad m = 8.5 \text{ kg}$$

(a) Choose a diagonal weighting matrix for the state so that the cost function penalizes $q_x x^2 + q_{xd}\dot{x}^2 + q_\theta \theta^2 + q_{\theta d}\dot{\theta}^2 + q_u u^2$. Solve for the optimal gains, e.g. with the `lqr` command in Matlab and compute the closed-loop eigenvalues for a variety of choices of weights (`lqr` returns these also). What do you notice as you (i) decrease q_u keeping all other weightings constant, (ii) decrease q_{qd} and $q_{\theta d}$ while keeping non-zero weight on q_x and q_θ? You should plot how the eigenvalues vary for the above, but you do need to list the gains or eigenvalues for each case.

(b) Can you set $q_{xd} = q_\theta = q_{\theta d} = 0$ so that only q_x and q_u are non-zero and find a solution? Can you set $q_x = q_{xd} = q_{\theta d} = 0$ so that only q_θ and q_u are non-zero and find a solution? Why or why not?

(c) For a reference state input $\underline{x}_d$ the control can be written as $u = -K(\underline{x} - \underline{x}_d)$. Plot the step response to a command input in position x_d for several different choices of weighting (e.g. explore the options in part (a) above). If the full state vector is

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \text{then} \qquad \underline{x}_d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_d$$